Nečas Center for Mathematical Modeling

A generalization of some regularity criteria to the Navier-Stokes equations involving one velocity component

Š. Axmann, M. Pokorný

Preprint no. 2014-11

http://ncmm.karlin.mff.cuni.cz/

A generalization of some regularity criteria to the Navier–Stokes equations involving one velocity component

Šimon Axmann and Milan Pokorný

To Yoshihiro Shibata

Mathematics Subject Classification (2010). Primary 35Q30; Secondary 76D05.

Keywords. Incompressible Navier–Stokes equations, global regularity, regularity criteria.

Abstract. We present generalizations of results concerning conditional global regularity of weak Leray–Hopf solutions to incompressible Navier–Stokes equations presented by Zhou and Pokorný in articles [15], [17], and [18]; see also [13]. We are able to replace the condition on one velocity component (or its gradient) by a corresponding condition imposed on a projection of the velocity (or its gradient) onto a more general vector field. Comparing to our other recent results from [1], the conditions imposed on the projection are more restrictive here, however due to the technique used in [1], there appeared a specific additional restriction on geometrical properties of the reference field, which could be omitted here.

1. Introduction

We consider the Cauchy problem for the instationary incompressible Navier–Stokes equations in the full three space dimensions

$$\left.\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} \\ \operatorname{div}\mathbf{v} = 0 \end{aligned}\right\} \text{ in } (0,T)\times\mathbb{R}^3, \qquad (1)$$

$$\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3,$$

where $\mathbf{v} : (0,T)\times\mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field, $p : (0,T)\times\mathbb{R}^3 \to \mathbb{R}$ is the pressure, $\mathbf{f} : (0,T)\times\mathbb{R}^3 \to \mathbb{R}^3$ is the given density of external forces, and $\nu > 0$ is given kinematic viscosity. For the sake of simplicity, we set $\nu = 1$

and $\mathbf{f} \equiv \mathbf{0}$ in our further considerations. Indeed, the actual value of viscosity does not play any role. It would be also possible to formulate some suitable assumptions on the regularity of $\mathbf{f}$ in such a way that our main results remain true. However, it would lead to unnecessary technicalities which we prefer to omit here.

The mathematical theory of Navier–Stokes equations has long, interesting history (see e.g. [16]). In the celebrated works of Leray [11], [12] and Hopf [6] the existence of weak solutions to system (1) in space $\mathbf{v} \in L^2(0,T,(W^{1,2}_{\text{div}}(\mathbb{R}^3))) \cap L^\infty(0,T,(L^2(\mathbb{R}^3)))$ for any given $\mathbf{v}_0 \in L^2_{\text{div}}(\mathbb{R}^3)$ was proved; they satisfy energy inequality. Further, for $\mathbf{v}_0 \in W^{1,2}_{\text{div}}(\mathbb{R}^3)$ the existence of (possibly short) time interval $(0,T^*)$ such that there exists a unique strong solution in space $\mathbf{v} \in L^2(0,T^*,(W^{2,2}(\mathbb{R}^3))) \cap L^\infty(0,T^*,(W^{1,2}_{\text{div}}(\mathbb{R}^3)))$ was established (see [10]). The uniqueness and regularity of Leray–Hopf weak solutions is still a challenging open problem [9]. For overview of known results see e.g. [4].

On the other hand, there were established many criteria ensuring the smoothness of the solution under additional assumptions concerning the velocity and its components, the gradient of the velocity and its components, the pressure, the vorticity, or other quantities.

During the last decade, an interesting progress was achieved in the field of regularity criteria concerning only one velocity component. The very first result in this direction is criterion proved by Neustupa and Penel [14], which ensures the regularity for $v_3 \in L^\infty\left(0,T,\left(L^\infty(\mathbb{R}^3)\right)\right)$. Similar result for the gradient of one velocity component $\left(\nabla v_3 \in L^t\left(0,T,\left(L^s(\mathbb{R}^3)\right)\right), \frac{2}{t} + \frac{3}{s} \leq 1,\ s \geq 3\right)$ is due to He [5]. These pioneering results were then improved by Neustupa et al. [13] $\left(v_3 \in L^t\left(0,T,\left(L^s(\mathbb{R}^3)\right)\right), \frac{2}{t} + \frac{3}{s} \leq \frac{1}{2},\ s \geq 2\right.$, as local criterion for suitable weak solution), and Pokorný [15] $\left(\nabla v_3 \in L^t\left(0,T,\left(L^s(\mathbb{R}^3)\right)\right), \frac{2}{t} + \frac{3}{s} \leq \frac{3}{2},\ s \geq 2.\right)$, observing the equation for vorticity; the same results were obtained also by Zhou [17], [18]. Further improvements were later done via several techniques by Kukavica and Ziane [8], Cao and Titi [2], and finally Zhou and Pokorný [19], [20]. Note that the results in [1] contain generalization of these criteria. However, due to the used technique (multiplicative Gagliardo–Nirenberg inequality which has to be generalized) we get additional geometrical restrictions on the field $\mathbf{b}$ which can be avoided in our present paper.

Notation

In the whole paper, the standard notation for Lebesgue spaces $L^p(\mathbb{R}^3)$ with the norm $\|\cdot\|_p$ will be used. For the sake of brevity, we will denote the norm on Bochner spaces $L^p\left(0,t,\left(L^q(\mathbb{R}^3)\right)\right)$ by $\|\cdot\|_{p,q}$, the lenght of the time interval will be everywhere clear from the context. We will also use the same notation for scalar spaces X and their vector analogues X^N. All generic constants will be denoted by C, although its value may differ from line to line, or even in the same formula. We will use Einstein summation convention over repeated indices.

2. Main results

As we have already mentioned above, our main goal is to generalize the results of Zhou and Pokorný from articles [15], [17], and [18] (the latter proved originally for suitable weak solution in [13]).

Theorem 1. *Let* $\mathbf{v}$ *be a weak Leray–Hopf solution to the Navier–Stokes equations corresponding to initial datum* $\mathbf{v}_0 \in W^{1,2}_{\mathrm{div}}(\mathbb{R}^3)$. *Assume moreover that there exist* $\delta > 0$, *and a vector field* $\mathbf{b}(t,\mathbf{x}) : (0,T) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ *such that*

$$\nabla \mathbf{b} \in L^\infty\left(0,T,\left(L^\infty(\mathbb{R}^3)\right)\right),$$
$$\frac{\partial \mathbf{b}}{\partial t},\ \nabla^2 \mathbf{b} \in L^\infty\left(0,T,\left(L^3(\mathbb{R}^3)\right)\right),$$

and $|\mathbf{b}(t,\mathbf{x})| \geq \delta$ *such that the projection of the velocity*

$$v_{\mathbf{b}}(t,\mathbf{x}) := \mathbf{b}(t,\mathbf{x}) \cdot \mathbf{v}(t,\mathbf{x})$$

satisfies either

$$v_{\mathbf{b}}(t,\mathbf{x}) \in L^t\left(0,T,\left(L^s(\mathbb{R}^3)\right)\right),\ \frac{2}{t} + \frac{3}{s} \leq \frac{1}{2},\ 6 \leq s \leq \infty$$

or

$$\nabla v_{\mathbf{b}}(t,\mathbf{x}) \in L^t\left(0,T,\left(L^s(\mathbb{R}^3)\right)\right),\ \frac{2}{t} + \frac{3}{s} \leq \frac{3}{2},\ 2 \leq s \leq \infty.$$

Then $\mathbf{v}$ *is actually a strong solution to the Navier–Stokes equations in the interval* $[0,T]$.

Since the proofs of both cases have lot of similarities, we will prove them simultaneously. It is well known that there exists a unique strong solution to (1) on (possibly short) time interval $[0,T^*)$, we will work with this strong solution and show that actually $T^* \geq T$. The result concerning so-called weak-strong uniqueness will then yield the desired result. Let us denote space $Y(\tau) := L^\infty(0,\tau;L^2(\mathbb{R}^3)) \cap L^2(0,\tau;W^{1,2}(\mathbb{R}^3))$. Our first step in Lemma 1 will be to derive a suitable estimate of $\mathbf{b}\cdot\boldsymbol{\omega}$ in the norm of space Y, then we will test equation (1) by an analogue of the quantity $-\Delta\mathbf{v}$ and get the desired estimate of $\nabla\mathbf{v}$ using Lemma 1.

Lemma 1. *Let* $\mathbf{v}$ *be a strong solution to the Navier–Stokes equations corresponding to the initial condition* $\mathbf{v}_0 \in W^{1,2}_{\mathrm{div}}(\mathbb{R}^3)$. *Suppose that the assumptions of Theorem* 1 *are satisfied. Let* $0 < \tau < T^*$. *Then* $\omega_{\mathbf{b}} := \mathbf{b}\cdot\boldsymbol{\omega}$ *can be estimated on* $(0,\tau)$ *by* $\|\omega_{\mathbf{b}}\|_{Y(\tau)}$ *as follows*

$$\|\omega_{\mathbf{b}}\|^2_{\infty,2} + \|\nabla\omega_{\mathbf{b}}\|^2_{2,2} \leq \|\omega_{\mathbf{b}}(0)\|^2_2 + C(\tau)\,\left(1 + \|\nabla\mathbf{v}\|_{Y(\tau)}\right). \qquad (2)$$

In particular, if $\tau \to 0^+$, *then* $C(\tau) \to 0$ *and if* $\tau \to (T^*)^-$, *then* $C(\tau)$ *remains bounded.*

3. Proof of the lemma

By possible decreasing the value of t we could easily achieve that $\frac{2}{t}+\frac{3}{s}=\frac{1}{2}$ (or $\frac{3}{2}$, respectively). Applying the curl operator on (1) we get

$$\frac{\partial \omega_i}{\partial t} + \mathbf{v}\cdot\nabla\omega_i = \boldsymbol{\omega}\cdot\nabla v_i + \Delta\omega_i, \qquad i=1,2,3.$$

Multiplying these equations by $b_i(t,x)$

$$b_i\frac{\partial \omega_i}{\partial t} + b_i\mathbf{v}\cdot\nabla\omega_i = b_i(\boldsymbol{\omega}\cdot\nabla v_i) + (\Delta\omega_i)b_i,$$

summing up $\sum\limits_{i=1}^{3}$, and multiplying the arisen equation by $\omega_{\mathbf{b}} = \sum\limits_{i=1}^{3} b_i\omega_i$, we get four terms which could be rewritten in the following way

$$\begin{aligned}
&\omega_{\mathbf{b}} b_i\frac{\partial \omega_i}{\partial t} = \omega_{\mathbf{b}}\frac{\partial \omega_{\mathbf{b}}}{\partial t} - \omega_{\mathbf{b}}\frac{\partial b_i}{\partial t}\omega_i,\\
&b_i(\mathbf{v}\cdot\nabla\omega_i)\omega_{\mathbf{b}} = (\mathbf{v}\cdot\nabla\omega_{\mathbf{b}})\omega_{\mathbf{b}} - (\mathbf{v}\cdot\nabla b_i)\omega_i\omega_{\mathbf{b}},\\
&b_i(\boldsymbol{\omega}\cdot\nabla v_i)\omega_{\mathbf{b}} = (\boldsymbol{\omega}\cdot\nabla v_{\mathbf{b}})\omega_{\mathbf{b}} - (\boldsymbol{\omega}\cdot\nabla b_i)v_i\omega_{\mathbf{b}},\\
&(\Delta\omega_i)b_i\omega_{\mathbf{b}} = (\Delta\omega_{\mathbf{b}} - 2\nabla b_i\cdot\nabla\omega_i - \omega_i\Delta b_i)\,\omega_{\mathbf{b}};
\end{aligned}$$

whence

$$\begin{aligned}
&\frac{\partial \omega_{\mathbf{b}}}{\partial t}\omega_{\mathbf{b}} + \mathbf{v}\cdot\nabla\omega_{\mathbf{b}}\omega_{\mathbf{b}} = \boldsymbol{\omega}\cdot\nabla v_{\mathbf{b}}\omega_{\mathbf{b}} + \Delta\omega_{\mathbf{b}}\omega_{\mathbf{b}}\\
&+\underbrace{\omega_i\frac{\partial b_i}{\partial t}\omega_{\mathbf{b}} + \omega_i(\mathbf{v}\cdot\nabla b_i)\omega_{\mathbf{b}} - v_i(\boldsymbol{\omega}\cdot\nabla b_i)\omega_{\mathbf{b}} - \Delta b_i\omega_i\omega_{\mathbf{b}} - 2(\nabla\omega_i\cdot\nabla b_i)\omega_{\mathbf{b}}}_{=:\mathrm{I}}.
\end{aligned}$$

Integration over the whole $\mathbb{R}^3$ with integration by parts gives us

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{\mathbf{b}}\|_2^2 + \|\nabla\omega_{\mathbf{b}}\|_2^2 \le -\underbrace{\int_{\mathbb{R}^3}\mathbf{v}\cdot\nabla\omega_{\mathbf{b}}\omega_{\mathbf{b}}\mathrm{d}\mathbf{x}}_{=0} + \int_{\mathbb{R}^3}\boldsymbol{\omega}\cdot\nabla v_{\mathbf{b}}\omega_{\mathbf{b}}\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3} I\mathrm{d}\mathbf{x}.$$

The lower order terms I could be easily estimated using again integration by parts and Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Big|\omega_i\frac{\partial b_i}{\partial t}\omega_{\mathbf{b}} - v_i(\boldsymbol{\omega}\cdot\nabla b_i)\omega_{\mathbf{b}} + \Delta b_i\omega_i\omega_{\mathbf{b}}\\
&\qquad\qquad + 2(\omega_i\nabla b_i)\cdot\nabla\omega_{\mathbf{b}} + \omega_i(\mathbf{v}\cdot\nabla b_i)\omega_{\mathbf{b}}\Big|\mathrm{d}\mathbf{x}\\
&\le C_\varepsilon\|\nabla\mathbf{v}\|_2^2\left[\left\|\frac{\partial\mathbf{b}}{\partial t}\right\|_3^2 + \|\Delta\mathbf{b}\|_3^2 + 2\|\nabla\mathbf{b}\|_\infty^2\right] + 3\varepsilon\|\nabla\omega_{\mathbf{b}}\|_2^2\\
&\qquad + 2\|\nabla\mathbf{b}\|_\infty\int_{\mathbb{R}^3}|\boldsymbol{\omega}|\,|\mathbf{v}|\,|\omega_{\mathbf{b}}|\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

In the last integral we will use Hölder's inequality, interpolation, and Young's inequality

$$\|\nabla \mathbf{b}\|_\infty \int_{\mathbb{R}^3} |\boldsymbol{\omega}|\,|\mathbf{v}|\,|\omega_{\mathbf{b}}|\,\mathrm{d}\mathbf{x} \leq C\,\|\nabla \mathbf{b}\|_\infty \|\nabla \mathbf{v}\|_2 \|\mathbf{v}\|_3 \|\omega_{\mathbf{b}}\|_6$$
$$\leq \varepsilon \|\nabla \omega_{\mathbf{b}}\|_2^2 + C_\varepsilon \|\nabla \mathbf{b}\|_\infty^2 \|\nabla \mathbf{v}\|_2^2 \|\mathbf{v}\|_3^2 ,$$

where

$$\int_0^t \|\mathbf{v}\|_3^2 \|\nabla \mathbf{v}\|_2^2 \,\mathrm{d}\tau \leq \|\nabla \mathbf{v}\|_{4,2}^2 \|\mathbf{v}\|_{4,3}^2 \leq \|\nabla \mathbf{v}\|_{2,2} \|\mathbf{v}\|_{4,3}^2 \|\nabla \mathbf{v}\|_{\infty,2} .$$

Now, we will estimate the leading terms, distinguishing two considered cases.

1. Assume the projection $v_{\mathbf{b}}$ has better integrability properties. Then

$$\int_{\mathbb{R}^3} \boldsymbol{\omega}\cdot\nabla v_{\mathbf{b}}\omega_{\mathbf{b}}\mathrm{d}\mathbf{x} \leq \left|-\int_{\mathbb{R}^3} v_{\mathbf{b}}\boldsymbol{\omega}\cdot\nabla\omega_{\mathbf{b}}\mathrm{d}\mathbf{x}\right| \leq \varepsilon \|\nabla \omega_{\mathbf{b}}\|_2^2 + C_\varepsilon \int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 {v_{\mathbf{b}}}^2 \mathrm{d}\mathbf{x}$$
$$\leq \varepsilon \|\nabla \omega_{\mathbf{b}}\|_2^2 + 2C_\varepsilon \int_{\mathbb{R}^3} |\nabla \mathbf{v}|^2 {v_{\mathbf{b}}}^2 \mathrm{d}\mathbf{x}$$

$$\int_{\mathbb{R}^3} {v_{\mathbf{b}}}^2 |\nabla \mathbf{v}|^2 \,\mathrm{d}\mathbf{x} \leq \|v_{\mathbf{b}}\|_s^2 \|\nabla \mathbf{v}\|_{\frac{2s}{s-2}}^2$$
$$\leq \|v_{\mathbf{b}}\|_s^2 \|\nabla \mathbf{v}\|_{\frac{2s}{s-4}} \|\nabla \mathbf{v}\|_2$$
$$\leq C\,\|v_{\mathbf{b}}\|_s^2 \|\nabla \mathbf{v}\|_2^{1-\frac{6}{s}} \|\Delta \mathbf{v}\|_2^{\frac{6}{s}} \|\nabla \mathbf{v}\|_2$$
$$= C\,\|v_{\mathbf{b}}\|_s^2 \|\nabla \mathbf{v}\|_2^{\frac{4}{t}} \|\Delta \mathbf{v}\|_2^{\frac{6}{s}} \|\nabla \mathbf{v}\|_2 ,$$

(recall $\frac{2}{t}+\frac{3}{s}=\frac{1}{2}$) which gives using assumptions on $\mathbf{b}$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{\mathbf{b}}\|_2^2 + \|\nabla \omega_{\mathbf{b}}\|_2^2 \leq C\,\|v_{\mathbf{b}}\|_s^2 \|\nabla \mathbf{v}\|_2^{\frac{4}{t}} \|\Delta \mathbf{v}\|_2^{\frac{6}{s}} \|\nabla \mathbf{v}\|_2$$
$$+ C(\mathbf{b})\,\|\nabla \mathbf{v}\|_2^2 (1 + \|\mathbf{v}\|_3^2).$$

Integrating over time interval $(0,\tau)$, with usage of Hölder's inequality then yields

$$\|\omega_{\mathbf{b}}(\tau)\|_2^2 + \int_0^\tau \|\nabla \omega_{\mathbf{b}}\|_{2,2}^2 \,\mathrm{d}\sigma$$
$$\leq \|\omega_{\mathbf{b}}(0)\|_2^2 + C\int_0^\tau \|v_{\mathbf{b}}\|_s^2 \|\nabla \mathbf{v}\|_2^{\frac{4}{t}} \|\Delta \mathbf{v}\|_2^{\frac{6}{s}} \|\nabla \mathbf{v}\|_2 \,\mathrm{d}\sigma$$
$$+ C(\mathbf{b}) \int_0^\tau (\|\nabla \mathbf{v}\|_2^2 + \|\mathbf{v}\|_3^2 \|\nabla \mathbf{v}\|_2^2)\mathrm{d}\sigma.$$

Recall that from the energy inequality we have estimate of $\mathbf{v}$ in the spaces $L^2(0,T,(W^{1,2}_{\mathrm{div}}(\mathbb{R}^3)))$ and $L^\infty(0,T,(L^2(\mathbb{R}^3)))$, hence due to the interpolation also in $L^4(0,T,(L^3(\mathbb{R}^3)))$. Thus

$$\|\omega_{\mathbf{b}}(\tau)\|_2^2 + \int_0^\tau \|\nabla\omega_{\mathbf{b}}\|_{2,2}^2 \,\mathrm{d}\sigma \le \|\omega_{\mathbf{b}}(0)\|_2^2 + C(\mathbf{b}) \left[\|v_{\mathbf{b}}\|_{t,s}^2 \|\nabla\mathbf{v}\|_{\infty,2}^{\frac{4}{t}} \|\Delta\mathbf{v}\|_{2,2}^{\frac{6}{s}} + 1\right] \|\nabla\mathbf{v}\|_{2,2}^2 + C(\mathbf{b}) \|\nabla\mathbf{v}\|_{2,2} \|\mathbf{v}\|_{4,3}^2 \|\nabla\mathbf{v}\|_{\infty,2} \quad (3)$$

which yields the conclusion of Lemma in the first case as $\frac{4}{t} + \frac{6}{s} = 1$.

2. For a given $2 \le s \le \infty$, we will find $2 \le p \le 6$, $2 \le q \le 3$ such that $\frac{1}{s} + \frac{1}{p} + \frac{1}{q} = 1$. We will use gradually Hölder's inequality, interpolation, and Young's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \boldsymbol{\omega}\cdot\nabla v_{\mathbf{b}}\omega_{\mathbf{b}}\mathrm{d}\mathbf{x} &\le \|\nabla v_{\mathbf{b}}\|_s \|\omega_{\mathbf{b}}\|_p \|\boldsymbol{\omega}\|_q \\ &\le \|\nabla v_{\mathbf{b}}\|_s \|\omega_{\mathbf{b}}\|_2^{\frac{6-p}{2p}} \|\omega_{\mathbf{b}}\|_6^{\frac{3p-6}{2p}} \|\boldsymbol{\omega}\|_2^{\frac{6-q}{2q}} \|\boldsymbol{\omega}\|_6^{\frac{3q-6}{2q}} \\ &\le \varepsilon \|\nabla\omega_{\mathbf{b}}\|_2^2 + C \|\nabla v_{\mathbf{b}}\|_s^{\frac{4p}{6+p}} \|\boldsymbol{\omega}\|_2^{2\frac{p}{q}\frac{6-q}{6+p}} \|\boldsymbol{\omega}\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}} \|\omega_{\mathbf{b}}\|_2^{2\frac{6-p}{6+p}}.\end{aligned}$$

Altogether we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega_{\mathbf{b}}(\tau)\|_2^2 + \|\nabla\omega_{\mathbf{b}}\|_2^2 \le C \|\nabla v_{\mathbf{b}}\|_s^{\frac{4p}{6+p}} \|\boldsymbol{\omega}\|_2^{2\frac{p}{q}\frac{6-q}{6+p}} \|\boldsymbol{\omega}\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}} \|\omega_{\mathbf{b}}\|_2^{2\frac{6-p}{6+p}} + C(\mathbf{b}) \|\nabla\mathbf{v}\|_2^2 (1 + \|\mathbf{v}\|_3^2),$$

thus using generalized Gronwall inequality in the form of Theorem 2 from [3] gives us $\left(\frac{p-6}{p+6} + 1 = \frac{2p}{6+p}\right)$

$$\|\omega_{\mathbf{b}}(\tau)\|_2^{\frac{4p}{6+p}} \le \|\omega_{\mathbf{b}}(0)\|_2^{\frac{4p}{6+p}} + C \|\boldsymbol{\omega}\|_{\infty,2}^{4\frac{p}{q}\frac{3-q}{6+p}} \int_0^\tau \|\nabla v_{\mathbf{b}}\|_s^{\frac{4p}{6+p}} \|\boldsymbol{\omega}\|_2^{\frac{2p}{6+p}} \|\boldsymbol{\omega}\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}} \,\mathrm{d}\sigma + C\Big(\int_0^\tau \|\nabla\mathbf{v}\|_2^2 (1 + \|\mathbf{v}\|_3^2)\mathrm{d}\sigma\Big)^{\frac{2p}{6+p}}.$$

Estimating the second term as above and using the Hölder inequality in the form

$$\int_0^\tau \|\nabla v_{\mathbf{b}}\|_s^{\frac{4p}{6+p}} \|\boldsymbol{\omega}\|_2^{\frac{2p}{6+p}} \|\boldsymbol{\omega}\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}} \,\mathrm{d}\sigma \le \|\nabla v_{\mathbf{b}}\|_{t,s}^{\frac{4p}{6+p}} \|\boldsymbol{\omega}\|_{2,2}^{\frac{2p}{6+p}} \|\boldsymbol{\omega}\|_{2,6}^{2\frac{p}{q}\frac{3q-6}{6+p}}$$

we conclude

$$\|\omega_{\mathbf{b}}\|_{\infty,2}^{2} \leq C_1 + C_2 \|\boldsymbol{\omega}\|_{Y(\tau)}.$$

This finishes the proof of Lemma 1.

4. Proof of the main theorem

Without loss of generality we may assume that $|\mathbf{b}(\tau,\mathbf{x})| \equiv 1$. Then for every time $\sigma \in (0,T)$, and point $\mathbf{x} \in \mathbb{R}^3$, there exists at least one component $b_j(\sigma,\mathbf{x})$ such that $|b_j(\sigma,\mathbf{x})| > \frac{1}{2}$. Since vector field $\mathbf{b}(\cdot,\cdot)$ is continuous on $(0,T)\times\mathbb{R}^3$, the sets $\Omega_{r,\sigma} = \left\{\mathbf{x}\in\mathbb{R}^3 \mid b_r(\sigma,\mathbf{x}) > \frac{1}{2}\right\}$ a $\Omega_{r+3,\sigma} = \left\{\mathbf{x}\in\mathbb{R}^3 \mid b_r(\sigma,\mathbf{x}) < -\frac{1}{2}\right\}$, $r=1,2,3$ compose at each particular time a covering of $\mathbb{R}^3$ by six open sets $\{\Omega_{r,\sigma}\}_{r=1}^{6}$. For simplicity, we set $b_{r+3} := -b_r$, $r=1,2,3$. Using the partition of unity (see e.g. [7]) we get functions $\varphi_{r,\sigma} \in C_0^\infty(\Omega_{r,\sigma})$, $0 \leq \varphi_{r,\sigma}(\mathbf{x}) \leq 1$ such that $\sum_r \varphi_{r,\sigma} = 1$, and $|\nabla\varphi_{r,\sigma}| \leq C(\mathbf{b})$. We will multiply the following equivalent form of Navier–Stokes equations (1)

$$\frac{\partial\mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} = -\boldsymbol{\omega}\times\mathbf{v} - \nabla\left(p + \frac{1}{2}|\mathbf{v}|^2\right) \tag{4}$$

by the test function $-\sum_{r=1}^{6} \partial_l(\varphi_{r,\sigma} b_r \partial_l \mathbf{v})$. Let work with each term separately, for illustration only with $r=1$:

$$\begin{aligned}
-\int_{\Omega_{1,\sigma}} \frac{\partial\mathbf{v}}{\partial t}\cdot\partial_l(\varphi_{1,\sigma}b_1\partial_l\mathbf{v})\mathrm{d}\mathbf{x} &= \int_{\Omega_{1,\sigma}} \frac{\partial}{\partial t}\partial_l\mathbf{v}\cdot\varphi_{1,\sigma}b_1\partial_l\mathbf{v}\mathrm{d}\mathbf{x}\\
&\geq \int_{\Omega_{1,\sigma}} \frac{1}{4}\varphi_{1,\sigma}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{v}|^2\,\mathrm{d}\mathbf{x}
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega_{1,\sigma}} \Delta\mathbf{v}\cdot\partial_l(\varphi_{1,\sigma}b_1\partial_l\mathbf{v})\mathrm{d}\mathbf{x} &= \int_{\Omega_{1,\sigma}} \Delta\mathbf{v}\cdot\big(\varphi_{1,\sigma}b_1\partial_l\partial_l\mathbf{v} + \partial_l(\varphi_{1,\sigma}b_1)\partial_l\mathbf{v}\big)\mathrm{d}\mathbf{x}\\
&\geq \int_{\Omega_{1,\sigma}} \frac{1}{2}\varphi_{1,\sigma}|\Delta\mathbf{v}|^2\,\mathrm{d}\mathbf{x} + \underbrace{\int_{\Omega_{1,\sigma}} \Delta\mathbf{v}\cdot\partial_l(\varphi_{1,\sigma}b_1)\partial_l\mathbf{v}\mathrm{d}\mathbf{x}}_{=Z_1^1}
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega_{1,\sigma}} \partial_k\left(p+\frac{1}{2}|\mathbf{v}|^2\right)\partial_l(\varphi_{1,\sigma}b_1\partial_l v_k)\mathrm{d}\mathbf{x} &= \int_{\Omega_{1,\sigma}} \partial_l\left(p+\frac{1}{2}|\mathbf{v}|^2\right)\partial_k(\varphi_{1,\sigma}b_1\partial_l v_k)\mathrm{d}\mathbf{x}\\
&= \underbrace{\int_{\Omega_{1,\sigma}} \partial_l p\,\partial_k(\varphi_{1,\sigma}b_1)\partial_l v_k\mathrm{d}\mathbf{x}}_{=Z_2^1} + \underbrace{\int_{\Omega_{1,\sigma}} \frac{1}{2}\partial_l|\mathbf{v}|^2\,\partial_k(\varphi_{1,\sigma}b_1)\partial_l v_k\mathrm{d}\mathbf{x}}_{=Z_3^1}
\end{aligned}$$

$$\int\limits_{\Omega_{1,\sigma}} (\boldsymbol{\omega} \times \mathbf{v}) \cdot \partial_l \left(\varphi_{1,\sigma} b_1 \partial_l \mathbf{v} \right) \mathrm{d}\mathbf{x}$$
$$= \int\limits_{\Omega_{1,\sigma}} \varphi_{1,\sigma} b_1 (\boldsymbol{\omega} \times \mathbf{v}) \cdot \Delta \mathbf{v} \mathrm{d}\mathbf{x} - \underbrace{\int\limits_{\Omega_{1,\sigma}} (\boldsymbol{\omega} \times \mathbf{v}) \cdot \partial_l (\varphi_{1,\sigma} b_1) \partial_l \mathbf{v} \mathrm{d}\mathbf{x}}_{= Z_4^1}$$

We rewrite the term in the first integral and get

$$\begin{aligned} b_1(\boldsymbol{\omega} \times \mathbf{v}) \cdot \Delta \mathbf{v} = b_1\omega_2 v_3 \Delta v_1 &\quad -b_1\omega_1 v_3 \Delta v_2 &\quad +b_1\omega_1 v_2 \Delta v_3 \\ -b_1\omega_3 v_2 \Delta v_1 &\quad +b_1\omega_3 v_1 \Delta v_2 &\quad -b_1\omega_2 v_1 \Delta v_3 \\ +b_2\omega_2 v_3 \Delta v_2 &\quad -b_2\omega_2 v_3 \Delta v_2 & \\ +b_3\omega_2 v_3 \Delta v_3 &\quad &\quad -b_3\omega_2 v_3 \Delta v_3 \\ -b_2\omega_3 v_2 \Delta v_2 &\quad +b_2\omega_3 v_2 \Delta v_2 & \\ -b_3\omega_3 v_2 \Delta v_3 &\quad &\quad +b_3\omega_3 v_2 \Delta v_3 \\ &\quad +b_3\omega_3 v_3 \Delta v_2 &\quad +b_2\omega_2 v_2 \Delta v_3 \\ &\quad -b_3\omega_3 v_3 \Delta v_2 &\quad -b_2\omega_2 v_2 \Delta v_3. \end{aligned}$$

Thus,

$$\begin{aligned} b_1(\boldsymbol{\omega} \times \mathbf{v}) \cdot \Delta \mathbf{v} = &\omega_2 v_3 \Delta v_{\mathbf{b}} - \omega_3 v_2 \Delta v_{\mathbf{b}} \\ &+ \omega_{\mathbf{b}} v_2 \Delta v_3 - \omega_{\mathbf{b}} v_3 \Delta v_2 \\ &+ \omega_3 v_{\mathbf{b}} \Delta v_2 - \omega_2 v_{\mathbf{b}} \Delta v_3 + \text{ lower order terms}(I_4^1,\ I_5^1). \end{aligned} \tag{5}$$

Observation. The above mentioned equality holds true without additional lower order terms, if the vector $\mathbf{b}(\sigma, \cdot)$ is constant in space, otherwise we use the following identity

$$\int\limits_{\Omega_{1,\sigma}} \varphi_{1,\sigma} (\omega_l v_m) b_l \Delta v_n \mathrm{d}\mathbf{x} = \int\limits_{\Omega_{1,\sigma}} \varphi_{1,\sigma} (\omega_l v_m) \Delta (b_l v_n) \mathrm{d}\mathbf{x}$$
$$- \int\limits_{\Omega_{1,\sigma}} \varphi_{1,\sigma} \big((\omega_l v_m) v_n \Delta b_l + 2\varphi_{1,\sigma} \nabla v_n \cdot \nabla b_l (\omega_l v_m) \big) \mathrm{d}\mathbf{x}.$$

Similarly, for b_2 we get

$$\begin{aligned} b_2(\boldsymbol{\omega} \times \mathbf{v}) \cdot \Delta \mathbf{v} = &\omega_3 v_1 \Delta v_{\mathbf{b}} - \omega_1 v_3 \Delta v_{\mathbf{b}} \\ &+ \omega_{\mathbf{b}} v_3 \Delta v_2 - \omega_{\mathbf{b}} v_1 \Delta v_3 \\ &+ \omega_1 v_{\mathbf{b}} \Delta v_3 - \omega_3 v_{\mathbf{b}} \Delta v_2 + \text{ lower order terms}. \end{aligned} \tag{6}$$

For the term with b_3 we use (as above) the shifts $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. The case $r = 4, 5, 6$ is trivial.

Summing up $\sum\limits_{r=1}^{6}$, and using the definitions of $\varphi_{r,\sigma}$ we get (recall, we use summation convention)

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla\mathbf{v}\right|^2\mathrm{d}\mathbf{x}+\frac{1}{2}\int_{\mathbb{R}^3}\left|\Delta\mathbf{v}\right|^2\mathrm{d}\mathbf{x}&\\ \le\sum_{r=1}^{6}\left(|Z_1^r|+|Z_2^r|+|Z_3^r|+|Z_4^r|+CI_4^r+CI_5^r\right)&\\ +\sum_{i=1}^{3}\varepsilon_{ijk}(I_1^{jk}+I_2^{jk}+I_3^{jk}),&\qquad(7)\end{aligned}$$

where

$$\begin{aligned}I_1^{jk}&=\int\limits_{\mathbb{R}^3}\omega_k v_{\mathbf{b}}\Delta v_j\mathrm{d}\mathbf{x}, & I_2^{jk}&=\int\limits_{\mathbb{R}^3}\omega_j v_k\Delta v_{\mathbf{b}}\mathrm{d}\mathbf{x},\\ I_3^{jk}&=\int\limits_{\mathbb{R}^3}\omega_{\mathbf{b}} v_j\Delta v_k\mathrm{d}\mathbf{x}, & I_4^r&=\int\limits_{\mathbb{R}^3}|\boldsymbol{\omega}|\left|\mathbf{v}\right|^2\left|\Delta(\varphi_{r,\sigma}\mathbf{b})\right|\mathrm{d}\mathbf{x},\\ I_5^r&=2\int\limits_{\mathbb{R}^3}|\nabla\mathbf{v}|\left|\nabla(\varphi_{r,\sigma}\mathbf{b})\right||\boldsymbol{\omega}|\left|\mathbf{v}\right|\mathrm{d}\mathbf{x}, & Z_1^r&=\int\limits_{\mathbb{R}^3}\Delta\mathbf{v}\cdot\partial_l(\varphi_{r,\sigma}b_r)\partial_l\mathbf{v}\mathrm{d}\mathbf{x}, \qquad (8)\\ Z_2^r&=\int\limits_{\mathbb{R}^3}\partial_l p\partial_k(\varphi_{r,\sigma}b_r)\partial_l v_k\mathrm{d}\mathbf{x}, & Z_3^r&=\int\limits_{\mathbb{R}^3}\frac{1}{2}\partial_l\left|\mathbf{v}\right|^2\partial_k(\varphi_{r,\sigma}b_r)\partial_l v_k\mathrm{d}\mathbf{x},\\ Z_4^r&=\int\limits_{\mathbb{R}^3}(\boldsymbol{\omega}\times\mathbf{v})\cdot\partial_l(\varphi_{r,\sigma}b_r)\partial_l\mathbf{v}\mathrm{d}\mathbf{x}\end{aligned}$$

(9)

and ε_{ijk} is the Levi-Civita tensor, i.e. it is zero unless all indeces are different, it is equal to $+1$ for a positive permutation of 123 and equal to -1 otherwise.

Now, we will estimate these integrals in order to finish the proof. At first, we will consider the case, in which we have the additional information about the projection v_b itself; we will proceed quite analogously with [18].

$$\begin{aligned}\int\limits_0^\tau\left|I_1^{jk}\right|\mathrm{d}\sigma\le\int\limits_0^\tau\int\limits_{\mathbb{R}^3}\left|\omega_k v_{\mathbf{b}}\Delta v_j\right|\mathrm{d}\mathbf{x}\mathrm{d}\sigma&\le\int\limits_0^\tau\|v_{\mathbf{b}}\|_s\|\omega_k\|_{\frac{2s}{s-2}}\|\Delta v_j\|_2\,\mathrm{d}\sigma\\ &\le C\int\limits_0^\tau\|v_{\mathbf{b}}\|_s\|\omega_k\|_2^{\frac{s-3}{s}}\|\nabla\omega_k\|_2^{\frac{3}{s}}\|\Delta v_j\|_2\,\mathrm{d}\sigma\\ &\le\varepsilon\|\Delta\mathbf{v}\|_{2,2}^2+C_\varepsilon\int\limits_0^\tau\|v_{\mathbf{b}}\|_s^{\frac{2s}{s-3}}\|\nabla\mathbf{v}\|_2^2\,\mathrm{d}\sigma\\ &\le\varepsilon\|\Delta\mathbf{v}\|_{2,2}^2+C_\varepsilon(T)\|v_{\mathbf{b}}\|_{t,s}^{\frac{2s}{s-3}}\|\nabla\mathbf{v}\|_{\infty,2}^2\qquad\left(t\ge\frac{2s}{s-3}\right).\end{aligned}$$

Next

$$\begin{aligned}I_2^{jk} =&\varepsilon_{jmn}\int\limits_{\mathbb{R}^3}\partial_m v_n v_k \Delta v_{\mathbf{b}}\mathrm{d}\mathbf{x}\\ =&-\varepsilon_{jmn}\int\limits_{\mathbb{R}^3}\partial_m\partial_l v_n v_k \partial_l v_{\mathbf{b}}\mathrm{d}\mathbf{x}-\varepsilon_{jmn}\int\limits_{\mathbb{R}^3}\partial_m v_n \partial_l v_k \partial_l v_{\mathbf{b}}\mathrm{d}\mathbf{x}\\ =&\underbrace{-\varepsilon_{jmn}\int\limits_{\mathbb{R}^3}\partial_m\partial_l v_n v_k \partial_l v_{\mathbf{b}}\mathrm{d}\mathbf{x}}_{J_1^{jk}}+\underbrace{\varepsilon_{jmn}\int\limits_{\mathbb{R}^3}\partial_m\partial_l v_n \partial_l v_k v_{\mathbf{b}}\mathrm{d}\mathbf{x}}_{J_2^{jk}}\\ &+\underbrace{\varepsilon_{jmn}\int\limits_{\mathbb{R}^3}\partial_l\partial_l v_k \partial_m v_n v_{\mathbf{b}}\mathrm{d}\mathbf{x}}_{J_3^{jk}}.\end{aligned}$$

Here,

$$\begin{aligned}\left|J_1^{jk}\right| \le& \int\limits_{\mathbb{R}^3}\left|\varepsilon_{jmn}\partial_m\partial_l v_n v_k \partial_l v_{\mathbf{b}}\right|\mathrm{d}\mathbf{x} \le \varepsilon\left\|\Delta\mathbf{v}\right\|_2^2 + C_\varepsilon\int\limits_{\mathbb{R}^3} v_k^2(\partial_l v_{\mathbf{b}})^2\mathrm{d}\mathbf{x}\\ \le&\varepsilon\left\|\Delta\mathbf{v}\right\|_2^2 - C_\varepsilon\int\limits_{\mathbb{R}^3} v_{\mathbf{b}}\partial_l\partial_l v_{\mathbf{b}} v_k^2\mathrm{d}\mathbf{x} - C_\varepsilon\int\limits_{\mathbb{R}^3} v_{\mathbf{b}}\partial_l v_{\mathbf{b}}\partial_l\left(v_k^2\right)\mathrm{d}\mathbf{x}.\end{aligned} \tag{10}$$

Let us estimate the first integral on the right hand side of (10):

$$\begin{aligned}\left|\int\limits_0^\tau\int\limits_{\mathbb{R}^3} v_{\mathbf{b}}\partial_l\partial_l v_{\mathbf{b}} v_k^2\mathrm{d}\mathbf{x}\mathrm{d}\sigma\right| \le& \underbrace{\int\limits_0^\tau\int\limits_{\mathbb{R}^3}\left|\mathbf{b}\right|\left|\Delta\mathbf{v}\right|\left|v_{\mathbf{b}}\right|\left|\mathbf{v}\right|^2\mathrm{d}\mathbf{x}\mathrm{d}\sigma}_{J_{11}}\\ &+ 2\underbrace{\int\limits_0^\tau\int\limits_{\mathbb{R}^3}\left|\nabla\mathbf{b}\right|\left|\nabla\mathbf{v}\right|\left|v_{\mathbf{b}}\right|\left|\mathbf{v}\right|^2\mathrm{d}\mathbf{x}\mathrm{d}\sigma}_{J_{12}} + \underbrace{\int\limits_0^t\int\limits_{\mathbb{R}^3}\left|\Delta\mathbf{b}\right|\left|v_{\mathbf{b}}\right|\left|\mathbf{v}\right|^3\mathrm{d}\mathbf{x}\mathrm{d}\tau}_{J_{13}}.\end{aligned}$$

Then

$$\begin{aligned}J_{11} \le& \int\limits_0^\tau\left\|\Delta\mathbf{v}\right\|_2\left\|v_{\mathbf{b}}\right\|_s\left\|\mathbf{v}\right\|_{\frac{4s}{s-2}}^2\mathrm{d}\sigma\\ \le& C\int\limits_0^\tau\left\|\Delta\mathbf{v}\right\|_2\left\|v_{\mathbf{b}}\right\|_s\left\|\mathbf{v}\right\|_{\frac{3s}{s-3}}\left\|\mathbf{v}\right\|_6\mathrm{d}\sigma\\ \le& C\left\|\Delta\mathbf{v}\right\|_{2,2}\left\|v_{\mathbf{b}}\right\|_{t,s}\left\|\mathbf{v}\right\|_{\frac{4s}{s+6},\frac{3s}{s-3}}\left\|\mathbf{v}\right\|_{\infty,6}.\end{aligned}$$

As $\frac{2}{4s/(s+6)} + \frac{3}{3s/(s-3)} = \frac{3}{2}$, we can interpolate

$$\|\mathbf{v}\|_{\frac{4s}{s+6},\frac{3s}{s-3}} \le C(s)\,\|\mathbf{v}\|_{\infty,2}^{\frac{s-6}{2s}}\,\|\nabla\mathbf{v}\|_{2,2}^{\frac{s+6}{2s}}\,.$$

Thus, using Young's inequality and energy inequality we get

$$J_{11} \le \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\,\|v_{\mathbf{b}}\|_{t,s}^2\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,.$$

Note that for fixed $\varepsilon > 0$, $C_\varepsilon \to 0$ for $\tau \to 0$, uniformly for $s \in [6,\infty]$. The lower order terms J_{12}, and J_{13} may be bounded as follows:

$$\begin{aligned}
J_{12} \le& \int_0^\tau \|v_{\mathbf{b}}\|_s\,\|\nabla\mathbf{v}\|_2\,\|\nabla\mathbf{b}\|_\infty\,\|\mathbf{v}\|_{\frac{4s}{s-2}}^2\,\mathrm{d}\sigma\\
\le& \int_0^\tau \|v_{\mathbf{b}}\|_s\,\|\nabla\mathbf{v}\|_2\,\|\nabla\mathbf{b}\|_\infty\,\|\mathbf{v}\|_{\frac{3s}{s-3}}\,\|\mathbf{v}\|_6\,\mathrm{d}\sigma\\
\le& \|v_{\mathbf{b}}\|_{t,s}\,\|\nabla\mathbf{v}\|_{\infty,2}\,\|\nabla\mathbf{b}\|_{2,\infty}\,\|\mathbf{v}\|_{\frac{4s}{s+6},\frac{3s}{s-3}}\,\|\mathbf{v}\|_{\infty,6}\\
\le& C\,\|v_{\mathbf{b}}\|_{t,s}\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,,
\end{aligned}$$

$$\begin{aligned}
J_{13} \le& \int_0^\tau \|v_{\mathbf{b}}\|_s\,\|\mathbf{v}\|_6^2\,\|\mathbf{v}\|_{\frac{3s}{s-3}}\,\left\|\nabla^2\mathbf{b}\right\|_3\,\mathrm{d}\sigma\\
\le& \|v_{\mathbf{b}}\|_{t,s}\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,\|\mathbf{v}\|_{\frac{4s}{s+6},\frac{3s}{s-3}}\,\left\|\nabla^2\mathbf{b}\right\|_{2,3}\\
\le& C\,\|v_{\mathbf{b}}\|_{t,s}\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,,
\end{aligned}$$

where we have used Hölder's inequality $\left(\frac{1}{t} + \frac{1}{2} + \frac{s+6}{4s} = 1\right)$, the assumptions on $\mathbf{b}(\cdot,\cdot)$, and the fact that from energy inequality we have estimate of the norm of $\mathbf{v}$ in space $L^{\frac{4s}{s+6}}(0,T,(L^{\frac{3s}{s-3}}(\mathbb{R}^3))^3)$. Note that $C = C(\tau) \to 0$ for $\tau \to 0$ uniformly for $s \in [6,\infty]$.

Further, we will estimate the last integral from (10):

$$\left|\int_0^\tau\int_{\mathbb{R}^3} v_{\mathbf{b}}\partial_l v_{\mathbf{b}}\partial_l\left(v_k^2\right)\mathrm{d}\mathbf{x}\mathrm{d}\sigma\right| \le \underbrace{\int_0^\tau\int_{\mathbb{R}^3} |v_{\mathbf{b}}|\,|\nabla\mathbf{v}|^2\,|\mathbf{v}|\,|\mathbf{b}|\,\mathrm{d}\mathbf{x}\mathrm{d}\sigma}_{=J_{14}} + \underbrace{\int_0^\tau\int_{\mathbb{R}^3} |v_{\mathbf{b}}|\,|\nabla\mathbf{b}|\,|\nabla\mathbf{v}|\,|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\mathrm{d}\sigma}_{J_{15}=J_{12}}$$

$$J_{14} \le C\int_0^\tau \|v_{\mathbf{b}}\|_s\,\|\nabla\mathbf{v}\|_3^2\,\|\mathbf{v}\|_{\frac{3s}{s-3}}\,\mathrm{d}\sigma \le C\,\|v_{\mathbf{b}}\|_{t,s}\,\|\nabla\mathbf{v}\|_{4,3}^2\,\|\mathbf{v}\|_{\frac{2t}{t-2},\frac{3s}{s-3}}$$

The interpolation inequalities

$$\|\mathbf{v}\|_{\frac{2t}{t-2},\frac{3s}{s-3}} \leq C\,\|\mathbf{v}\|_{\infty,2}^{2/t}\,\|\mathbf{v}\|_{2,6}^{(t-2)/t}, \text{ and } \|\nabla\mathbf{v}\|_{4,3} \leq C\,\|\nabla\mathbf{v}\|_{\infty,2}^{\frac{1}{2}}\,\|\Delta\mathbf{v}\|_{2,2}^{\frac{1}{2}},$$

and Young's inequality yield

$$\left|\int_0^\tau \int_{\mathbb{R}^3} v_{\mathbf{b}}\partial_l v_{\mathbf{b}}\partial_l\left(v_k^2\right)\mathrm{d}\mathbf{x}\mathrm{d}\sigma\right| \leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\,\|v_{\mathbf{b}}\|_{t,s}^2\,\|\nabla\mathbf{v}\|_{\infty,2}^2,$$

which implies the bound on J_1^{jk}

$$\left|\int_0^\tau J_1^{jk}\mathrm{d}\sigma\right| \leq 3\varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\left(\|v_{\mathbf{b}}\|_{t,s}^2 + \|v_{\mathbf{b}}\|_{t,s}\right)\|\nabla\mathbf{v}\|_{\infty,2}^2.$$

Further,

$$\begin{aligned}\int_0^\tau \left|J_2^{jk}+J_3^{jk}\right|\mathrm{d}\sigma \leq & \int_0^\tau \Big(\int_{\mathbb{R}^3}|\partial_l\partial_m v_n\partial_l v_k v_{\mathbf{b}}|\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3}|\partial_l\partial_l v_k\partial_m v_n v_{\mathbf{b}}|\,\mathrm{d}\mathbf{x}\Big)\mathrm{d}\sigma\\ \leq & 2\int_0^\tau \|\Delta\mathbf{v}\|_2\,\|v_{\mathbf{b}}\|_s\,\|\nabla\mathbf{v}\|_{\frac{2s}{s-2}}\,\mathrm{d}\sigma\\ \leq & C\int_0^\tau \|v_{\mathbf{b}}\|_s\,\|\nabla\mathbf{v}\|_2^{\frac{s-3}{s}}\,\|\Delta\mathbf{v}\|_2^{\frac{3}{s}}\,\|\Delta\mathbf{v}\|_2\,\mathrm{d}\sigma\\ \leq & \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\,\|v_{\mathbf{b}}\|_{t,s}\,\|\nabla\mathbf{v}\|_{\infty,2}^2,\end{aligned}$$

$$\begin{aligned}\int_0^\tau \left|I_3^{jk}\right|\mathrm{d}\sigma \leq \int_0^\tau\int_{\mathbb{R}^3}|\omega_{\mathbf{b}}|\,|v_j|\,|\Delta v_k|\,\mathrm{d}\mathbf{x}\mathrm{d}\sigma &\leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\int_0^\tau \|\mathbf{v}\|_3^2\,\|\omega_{\mathbf{b}}\|_6^2\,\mathrm{d}\sigma\\ &\leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\,\|\mathbf{v}\|_{\infty,3}^2\,\|\omega_{\mathbf{b}}\|_{2,6}^2.\end{aligned}$$

Using $\|\mathbf{v}\|_{\infty,3}^2 \leq \|\mathbf{v}\|_{\infty,2}\,\|\mathbf{v}\|_{\infty,6} \leq C\,\|\nabla\mathbf{v}\|_{\infty,2}$, and the information which comes from (3), we get

$$\begin{aligned}\|\mathbf{v}\|_{\infty,3}^2\,\|\omega_{\mathbf{b}}\|_{2,6}^2 \leq & C\,\|\nabla\mathbf{v}\|_{\infty,2}\,(1+\|\nabla\mathbf{v}\|_{Y(\tau)})\\ & \leq C(\tau)(\|\nabla\mathbf{v}\|_{\infty,2} + \|\nabla\mathbf{v}\|_{\infty,2}^2 + \|\Delta\mathbf{v}\|_{2,2}^2) + C_0\,\|\mathbf{v}_0\|_{1,2}^2.\end{aligned}$$

Recall that $C \to 0$ for $\tau \to 0^+$, uniformly for $s \in [6,\infty]$. It remains to deduce suitable estimates of the lower order terms with derivatives of $\mathbf{b}(\cdot,\cdot)$.

$$\int_0^\tau |I_4^r|\,\mathrm{d}\sigma \leq \int_0^\tau \int_{\mathbb{R}^3} |\boldsymbol{\omega}|\,|\mathbf{v}|^2\,|\Delta(\varphi_{r,\sigma}\mathbf{b})|\,\mathrm{d}\mathbf{x}\mathrm{d}\sigma$$
$$\leq \int_0^\tau \|\nabla\mathbf{v}\|_6\,\|\mathbf{v}\|_3\,\|\mathbf{v}\|_6\,\|\Delta(\varphi_{r,\sigma}\mathbf{b})\|_3\,\mathrm{d}\sigma$$
$$\leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon \int_0^\tau \|\nabla\mathbf{v}\|_2^2\,\|\mathbf{v}\|_3^2\,\|\Delta(\varphi_{r,\sigma}\mathbf{b})\|_3^2\,\mathrm{d}\sigma$$
$$\leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,\|\mathbf{v}\|_{4,3}^2\,\|\Delta(\varphi_{r,\sigma}\mathbf{b})\|_{4,3}^2$$

$$\int_0^\tau |Z_1^r|\,\mathrm{d}\sigma \leq \int_0^\tau \int_{\mathbb{R}^3} |\Delta\mathbf{v}|\,|\nabla(\varphi_{r,\sigma}\mathbf{b})|\,|\nabla\mathbf{v}|\,\mathrm{d}\mathbf{x}\mathrm{d}\sigma$$
$$\leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2} + C_\varepsilon\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,\|\nabla(\varphi_{r,\sigma}\mathbf{b})\|_{2,\infty}^2$$

$$\int_0^\tau |I_5^r + Z_2^r + Z_3^r + Z_4^r|\,\mathrm{d}\sigma$$
$$\leq \int_0^\tau \int_{\mathbb{R}^3} \Big(|\nabla\mathbf{v}|\,|\nabla(\varphi_{r,\sigma}\mathbf{b})|\,|\boldsymbol{\omega}|\,|\mathbf{v}| + |\nabla p||\nabla(\varphi_{r,\sigma}\mathbf{b})||\nabla\mathbf{v}|$$
$$+ |\nabla\mathbf{v}|^2\,|\mathbf{v}||\nabla(\varphi_{r,\sigma}\mathbf{b})| + |\boldsymbol{\omega}|\,|\mathbf{v}|\,|\nabla\mathbf{v}|\,|\nabla(\varphi_{r,\sigma}\mathbf{b})|\Big)\mathrm{d}\mathbf{x}\mathrm{d}\sigma$$
$$\leq C\int_0^\tau \|\nabla\mathbf{v}\|_6\,\|\nabla\mathbf{v}\|_2\,\|\mathbf{v}\|_3\,(\|\nabla\mathbf{b}\|_\infty + 1)\,\mathrm{d}\sigma$$
$$\leq C\,\|\Delta\mathbf{v}\|_{2,2}\,\|\nabla\mathbf{v}\|_{\infty,2}\,\|\mathbf{v}\|_{4,3}\,\big(\,\|\nabla\mathbf{b}\|_{4,\infty} + (T^*)^{\frac{1}{4}}\big)$$
$$\leq \varepsilon\,\|\Delta\mathbf{v}\|_{2,2}^2 + C_\varepsilon\,\|\nabla\mathbf{v}\|_{\infty,2}^2\,\|\mathbf{v}\|_{4,3}^2$$

Collecting all the above estimates together, we see that

$$\|\nabla\mathbf{v}\|_{\infty,2}^2 + \|\Delta\mathbf{v}\|_{2,2}^2 \leq C_0\,\|\mathbf{v}_0\|_{1,2}^2 + C(\|\nabla\mathbf{v}\|_{\infty,2}^2 + \|\Delta\mathbf{v}\|_{2,2}^2 + 1), \qquad (11)$$

where $C \to 0$ for $\tau \to 0^+$, uniformly for $s \in [6,\infty]$. Therefore, taking τ sufficiently small, we get

$$\|\nabla\mathbf{v}\|_{\infty,2}^2 + \|\Delta\mathbf{v}\|_{2,2}^2 \leq 4C_0\,\|\mathbf{v}_0\|_{1,2}^2\,.$$

Repeating the same estimates on $(\tau, 2\tau)$ we get that

$$\|\nabla\mathbf{v}\|_{L^\infty(\tau,2\tau;L^2(\mathbb{R}^3))}^2 + \|\Delta\mathbf{v}\|_{L^2(\tau,2\tau;L^2(\mathbb{R}^3))}^2 \leq 4C_0\,\|\mathbf{v}(\tau)\|_{1,2}^2\,.$$

Therefore, after finite number of steps, we get that the regular solution exists on the whole time interval $(0,T)$.

Let us move to the case where we have information about the gradient of the projection and let us estimate all terms from (8). We start with the term I_2^{jk}:

$$I_2^{jk} = \varepsilon_{jmn} \int\limits_{\mathbb{R}^3} \partial_m v_n v_k \Delta v_{\mathbf{b}} \mathrm{d}\mathbf{x}$$
$$= \underbrace{-\varepsilon_{jmn} \int_{\mathbb{R}^3} \partial_m \partial_l v_n v_k \partial_l v_{\mathbf{b}} \mathrm{d}\mathbf{x}}_{J_1^{jk}} \underbrace{-\varepsilon_{jmn} \int_{\mathbb{R}^3} \partial_m v_n \partial_l v_k \partial_l v_{\mathbf{b}} \mathrm{d}\mathbf{x}}_{J_2^{jk}}$$

$$\left| J_1^{jk} \right| \leq \varepsilon \left\| \Delta \mathbf{v} \right\|_2^2 + C_\varepsilon \int_{\mathbb{R}^3} |\mathbf{v}|^2 \left| \nabla v_{\mathbf{b}} \right|^2 \mathrm{d}\mathbf{x} \tag{12}$$

In the estimate of the right hand side of (12), we will distinguish between two possible cases. For $2 \leq s \leq 3$, we get

$$\begin{aligned}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \left| \nabla v_{\mathbf{b}} \right|^2 \mathrm{d}\mathbf{x} \leq & \left\| \nabla v_{\mathbf{b}} \right\|_s^2 \left\| \mathbf{v} \right\|_{\frac{2s}{s-2}}^2 \\ \leq & \left\| \nabla v_{\mathbf{b}} \right\|_s^2 \left\| \nabla \mathbf{v} \right\|_2^{\frac{4s-6}{s}} \left\| \nabla^2 \mathbf{v} \right\|_2^{\frac{6-2s}{s}} \\ \leq & \varepsilon \left\| \nabla^2 \mathbf{v} \right\|_2^2 + C_\varepsilon \left\| \nabla v_{\mathbf{b}} \right\|_s^{\frac{2s}{2s-3}} \left\| \nabla \mathbf{v} \right\|_2^2 ,\end{aligned}$$

while for $s > 3$, we will proceed in the following way

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 \left| \nabla v_{\mathbf{b}} \right|^2 \mathrm{d}\mathbf{x} \leq \left\| \nabla v_{\mathbf{b}} \right\|_s^{\frac{6s}{5s-6}} \left\| \nabla v_{\mathbf{b}} \right\|_2^{\frac{4s-12}{5s-6}} \left\| \mathbf{v} \right\|_{\frac{2}{3}\frac{5s-6}{s-2}}^2 .$$

Further, due to $2 \leq \frac{2}{3}\frac{5s-6}{s-2} \leq 6$, we can interpolate

$$\left\| \mathbf{v} \right\|_{\frac{2}{3}\frac{5s-6}{s-2}}^2 \leq C \left\| \mathbf{v} \right\|_2^{\frac{4s-12}{5s-6}} \left\| \nabla \mathbf{v} \right\|_2^{\frac{6s}{5s-6}} .$$

Moreover,

$$\left\| \nabla v_{\mathbf{b}} \right\|_2^{\frac{4s-12}{5s-6}} \leq \left(\left\| \nabla \mathbf{v} \right\|_2 + \left\| \mathbf{v} \right\|_2 \left\| \nabla \mathbf{b} \right\|_\infty \right)^{\frac{4s-12}{5s-6}} ,$$

and using Young's inequality we have

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 \left| \nabla v_{\mathbf{b}} \right|^2 \mathrm{d}\mathbf{x} \leq C \left\| \nabla v_{\mathbf{b}} \right\|_s^{\frac{6s}{5s-6}} \left\| \mathbf{v} \right\|_2^{\frac{4s-12}{5s-6}} \left(\left\| \nabla \mathbf{v} \right\|_2^2 + C_0(\mathbf{b}) \right) .$$

The integrals J_2^{jk}, and I_3^{jk} can be estimated in a straightforward way, analogously as in [15]

$$\left| J_2^{jk} \right| \leq \left\| \nabla v_{\mathbf{b}} \right\|_s \left\| \nabla \mathbf{v} \right\|_{\frac{2s}{s-1}}^2 \leq \varepsilon \left\| \nabla^2 \mathbf{v} \right\|_2^2 + C_\varepsilon \left\| \nabla v_{\mathbf{b}} \right\|_s^{\frac{2s}{2s-3}} \left\| \nabla \mathbf{v} \right\|_2^2 ,$$

$$\left| I_3^{jk} \right| \leq \left\| \nabla^2 \mathbf{v} \right\|_2 \left\| \omega_{\mathbf{b}} \right\|_3 \left\| \mathbf{v} \right\|_6 \leq \varepsilon \left\| \nabla^2 \mathbf{v} \right\|_2^2 + \varepsilon \left\| \omega_{\mathbf{b}} \right\|_3^4 + C_\varepsilon \left\| \nabla \mathbf{v} \right\|_2^4 .$$

We now return to the term I_1^{jk}. We have (below, δ_{ij} denotes the Kronecker symbol)

$$\begin{aligned}\varepsilon_{ijk} I_1^{jk} &= \varepsilon_{ijk} \int_{\mathbb{R}^3} \omega_k v_{\mathbf{b}} \Delta v_j \mathrm{d}\mathbf{x} = \varepsilon_{ijk}\varepsilon_{klm} \int_{\mathbb{R}^3} \partial_l v_m v_{\mathbf{b}} \Delta v_j \mathrm{d}\mathbf{x} \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \int_{\mathbb{R}^3} \partial_l v_m v_{\mathbf{b}} \Delta v_j \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} (\partial_i v_j v_{\mathbf{b}} \Delta v_j - \partial_j v_i v_{\mathbf{b}} \Delta v_j) \mathrm{d}\mathbf{x} \\ &= - \int_{\mathbb{R}^3} \partial_i v_j \partial_l v_{\mathbf{b}} \partial_l v_j \mathrm{d}\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} (\partial_l v_j)^2 \partial_i v_{\mathbf{b}} \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3} v_i \partial_j v_{\mathbf{b}} \Delta v_j \mathrm{d}\mathbf{x}.\end{aligned}$$

Therefore these terms can be treated exactly as terms above coming from I_2^{jk}. Next, we have to estimate the lower order terms. Since they can be treated exactly as in the previous case (additional information about $v_{\mathbf{b}}$), we skip the details.

Altogether we get

$$\begin{aligned}\|\nabla \mathbf{v}\|_{\infty,2}^2 + \left\|\nabla^2 \mathbf{v}\right\|_{2,2}^2 \leq \varepsilon \int_0^\tau \|\omega_{\mathbf{b}}\|_3^4 \,\mathrm{d}\sigma \\ + C_\varepsilon \int_0^\tau \Big\{ g(s) \|\nabla v_{\mathbf{b}}\|_s^{\frac{2s}{2s-3}} C_0(\mathbf{b}) + \|\nabla v_{\mathbf{b}}\|_s^{\frac{6s}{5s-6}} \\ + \|\Delta \mathbf{b}\|_6^2 + \|\mathbf{v}\|_3^4 + \|\nabla \mathbf{b}\|_\infty^4 + \|\nabla \mathbf{v}\|_2^2 \Big\} \|\nabla \mathbf{v}\|_2^2 \,\mathrm{d}\sigma,\end{aligned}$$

where $g(s) = 0$, for $2 \leq s \leq 3$, and $g(s) = 1$, for $s > 3$. Note that both $\frac{2s}{2s-3}$ and $\frac{6s}{5s-6}$ are less than $t = \frac{4s}{3s-6}$. Using the estimate from Lemma 1 we obtain

$$\int_0^\tau \|\omega_{\mathbf{b}}\|_3^4 \,\mathrm{d}\sigma \leq C(1 + \|\nabla \mathbf{v}\|_{Y(\tau)}^2).$$

Choosing ε sufficiently small, we can use Gronwall's inequality in order to conclude that

$$\|\nabla \mathbf{v}\|_{\infty,2}^2 + \left\|\nabla^2 \mathbf{v}\right\|_{2,2}^2 \leq C(\mathbf{v}_0, \|\nabla v_{\mathbf{b}}\|_{t,s}).$$

As this inequality holds for any $\tau < T^*$ and C is independent of τ, the proof of Theorem 1 is complete.

Acknowledgment

The work of the first author was supported by the grant SVV-2013-267316. The work of the second author was partially supported by the grant No. 201/09/0917 of the Grant Agency of the Czech Republic.

References

[1] Š. Axmann, M. Pokorný, *A note on regularity criteria for the solutions to Navier-Stokes equations involving one velocity component.* In preparation.

[2] C. Cao, E.S. Titi, *Regularity criteria for the three-dimensional Navier–Stokes equations.* Indiana University Mathematics Journal **57** (2008), No. 6, 2643–2660.

[3] H. El-Owaidy et.al., *On some new integral inequalities of Gronwall–Bellman type.* Applied Mathematics and Computation, **106** (1999), 289–303.

[4] G.P. Galdi, *An Introduction to the Navier–Stokes initial–boundary value problem.* In *Fundamental directions in mathematical fluid mechanics. Advances in Mathematical Fluid Mechanics.* Basel: Birkhuser, 2000, 1–70.

[5] C. He, *Regularity for solutions to the Navier–Stokes equations with one velocity component regular.* Electronic Journal of Differential Equations (2002), No. 29, 1–13.

[6] E. Hopf, *Über die Anfangswertaufgabe fr die hydrodynamischen Grundgleichungen.* Mathematische Nachrichten **4** (1951), No. 1, 213–231.

[7] A. Kufner, O. John, S. Fučík, *Function spaces.* 1st edition, Prague, Academia, 1977.

[8] I. Kukavica, M. Ziane, *One component regularity for the Navier–Stokes equations.* Nonlinearity **19** (2006), No. 2, 453–469.

[9] O.A. Ladyzhenskaya, *Sixth problem of the millenium: Navier–Stokes equations, existence and smoothness.* Russian Mathematical Surveys **58** (2003), No. 2, 251–286.

[10] O.A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow.* 2nd edition. New York-London-Paris: Gordon and Breach Science Publishers, 1969.

[11] J. Leray, *Étude de diverses équations intgrales non linéaires et de quelques problèmes que pose l'hydrodynamique.* Journal de Mathématiques Pures et Appliquées **12** (1933), No. 2, 1–82.

[12] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace.* Acta Mathematica **63** (1934), No. 1, 193–248.

[13] J. Neustupa, A. Novotný, P. Penel, *An interior regularity of weak solution to the Navier–Stokes equations in dependence on one component of velocity.* Quaderni di Matematica, 2002, vol. 10 Topics in mathematical fluid mechanics, 163–183.

[14] J. Neustupa, P. Penel, *Regularity of a suitable weak solution to the Navier–Stokes equations as a consequence of regularity of one velocity component.* In: *Applied Nonlinear Analysis*, edited by Sequeira et al., New York: Kluwer Academic/Plenum Pulishers, 1999, 391–402.

[15] M. Pokorný, *On the result of He concerning the smoothness of solutions to the Navier–Stokes equations.* Electronic Journal of Differential Equations (2003), No. 11, 1–8.

[16] R. Temam, *Some developments on Navier–Stokes equations in the second half of the 20th century.* In: *Development of mathematics 1950–2000.* Basel: Birkhuser, 2000, 1049–1106.

[17] Y. Zhou, *A new regularity criterion for the Navier–Stokes equations in terms of the gradient of one velocity component.* Methods and Applications of Analysis **9** (2002), No. 4, 563–578.

[18] Y. Zhou, *A new regularity criterion for weak solutions to the Navier–Stokes equations.* Journal de Mathématiques Pures et Appliquées **84** (2005), No. 11, 1496–1514.

[19] Y. Zhou, M. Pokorný, *On a regularity criterion for the Navier–Stokes equations involving gradient of one velocity component.* Journal of Mathematical Physics **50** (2009), No. 12, 123514.

[20] Y. Zhou, M. Pokorný, *On the regularity criterion of the Navier–Stokes equations via one velocity component.* Nonlinearity **23** (2010), No. 5, 1097–1107.

Šimon Axmann
Mathematical Institute of Charles University
Faculty of Mathematics and Physics
Sokolovská 83
186 75 Praha 8
Czech Republic
e-mail: axmas7am@karlin.mff.cuni.cz

Milan Pokorný
Mathematical Institute of Charles University
Faculty of Mathematics and Physics
Sokolovská 83
186 75 Praha 8
Czech Republic
e-mail: pokorny@karlin.mff.cuni.cz