Global gradient estimates in elliptic problems under minimal data and domain regularity

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1. A.C. & V.Maz'ya Global Lipschitz regularity for a class of quasilinear elliptic equations, Comm. Part. Diff. Equat. (2011)
2. A.C. & V.Maz'ya Global boundedness of the gradient for a class of nonlinear elliptic systems, Arch. Ration. Mech. Anal. (2014)
3. A.C. & V.Maz'ya Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems, J. Europ. Math. Soc. (2014)

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is a Carathéodory function, and $\exists\; p > 1$ and $C > 0$ s.t., for a.e. $x \in \Omega$:

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$$[a(x, \xi) - a(x, \eta)] \cdot (\xi - \eta) > 0 \quad \text{for } \xi, \eta \in \mathbb{R}^n \text{ with } \xi \neq \eta .$$

Model case: p-Laplace Dirichlet problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

If $f \in (W_0^{1,p}(\Omega))' \cap L^1(\Omega)$, then weak solutions $u \in W_0^{1,p}(\Omega)$ are well defined; namely

$$\int_\Omega a(x, \nabla u) \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx$$

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If f is just in $L^1(\Omega)$, solutions u to the Dirichlet problem (1) can be defined as limits of solutions to approximating problems with smooth right-hand sides.

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Covers a full range of norm bounds for solutions and their gradient in terms of norms of the datum f.

The decreasing rearrangement $u^*: [0, \infty) \to [0, \infty]$ of a measurable function u in Ω is defined as

$$u^*(s) = \sup\{t \geq 0 : |\{|u(x)| > t\}| > s\} \qquad \text{for } s \in [0, \infty).$$

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We also set $u^{**}(s) = \dfrac{1}{s} \displaystyle\int_0^s u^*(r)\, dr$.

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In formulas,

$$f^\bigstar(x) = f^*(\omega_n |x|^n) \quad \text{for } x \in \Omega^\bigstar,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Theorem [Talenti]

Let u be the solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

and let v be the solution to the symmetrized problem

$$\begin{cases} -\mathrm{div}(|\nabla v|^{p-2}\nabla v) = f^\bigstar(x) & \text{in } \Omega^\bigstar \\ v = 0 & \text{on } \partial\Omega^\bigstar\,. \end{cases}$$

Let $0 < q \leq p$. Then

$$\int_{\Omega} |\nabla u|^q dx \leq \int_{\Omega^\bigstar} |\nabla v|^q dx;$$

equivalently,

$$\|\nabla u\|_{L^q(\Omega)} \leq (n\omega_n^{1/n})^{-\frac{1}{p-1}} \left(\int_0^{|\Omega|} r^{-\frac{q}{n'(p-1)}} \left(\int_0^r f^*(\varrho) d\varrho \right)^{\frac{q}{p-1}} dr \right)^{\frac{1}{q}},$$

where $n' = \dfrac{n}{n-1}$.

Via this estimate, bounds for the norms $\|\nabla u\|_{L^q(\Omega)}$, with $q \leq p$, in terms of rearrangement invariant norms of the datum f are reduced to one-dimensional Hardy-type inequalities.

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A rearrangement invariant (briefly, r.i.) space $X(\Omega)$ is a Banach function space such that

$$\|w\|_{X(\Omega)} = \|z\|_{X(\Omega)} \quad \text{if } w^* = z^*.$$

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If $X(\Omega)$ is an r.i. space, then there exists a representation space $\overline{X}(0, |\Omega|)$ s.t.

$$\|w\|_{X(\Omega)} = \|w^*\|_{\overline{X}(0,|\Omega|)} \quad \forall w \in X(\Omega).$$

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- Orlicz spaces $L^A(\Omega)$:

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\big(|u(x)|/\lambda\big)\,dx \le 1\right\}.$$

Corollary

Let $0 < q \le p$ and let $X(\Omega)$ be an r.i. space such that

$$\left\| r^{-\frac{1}{n'(p-1)}} \left(\int_0^r \varphi(\varrho) d\varrho \right)^{\frac{1}{p-1}} \right\|_{L^q(0,|\Omega|)} \le C \|\varphi\|_{\overline{X}(0,|\Omega|)}^{\frac{1}{p-1}},$$

for some constant C and every nonnegative and non-increasing function $\varphi \in \overline{X}(0,|\Omega|)$. If $f \in X(\Omega)$ and u is the solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \,, \end{cases}$$

then

$$\|\nabla u\|_{L^q(\Omega)} \le C (n\omega_n^{1/n})^{-\frac{1}{p-1}} \|f\|_{X(\Omega)}^{\frac{1}{p-1}}.$$

Estimates for $\|\nabla u\|_{L^q(\Omega)}$ with $q > p$, cannot hold without additional smoothness assumptions on the function $a(x, \xi)$ and on Ω.

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• Limiting cases (e.g. weak type estimates, namely estimates in Marcikiewicz spaces): [Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vazquez, 1995], [Dolzmann, Hungerbühler, Müller, 2000].

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• Local estimates in Lorentz spaces [Mingione 2010].

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Theorem [Alvino, C. , Maz'ya, Mercaldo, 2010]

Let u be the weak solution to the Dirichlet problem

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Then

$$|\nabla u|^*(s) \le C(n,p) \left(\frac{1}{s} \int_{\frac{s}{2}}^{|\Omega|} r^{-\frac{p'}{n'}} \left(\int_0^r f^*(\rho)\, d\rho \right)^{p'} dr \right)^{\frac{1}{p}}$$

for $s \in (0, |\Omega|)$.

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Counterparts of the estimates for $|\nabla u|$ hold for solutions to Neumann problems

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Optimal norms in estimates on irregular domains require the use of isocapacitary inequalities instead of isoperimetric inequalities [C. , Maz'ya], [Alvino, C. , Maz'ya, Mercaldo].

Estimates for norms $\|\nabla u\|_{Y(\Omega)}$ stronger than $\|\nabla u\|_{L^p(\Omega)}$, require smoothness of the function $a(x, \xi)$ and regularity of Ω.

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$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4}$$

where $\int_\Omega f(x)\,dx = 0$.

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If $n \geq 3$, u is the Newtonian potential of f, namely

$$u(x) = C(n) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy \qquad \text{for } x \in \mathbb{R}^n.$$

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Hence,

$$|\nabla u(x)| \leq C'(n) \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\, dy \qquad \text{for } x \in \mathbb{R}^n.$$

A rearrangement inequality for convolutions by O'Neil implies that

$$|\nabla u|^*(s) \leq C' \int_s^\infty f^{**}(r) r^{-\frac{1}{n'}} dr \quad \text{for } s > 0. \tag{5}$$

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Is there an analogue of (5) for nonlinear problems?

First step, of independent interest: maximal integrability property of $|\nabla u|$, namely boundedness of $|\nabla u|$.

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In case of systems, global $C^{1,\alpha}$-regularity, with $\partial\Omega \in C^{1,\beta}$, is established in [Chen, Di Benedetto 1989] for $f \in L^\infty(\Omega)$, and in [Beirão da Veiga, Crispo] for $p < 2$ (“close ” to 2) and $f \in L^q(\Omega)$, with $q > n$.

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• Pb.: minimal integrability of f and minimal regularity of Ω for $|\nabla u| \in L^\infty(\Omega)$, i.e. u Lipschitz continuous.

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$L^{n,1}(\Omega)$ is a kind of borderline space. Recall that, if $|\Omega| < \infty$ and $q > n$, then

$$L^q(\Omega) \subsetneqq L^{n,1}(\Omega) \subsetneqq L^n(\Omega).$$

Theorem [C., Maz'ya]

Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $f \in L^{n,1}(\Omega)$. Let u be a weak solution to either the Dirichlet or the Neumann p-Laplacian problem. Then there exists a constant $C = C(p, \Omega)$ such that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}}. \qquad (6)$$

In particular, u is Lipschitz continuous on $\overline{\Omega}$.

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Counterexamples show that, even for the scalar Laplace operator, a solution $u \notin C^1(\overline{\Omega})$ can exist in a convex domain with $\partial\Omega \in C^1$.

Independent result, in the same spirit, by [Duzaar, Mingione, 2010] for local solutions (approach via nonlinear potentials)

• The spaces $W^2L^{n-1,1}$ and $L^{n,1}$ are independent of p, and they are essentially optimal. In particular, the space $L^{n,1}$ is the same as for the Laplace equation in B.

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• The result is new even for

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

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by Δu, and integrate over the level sets of the gradient

$$\int_{\{|\nabla u|>t\}} \Delta u f\,dx = -\int_{\{|\nabla u|>t\}} \Delta u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)dx \quad \text{for } t > 0.$$

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• Estimate

$$\Delta u \,\mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

by an expression in divergence form, integrate by parts, use the boundary condition.

- Use:

- the coarea formula applied to $|\nabla u|$, namely

$$\int_\Omega \phi(x)|\nabla|\nabla u||\,dx = \int_0^\infty \int_{\{|\nabla u|=t\}} \phi(x) d\mathcal{H}^{n-1}(x)\,dt\,,$$

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to derive a differential inequality for the distribution function of $|\nabla u|$

$$\nu(t) = |\{|\nabla u|>t\}|.$$

Obtain

$$
\begin{aligned}
t^{2p-2} \leq\ & C\|\nabla u\|_{L^\infty(\Omega)}^{p}(-\nu'(t))\nu(t)^{-1/n'}\phi(\nu(t)) \\
& + C\|\nabla u\|_{L^\infty(\Omega)}(-\nu'(t))\nu(t)^{-2/n'}\int_0^{\nu(t)} f^*(r)^2 dr \\
& + C\|\nabla u\|_{L^\infty(\Omega)}^{2p-1}(-\nu'(t))\nu(t)^{-1/n'}k^{**}\left(C'\nu(t)^{1/n'}\right)
\end{aligned} \tag{7}
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for a.e. $t > \operatorname{med}(|\nabla u|)$.

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- $$\phi(s) = \left(\frac{d}{ds}\int_{\{|\nabla u|>|\nabla u|^*(s)\}} f^2 dx\right)^{1/2}$$ for a.e. $s \in (0, \mathcal{H}^n(M))$,

a so-called pseudo-rearrangement of f^2,

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a so-called pseudo-rearrangement of f^2,

- k stands for the curvature of Ω.

• Integrate the differential inequality (7) and use again properties of rearrangements to conclude that

$$\|\nabla u\|_{L^\infty(\Omega)} \le C\|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}}.$$

□

Pb.: Rearrangement estimate for $|\nabla u|$.

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Theorem [C., Maz'ya]

Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $2 \leq p < n$, $f \in L^1(\Omega)$, and let u be the solution to either the Dirichlet problem or the Neumann p-Laplacian problem. Then there exists a constant $C = C(p, \Omega)$ such that

$$|\nabla u|^*(s)^{p-1} \leq C \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}} dr \quad \text{for } s \in (0, |\Omega|). \tag{8}$$

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Recall that for the Laplace equation in $\mathbb{R}^n$

$$|\nabla u|^*(s) \leq C \int_s^{\infty} f^{**}(r) r^{-\frac{1}{n'}} dr \quad \text{for } s > 0.$$

Important consequence of the estimate

$$|\nabla u|^*(s)^{p-1} \le C \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}} \, dr.$$

It translates verbatim the linear theory of integrability of $|\nabla u|$ for the Laplace equation to the theory of integrability of $|\nabla u|^{p-1}$ for the nonlinear p-Laplace equation.

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Pointwise estimates, for local solutions, involving Riesz potentials are established in [Kuusi-Mingione, 2011].

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An idea would be to interpolate between these two estimates, on making use of Peetre K-functional.

Pb.: the map

$$f \mapsto \nabla u$$

is not linear!

However:

- One can prove the stability estimate

$$\|\nabla u - \nabla v\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C\|f-g\|_{L^1(\Omega)}^{\frac{1}{p-1}}, \qquad (11)$$

where v is the solution to the same problem, with the right-hand side f replaced by g.

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where v is the solution to the same problem, with the right-hand side f replaced by g.

• Use a nonlinear interpolation argument, again relying upon Peetre K-functional, between inequalities (9) and (11) to conclude that

$$|\nabla u|^*(s)^{p-1} \leq C \int_s^{|\Omega|} f^{**}(r) r^{-\frac{1}{n'}} dr \quad \text{for } s \in (0, |\Omega|).$$

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Distinctive feature of the rearrangement estimate: it is independent of specific function spaces. It reduces any inequality between r.i. (quasi-)norms of $|\nabla u|$ and f to one-dimensional Hardy-type inequalities involving the corresponding representation quasi-norms.

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Corollary

Let Ω be as above. Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces on Ω. Assume that there exists a constant C such that

$$\left\| \int_s^{|\Omega|} \int_0^r \varphi(\rho)\, d\rho\, r^{-\frac{1}{n'}-1} dr \right\|_{\overline{Y}(0,|\Omega|)} \leq C \|\varphi\|_{\overline{X}(0,|\Omega|)}.$$

for every $\varphi \in \overline{X}(0,|\Omega|)$. If $f \in X(\Omega)$ and u is the solution to either the Dirichlet problem or the Neumann p-Laplacian problem, then there exists a constant C' such that

$$\|\, |\nabla u|^{p-1} \,\|_{Y(\Omega)} \leq C' \|f\|_{X(\Omega)}.$$

Applications.

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(i) If $1 \leq r \leq \infty$ and $1 < q < n$, then

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$$\|\nabla u\|_{L^{\frac{qn(p-1)}{n-q},r(p-1)}(\Omega)} \leq C\|f\|_{L^{q,r}(\Omega)}^{\frac{1}{p-1}}.$$

(ii) If $q = 1$ and $r = 1$, then $L^{1,1}(\Omega) = L^1(\Omega)$

$$\|\nabla u\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p-1}}.$$

(iii) If $q = n$ and $r > 1$, then

$$\|\nabla u\|_{L^{\infty,r(p-1)}(\log L)^{-\frac{1}{p-1}}(\Omega)} \leq C\|f\|_{L^{n,r}(\Omega)}^{\frac{1}{p-1}},$$

where

$$\|\nabla u\|_{L^{\infty,r(p-1)}(\log L)^{-\frac{1}{p-1}}(\Omega)} = \left(\int_0^{|\Omega|} |\nabla u|^*(s)^{r(p-1)} \frac{ds}{s\log^r(1/s)}\right)^{\frac{1}{r(p-1)}}.$$

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(iv) If either $q > n$, or $q = n$ and $r = 1$, then

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2. Bounds in Orlicz spaces.

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and the Sobolev conjugate A_n of A

$$A_n(t) = A\big(H^{-1}(t)\big) \qquad \text{for } t \geq 0$$

[C. 1996].

Assume that there exists $c > 0$ s.t.

$$B(t) \le A_n(ct) \quad \text{and} \quad \widetilde{A}(t) \le \big(\widetilde{B}\big)_n(ct) \quad \text{for } t > 0.$$

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Then there exist a constant C such that

$$\| |\nabla u|^{p-1} \|_{L^B(\Omega)} \le C \|f\|_{L^A(\Omega)}.$$

For example, if either $q > 1$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \geq 0$ denote by

$$L^q \log^\alpha L(\Omega)$$

the Orlicz space associated with

$$A(t) \approx t^q \log^\alpha t \quad \text{near infinity.}$$

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If either $q > n$ or $q = n$ and and $\alpha > n-1$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q \log^\alpha L(\Omega)}^{\frac{1}{p-1}}.$$