On de Casteljau-type algorithms for rational Bezier curves

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Abstract

We consider the space of rational functions of degree n with a common denominator. It is shown that the corresponding rational Bézier curves admit up to $1 + n!$ different de Casteljau-type algorithms, depending on the ordering of the elementary factors of the polynomial. Our observations generalize recent results of Han, Chu and Qiu [2], which cover the case of denominators of the form $\prod_{i=1}^{n}(1 - t + q^{i-1}t)$ where q is a positive constant, to rational curves with general denominators.

Keywords: rational Bernstein functions, De Casteljau algorithm, rational Bézier curves, Lupaş numbers, degree elevation

1. Introduction

Rational Bézier and B-spline curves and surfaces are one of the the standard representations for free-form geometry in Computer Aided Design and Geometric Modeling [1, 3, 6]. The use of rational representations allows the exact description of conic sections and quadric surfaces (including spheres and cylinders), which are of fundamental interest for various applications.

Bernstein polynomials and B-splines form bases with optimal properties for the spaces of polynomials and spline functions of given degree (and knots in the case of spline function). The spaces of rational (spline) functions with a common denominator are spanned by basis functions with similar properties, which are constructed by collecting rational Bernstein functions or NURBS (Non-Uniform Rational B-splines) basis functions.

In a recent paper, Han, Chu and Qiu [2] consider rational functions with denominators of the form $\prod_{i=1}^{n}(1 - t + q^{i-1}t)$ where q is positive a real constant. Based on an operator that has been introduced by Lupaş [4], they introduce a system of rational basis functions that shares many properties with Bernstein polynomials.

The present paper shows how these observations can be extended to spaces of rational functions with more general denominators, simply by using rational Bernstein functions.

More precisely, we consider nested spaces of rational functions, obtained by successively multiplying the denominator with linear factors and raising the degree of the denominator. We derive recurrence formulas for the weights and basis functions of these spaces. Based on these recurrences it is observed that each ordering of the denominator factors provides a de Casteljau-type algorithm for curves expressed with respect to this basis.

2. Rational Bernstein functions

We consider an infinite sequence of linear factors

$$L_i(t) = a_i(1 - t) + b_i t, \quad i \in \mathbb{Z}_+, \tag{1}$$

which are defined by the real coefficients a_i and b_i, $(a_i, b_i) \neq (0, 0)$. If all coefficients are positive, then these factors do not possess roots in the interval $[0, 1]$. Some of these factors may degenerate to constants. This is the case if the coefficients satisfy $a_i = b_i$.

For any positive integer n, we denote the product of the first n factors by

$$\omega^n(t) = L_1(t) \cdot \ldots \cdot L_n(t). \tag{2}$$

The product is a polynomial of degree at most n. It possesses a unique representation

$$\omega^n(t) = \sum_{i=0}^{n} w_i^n \beta_i^n(t). \tag{3}$$

with respect to the Bernstein polynomials $\beta_i^n(t) = \binom{n}{i} t^i (1 - t)^{n-i}$ of degree n. Following the usual approach in Computer Aided Geometric Design [1, 3, 6], the coefficients of this representation are called the *weights*, and we use them do define the *rational Bernstein functions*

$$\rho_i^n(t) = \frac{w_i^n \beta_i^n(t)}{\omega^n(t)}. \tag{4}$$

If all weights are non-zero, then these functions span the space of rational functions of degree n with denominator ω^n,

$$R^n = \text{span}\{\rho_i^n \mid i = 0, \ldots, n\} = \{P/\omega^n \mid P \in \Pi^n\}, \tag{5}$$

where Π^n is the space of polynomials of degree n. These spaces are nested, i.e. $R^{n-1} \subset R^n$.

We extend these definitions to include the case $n = 0$ by defining

$$\omega^0 = \rho_0^0 = 1.$$

Consequently, R^0 is the linear space of constant functions. Moreover, the functions in (4) are defined for all integers i by setting of $\rho_i^n = 0$ whenever $i < 0$ or $i > n$.

The rational Bernstein functions possess several useful properties, which are similar to the properties of the "Lupaş q-analogues of the Bernstein functions" [2]:

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Proposition 1.

(i) *Non-negativity: If all coefficients a_i, b_i are positive, then $\rho_i^n(t) \geq 0$ for $t \in [0,1]$.*

(ii) *Partition of unity: $\sum_{i=0}^n \rho_i^n(t) = 1$ almost everywhere[1].*

(iii) *Endpoint interpolation: If all coefficients a_i, b_i are non-zero, then $\rho_i^n(0) = \delta_{i0}$ and $\rho_i^n(1) = \delta_{in}$.*

(iv) *Inverse property: $\rho_i^n(t) = \hat{\rho}_i^n(1-t)$, where $\hat{\rho}$ are the basis functions defined in an analogous way using the linear factors $\hat{L}_i(t) = b_i(1-t) + a_i t$.*

(v) *Reducibility: We obtain the classical polynomial Bernstein basis when $a_i = b_i = 1$.*

The proofs of these observations follow directly from the definition of the rational Bernstein functions.

3. Recurrence relations

Before establishing recurrence relations, we need to analyze the weights in more detail.

Proposition 2. *The weights take the form*

$$w_i^n = \frac{1}{\binom{n}{i}} \left(\sum_{\substack{K \cup L = \{1,\ldots,n\} \\ |K|=(n-i), |L|=i}} \prod_{k \in K} a_k \prod_{l \in L} b_k \right). \tag{6}$$

They satisfy the recurrence formula

$$w_i^n = a_n \frac{(n-i)}{n} w_i^{n-1} + b_n \frac{i}{n} w_{i-1}^{n-1}. \tag{7}$$

PROOF. The recurrence of the denominators

$$\omega_n = \omega_{n-1} L_n \tag{8}$$

implies the equation

$$\sum_{i=0}^n w_i^n \beta_i^n(t) = \left[\sum_{i=0}^{n-i} w_i^{n-1} B_i^{n-1}(t) \right] [a_n(1-t) + b_n t], \tag{9}$$

from which we obtain

$$w_i^n \beta_i^n(t) = a_n(1-t) w_i^{n-1} \beta_i^{n-1}(t) + b_n t w_{i-1}^{n-1} \beta_{i-1}^{n-1}(t). \tag{10}$$

Dividing both sides by $\beta_i^n(t)$ gives (7). The explicit formula (6) can be proved directly by expanding the product and comparing coefficients. □

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[1]except for the roots of ω^n

In particular, if all coefficients a_i and b_i are positive, then so are the weights.

Based on these observations we derive a recurrence relation for the rational Bernstein functions.

Proposition 3. *The rational Bernstein functions satisfy the recurrence formula*

$$\rho_i^n(t) = \frac{\alpha_n(1-t)}{L_n(t)}\rho_i^{n-1}(t) + \frac{b_n t}{L_n(t)}\rho_{i-1}^{n-1}(t).\tag{11}$$

PROOF. Combining (8), (10) and (4) confirms (11). □

This recurrence will be used in the next section to derive a de Casteljau-type algorithm for evaluating rational Bézier curves.

Note that there are infinitely many formulas expressing ρ_i^n as a (non-constant) linear combination of ρ_i^{n-1} and ρ_{i-1}^{n-1}. More precisely we have

$$\rho_i^n(t) \;=\; \frac{n}{n-i}\,\frac{(1-t)}{L_n(t)}\,\frac{w_i^n}{w_i^{n-1}}\,\rho_i^{n-1}(t),\tag{12}$$

$$\rho_i^n(t) \;=\; \frac{n}{i}\,\frac{t}{L_n(t)}\,\frac{w_i^n}{w_{i-1}^{n-1}}\,\rho_{i-1}^{n-1}(t)\tag{13}$$

and any affine combination of (12) and (13) provides a valid formula. In order to obtain a de Casteljau-type algorithm, however, the coefficients in the formula need to be independent of i, and the recurrence (11) is the only one with this property.

Another formula expresses each rational Bernstein function of degree n in terms of two functions of degree $n+1$, thereby confirming the nested nature of the spaces R^n.

Proposition 4. *The rational Bernstein functions satisfy*

$$\rho_i^n(t) = a_{n+1}\frac{n+1-i}{n+1}\,\frac{w_i^n}{w_i^{n+1}}\,\rho_i^{n+1}(t) + b_{n+1}\frac{i+1}{n+1}\,\frac{w_i^n}{w_{i+1}^{n+1}}\,\rho_{i+1}^{n+1}(t).\tag{14}$$

PROOF. Expressing $\rho_{i+1}^{n+1}(t)$ using equation (13) and $\rho_i^{n+1}(t)$ using equation (12) leads to the formula. □

This result allows to formulate an algorithm for *degree elevation*. Due to the linear independence of the rational Bernstein functions, there exists only one formula of this kind.

4. De Casteljau-type algorithms

Given the control points $P_0,\ldots,P_n \in \mathbb{R}^d$ for some dimension d, we define a rational Bézier curve in $\mathbb{R}^d$,

$$c(t) = \sum_{i=0}^{n} P_i \rho_i^n(t).\tag{15}$$

Definition 5. For any given value of $t \in [0, 1]$, the *de Casteljau-type algorithm* defines recursively the points

$$P_i^0 = P_i, \text{ for } i = 0, \ldots n; \tag{16}$$

$$P_i^j = \frac{\alpha_j(1-t)}{L_j} P_i^{j-1} + \frac{b_j t}{L_j} P_{i+1}^{j-1}, \text{ for } j = 1, \ldots, n \text{ and } i = 0, \ldots, (n-j). \tag{17}$$

Proposition 6. *The points defined in the de Casteljau-type algorithm satisfy*

$$P_i^j = \sum_{k=0}^{j} P_{i+k} \rho_k^j(t). \tag{18}$$

In particular we have $P_0^n = c(t)$.

PROOF. We proceed by mathematical induction. For $j = 0$ we get (18) by the convention $\rho_0^0(t) \equiv 1$. For the induction step we get

$$P_i^j = \frac{\alpha_j(1-t)}{L_j} P_i^{j-1} + \frac{b_j t}{L_j} P_{i+1}^{j-1} = \tag{19}$$

$$= \frac{\alpha_j(1-t)}{L_j} \left(\sum_{k=0}^{j-1} P_{i+k} \rho_k^{j-1}(t) \right) + \frac{b_j t}{L_j} \left(\sum_{k=0}^{j-1} P_{i+k+1} \rho_k^{j-1}(t) \right) = \tag{20}$$

$$= \sum_{k=0}^{j} P_{i+k} \left(\frac{\alpha_j(1-t)}{L_j} \rho_k^{j-1}(t) + \frac{b_j t}{L_j} \rho_{k-1}^{j-1}(t) \right) = \sum_{k=0}^{j} P_{i+k} \rho_k^j(t), \tag{21}$$

where the last equality follows from (11). □

The maximum number of different de Casteljau-type algorithms of this form is $n!$ (This is a factorial, not an exclamation mark!). Indeed, if all linear factors are different, then their permutations define the different algorithms. Note that all these de Casteljau-type algorithms are different from the standard rational de Casteljau algorithm, see Example 9.

For each step (17) of these algorithms, the ratio used to generate the new point from the two existing ones is the same for all i. This is different from the standard de Casteljau algorithm (see Figure 3), where a different ratio is used in each linear combination.

Our approach can be extended to quadratic elementary factors of the denominator as follows. Consider linear factors with complex coefficients. If two consecutive linear factors are conjugate complex, then their product is real and the composition of the corresponding two steps in the de Casteljau-type algorithms gives linear combinations with real coefficients. This leads to a de Casteljau-type algorithm also for quadratic elementary factors, since these can be split into two adjacent conjugate complex linear factors. Consequently, we can extend this approach to rational curves with any denominator.

5. Examples

We present three examples that illustrate the findings of this paper.

Example 7. For the special choice $a_i = a$, $b_i = b$ we obtain $w_i^n = a^{n-i}b^i$. In this case, the rational basis functions are the Bernstein polynomials composed with a rational reparametrization of degree 1 that maps the boundaries of the interval $[0, 1]$ onto itself. More precisely we get

$$\rho_i^n(t) = \beta_i^n \left(\frac{bt}{a(1-t) + bt} \right). \tag{22}$$

Example 8. For the special choice $a_i = 1$, $b_i = q^{i-1}$, where q is a positive real number we get the "Lupaş q-analogues of the Bernstein functions", which were considered earlier in [2]. The authors of that paper observed that the weights admit a particularly nice closed-form representation in this case.

Example 9. Consider three linear factors

$$L_1(t) = 3(1-t) + t, \quad L_2(t) = 6(1-t) + 5t, \quad L_3(t) = 1(1-t) + 3t. \tag{23}$$

We obtain the weights

$$w_0^1 = 6, w_1^1 = \frac{2}{3} \tag{24}$$

$$w_0^2 = 36, w_1^2 = 14, w_2^2 = \frac{10}{3} \tag{25}$$

$$w_0^3 = 216, w_1^3 = 300, w_2^3 = 272, w_3^3 = 180. \tag{26}$$

The corresponding cubic rational basis functions $\rho_i^3(t)$ are displayed in Figure 1. We consider a curve with control points

$$P_0 = [0, 0], P_1 = [-1, 1], P_2 = [2, 3], P_3 = [1, 0]. \tag{27}$$

The de Casteljau-type algorithm for $t = 1/2$ generates the points P_j^i

$$
\begin{array}{c|cccc}
i \backslash j & 0 & 1 & 2 & 3 \\
\hline
0 & [0, 0] & [-1, 1] & [2, 3] & [1, 0] \\
1 & [-\frac{1}{4}, \frac{1}{4}] & [-\frac{1}{4}, \frac{3}{2}] & [\frac{7}{4}, \frac{9}{4}] & \\
2 & [-\frac{1}{4}, \frac{9}{11}] & [\frac{29}{44}, \frac{81}{44}] & & \\
3 & [\frac{19}{44}, \frac{279}{176}], & & &
\end{array}
\tag{28}
$$

which are displayed on Figure 2, top left. The five additional permutations of the factors L_1, L_2 and L_3 lead to five further de Casteljau-type algorithms that generate the same curve point.

Note that these algorithms do not provide the tangent property of the classical de Casteljau algorithm, i.e., the line connecting the last two points is generally not tangent to the curve. Similarly, these algorithms do not have a subdivision property and cannot be used to split the curve, as they are not based on the blossoming approach, cf. [5].
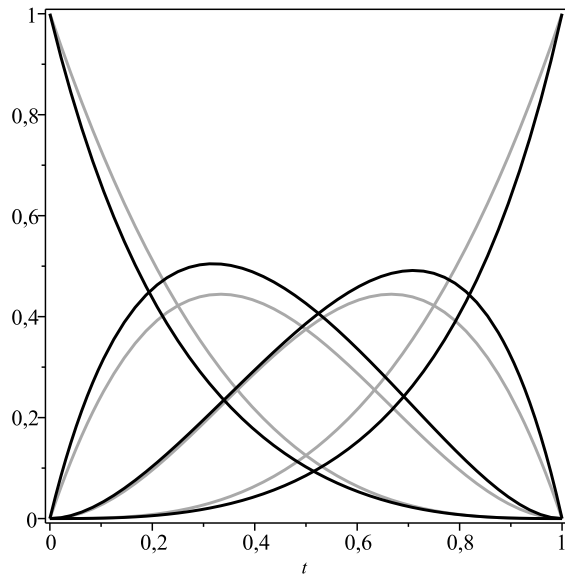
Figure 1: The rational Bernstein functions of degree three from Example 9 (black) compared to the Bernstein polynomials (gray).
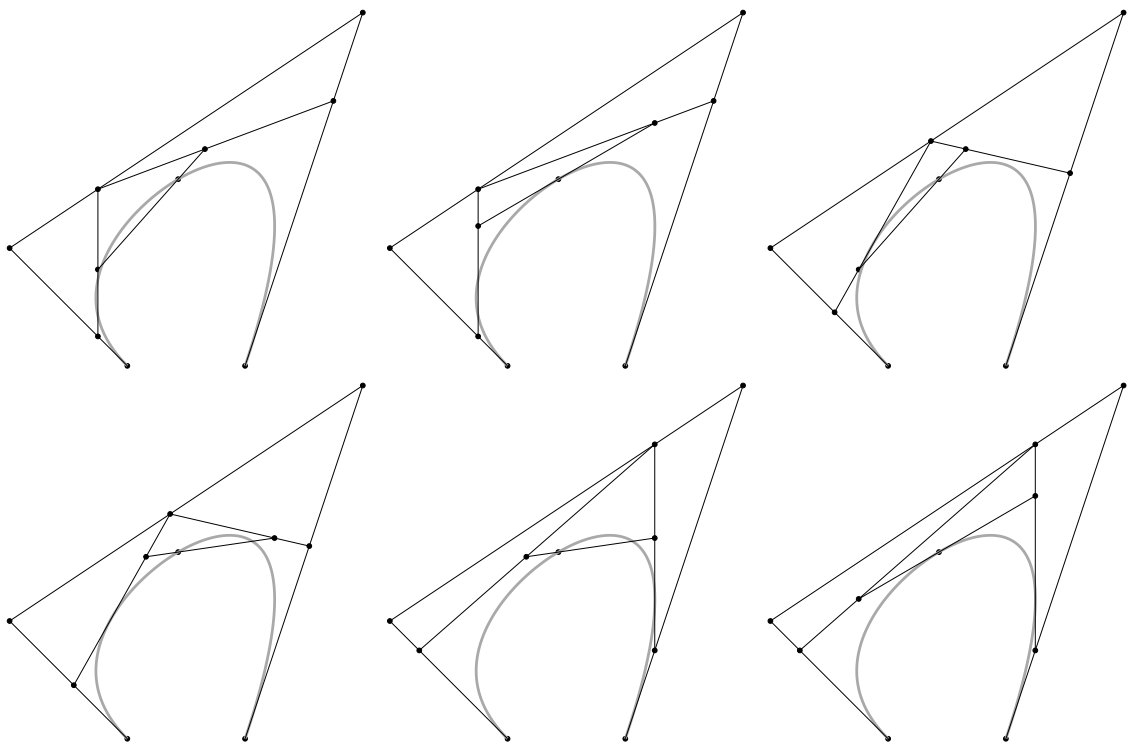

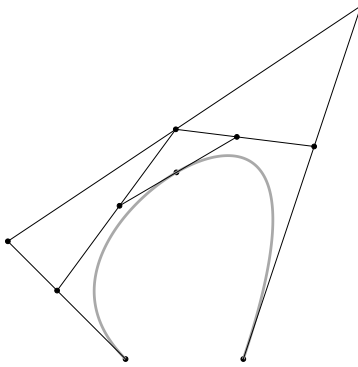Figure 2: Six different de Casteljau-type algorithms for value $t = 1/2$.

Figure 3: The standard rational de Casteljau algorithm for value $t = 1/2$.

6. Conclusion

We have analyzed nested spaces of rational functions, obtained by successively multiplying the denominator with linear factors. We were able to determine the recurrence formulas for weights and basis functions of these spaces. Each ordering of the denominator factors provides a de Casteljau-type algorithm for curves expressed with respect to these rational basis.

The algorithms can be extended in a straightforward way to the case of tensor-product patches. Indeed, in this case each variable is handled separately. Future research could be devoted to triangular rational patches with denominators that have only linear elementary factors, and to rational spline curves and surfaces.

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