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Lipschitz truncation

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Approximate a Sobolev function in a suitable way by Lipschitz functions.

Requirement: Change the function only on a small set.

Cannot use convolution as it changes the function on a large set!

Acerbi-Fusco 1984

Method first appeared in:

Theorem (Acerbi-Fusco '84)

If $f = f(x, s, \xi)$ is Caratheodory, quasi-convex in ξ and for some $p > 1$

$$0 \leq f(x, s, q) \lesssim a(x) + |s|^p + |q|^p,$$

then $f \mapsto \int f(x, u(x), \nabla u(x))\, dx$ *is $W^{1,p}$-weakly seq. lower continuous.*

Idea:

- First show $W^{1,\infty}$-weak-*-sequentially lower continuity.
- Now approximate $W^{1,p}$ functions by $W^{1,\infty}$ functions.

Lipschitz truncation – cutting the gradients

Maximal function: $$(Mf)(x) = \sup_{B \ni x} \fint_B |f|\,dy.$$

Majorant: $|f| \leq Mf$

Bounded: $\|Mf\|_p \lesssim \|f\|_p$ for $p > 1$ and $\sup_{\lambda>0} \big(\lambda |\{Mf > \lambda\}|\big) \lesssim \|f\|_1$.

For $\mathbf{w} \in W_0^{1,1}(\Omega)$ we have

$$|\mathbf{w}(x) - \mathbf{w}(y)| \lesssim |x-y| \big(M(\nabla \mathbf{w})(x) + M(\nabla \mathbf{w})(y)\big),$$

- $\mathbf{w}$ is Lipschitz outside the small, open bad set $\{M(\nabla \mathbf{w}) > \lambda\}$.
- Cut out the bad set and

  extend to $\mathbf{w}_\lambda \in W_0^{1,\infty}(\Omega)$ with $\|\nabla \mathbf{w}_\lambda\|_\infty \lesssim \lambda$.

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Applied to weak sequences

- Start with $\mathbf{w}^n \rightharpoonup 0$ in $W^{1,p}$.

  Then Lipschitz truncations satisfy for fixed $\lambda > 0$

  - $\|\nabla \mathbf{w}_\lambda^n\|_\infty \lesssim \lambda$,
  - $\mathbf{w}_\lambda^n \overset{*}{\rightharpoonup}$??? in $W^{1,\infty}$ (subsequence).

  This makes some technical problems in [Acerbi-Fusco 1984].

- Landes showed in 1996 that $\mathbf{w}_\lambda^n \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}$:

  use the bad set $\{M(\nabla \mathbf{w}) > \lambda\} \cup \{M\mathbf{w} > \theta_n\}$ with $\theta_n \to 0$ slowly

  and $\|\mathbf{w}_\lambda\|_\infty \lesssim \theta_n$.

Applied to weak sequences

- Start with $\mathbf{w}^n \rightharpoonup 0$ in $W^{1,p}$.

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  - $\|\nabla \mathbf{w}^n_\lambda\|_\infty \lesssim \lambda$,
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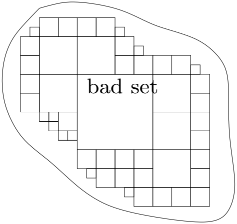
  and $\|\mathbf{w}_\lambda\|_\infty \lesssim \theta_n$.

Lipschitz truncation – today

$\mathbf{w} \in W^{1,1}$ is Lipschitz outside bad set $\mathtt{Bad}_\lambda := \{M(\nabla \mathbf{w}) > \lambda\}$.

Whitney covering $\mathtt{Bad}_\lambda = \bigcup_i Q_i$ with partition of unity φ_i



$$\mathbf{w}_\lambda := \begin{cases} \mathbf{w} & \text{on good set,} \\ \sum_i \varphi_i \langle \mathbf{w} \rangle_{Q_i} & \text{on bad set.} \end{cases}$$

Rewrite as $$\mathbf{w} = \mathbf{w}_\lambda + \sum_i \varphi_i (\mathbf{w} - \langle \mathbf{w} \rangle_{Q_i})$$ since $\sum_i \varphi_i = 1$ on $\mathtt{Bad}_\lambda$.

Basic properties

Well defined

We have $\mathbf{w}_\lambda \in W^{1,1}$

Use $\mathbf{w} = \mathbf{w}_\lambda + \sum_i \varphi_i(\mathbf{w} - \langle \mathbf{w} \rangle_{Q_i})$. Sum converges in $W^{1,1}$.

Stability

$\|\mathbf{w}_\lambda\|_p \lesssim \|\mathbf{w}\|_p$ and $\|\nabla \mathbf{w}_\lambda\|_p \lesssim \|\nabla \mathbf{w}\|_p$ for $1 \leq p \leq \infty$.

Lipschitz property

$M(\nabla \mathbf{w}_\lambda) \lesssim \lambda$. In particular, $\|\nabla \mathbf{w}_\lambda\|_\infty \lesssim \lambda$.

Calderón-Zygmund decomposition

Lipschitz truncation

We can decompose $\mathbf{w} \in W^{1,1}$ into

$$\mathbf{w} = \mathbf{w}_\lambda + \sum_i \varphi_i(\mathbf{w} - \langle \mathbf{w} \rangle_{Q_i})$$

with $\mathbf{w}_\lambda \in W^{1,\infty}$.

Calderón-Zygmund decomposition

We can decompose $f \in L^1$ into

$$f = g + \sum_i \varphi_i(f - \langle f \rangle_{Q_i})$$

with $g \in L^\infty$.

Additional smallness

Let $1 < p < \infty$.

Note that $\{\mathbf{w} \neq \mathbf{w}_\lambda\} \subset \{M(\nabla \mathbf{w}) > \lambda\}$.

Weak type estimate: $\lambda^p |\{M(\nabla \mathbf{w}) > \lambda\}| \leq c\, \|\nabla \mathbf{w}\|_p^p$.

Strong type estimate: $\sum_j (2^j)^p |\{M(\nabla \mathbf{w}) > 2^j\}| \leq c\, \|\nabla \mathbf{w}\|_p^p$.

Most summands are small.

Smallness [D., Malek, Steinhauer '08; FMS '03]

There exists $\lambda \in [2^{2^j}, 2^{2^{j+1}}]$ with

$$\|\chi_{\{\mathbf{w} \neq \mathbf{w}_\lambda\}} \nabla \mathbf{w}_\lambda\|_p^p \leq c\, \lambda^p |\{M(\nabla \mathbf{w}) > \lambda\}| \leq c\, 2^{-j}\, \|\nabla \mathbf{w}\|_p^p.$$

Preserving zero boundary values

Theorem

Lipschitz truncation can preserve zero boundary values!

Recall $\mathbf{w} = \mathbf{w}_\lambda + \sum_i \varphi_i(\mathbf{w} - \langle \mathbf{w} \rangle_i)$.

away from $\partial\Omega$: $\mathbf{w}_i := \langle \mathbf{w} \rangle_{Q_i}$

close to $\partial\Omega$: $\mathbf{w}_i := 0$

Need assumptions on Ω for Poincaré:
fat complement.

Applied to weak sequences

Theorem (Diening, Málek, Steinhauer '07, +Breit '11)

For $\mathbf{w}^n \rightharpoonup 0 \in W_0^{1,p}$ *and* $p > 1$ *exists* $\mathbf{w}_k^n \in W_0^{1,\infty}$ *such that*

- $\mathbf{w}_k^n \overset{n}{\to} 0$ *strongly in* L^∞,
- $\nabla \mathbf{w}_k^n \overset{n}{\to} 0$ **-weakly in* L^∞,
- $\|\nabla \mathbf{w}_k^n \, \chi_{\{\mathbf{w}_n \neq \mathbf{w}_k^n\}}\|_p^p \lesssim \lambda^p \, |\mathtt{Bad}_k^n| \lesssim 2^{-k}$.

Reproving the result of Acerbi-Fusco

Let $\mathbf{u} \in W^{1,p}(\Omega)$ and $\mathbf{w}^n \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$. Then

$$
\begin{aligned}
F(\mathbf{u},\Omega) &:= \int_\Omega f(x,\mathbf{u},\nabla \mathbf{u})\,dx \\
&\leq \liminf_n F(\mathbf{u}+\mathbf{w}_k^n) \qquad \text{by } \mathbf{w}_k^n \overset{*}{\rightharpoonup} 0 \text{ in } W_0^{1,\infty}(\Omega) \text{ for } n \to \infty \\
&\leq \lim_n F(\mathbf{u}+\mathbf{w}_k^n, \mathtt{Bad}_k^n) + \liminf_n F(\mathbf{u}+\mathbf{w}^n, \mathtt{Good}_k^n) \\
&\leq \lim_n F(\mathbf{u}+\mathbf{w}_k^n, \mathtt{Bad}_k^n) + \liminf_n F(\mathbf{u}+\mathbf{w}^n, \Omega) \qquad \text{using } f \geq 0 \\
&\leq \lim_n \int_{\mathtt{Bad}_k^n} |\nabla u|^p + \lambda^p\,dx + \liminf_n F(\mathbf{u}+\mathbf{w}^n,\Omega) \qquad \text{growth cond.} \\
&\leq \varepsilon_k + 2^{-k} + \liminf_n F(\mathbf{u}+\mathbf{w}^n,\Omega).
\end{aligned}
$$

Now $j \to \infty$ proves the $W^{1,p}$-weak-seq. lsc property of F.

Power law fluids

Lipschitz truncation can be used to prove:

Theorem (Frehse, Málek, Steinhauer '03; +Diening '07)

There exists a weak solution for $p > \frac{6}{5}$ in $\mathbb{R}^3$ to

$$-\mathrm{div}\big(|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u})\big) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{f},$$
$$\mathrm{div}\mathbf{u} = 0.$$

- Convection $(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathrm{div}(\mathbf{u}\otimes\mathbf{u})$, requires $W^{1,p} \hookrightarrow L^2$, i.e. $p \geq \frac{6}{5}$.
- Use Lipschitz truncation & pointwise monotonicity.
- Use Bogovskiĭ correction to overcome $\mathrm{div}\mathbf{u}_\lambda \neq 0$.
- $p(\cdot)$ is possible (electrorheological fluids).

Prantl-Eyring fluids

Stationary problems

Prandtl-Eyring fluids: constitutive law reads

$$\mathbf{S}(\varepsilon(\mathbf{u})) = \frac{\log(1 + |\varepsilon(\mathbf{u})|)}{|\varepsilon(\mathbf{u})|}\varepsilon(\mathbf{u}).$$

- Natural spaces $L^{t \ln t}$ (almost $p = 1$).
- Critical already for $n = 2$ due to convection $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$.
- Problem: Bogovskiĭ-operator unbounded on $L^{t \ln t}$.

$\Rightarrow$ Need solenoidal Lipschitz truncation, i.e. $\operatorname{div}(\mathbf{u}_\lambda) = 0$.

Solenoidal Lipschitz truncation; stationary

Curl representation for $\mathbb{R}^3$

For $\mathbf{u} \in W^{1,p}_{0,\mathrm{div}}$ exists $\boldsymbol{\omega} := \mathrm{curl}^{-1}\mathbf{u} \in W^{2,p}_{\mathrm{div}}(\Omega)$.

Use $W^{2,\infty}$-truncation for $\boldsymbol{\omega}$ (Lipschitz truncation of second order)

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\lambda + \sum_i \varphi_i(\boldsymbol{\omega} - \boldsymbol{\omega}_i).$$

with $\boldsymbol{\omega}_i$ local, linear approximations of $\boldsymbol{\omega}$.

Solenoidal Lipschitz truncation [Breit, Diening, Schwarzacher '12]

$\mathbf{u}_\lambda := \mathrm{curl}(\boldsymbol{\omega}_\lambda)$ is solenoidal and behaves like Lipschitz truncation.

Also: [Breit, Diening, Fuchs '11; Diening, Kreuzer, Süli '12]

Almost harmonic (1/3)

Definition

$u \in W^{1,1}(B)$ is almost harmonic on B if for suitable small $\delta > 0$

$$\left| \fint_B \mathcal{A} \nabla u \nabla \xi \, dx \right| \leq \delta \fint_B |\nabla u| \, dx \, \|\nabla \xi\|_\infty \qquad \text{for } \xi \in C_0^\infty(B).$$

Can use $\xi \in W_0^{1,\infty}(B)$.

Theorem (Approximation Lemma)

If u is almost harmonic, then there exists harmonic h close to u.

Giaquinta, Simon, Duzaar, Mingione, ...

Important for partial regularity!

Almost harmonic (2/3)

Theorem (Classical approximation Lemma)

If u is almost harmonic, then there exists harmonic h L^2-close to u

Classical: Contradiction argument with harmonic limit and compactness.

New direct approach:
$$\begin{aligned} -\Delta h &= 0 && \text{on } \Omega \\ h &= u && \text{on } \partial\Omega. \end{aligned}$$

Use $W_0^{1,\infty}$-truncation of $u - h$ as test function.

Theorem (Diening, Stroffolini, Verde)

$$\left(\fint_B |\nabla(u-h)|^{2\theta}\,dx \right)^{\frac{1}{\theta}} \leq \varepsilon(\delta,\theta) \fint_B |\nabla u|^2\,dx \qquad \text{for } \theta \in (0,1)$$

Almost harmonic (3/3)

What about the quasi-convex case?

Consider $-\mathrm{div}(\mathcal{A}\nabla u)$ with $\mathcal{A}$ Legendre-Hadamard elliptic tensor

Theorem (Diening, Lengeler, Stroffolini, Verde, Cruz-Uribe)

$$\fint_B |\nabla(u-h)|^q \, dx \leq c\,\delta^{\frac{q(s-1)}{sq-1}} \left(\fint_B |\nabla u|^{qs}\, dx\right)^{\frac{1}{s}} \qquad q \in (1,\infty), s > 1$$

Idea:

$$\|\nabla(u-h)\|_q \sim \sup_{\xi \in W_0^{1,q'}(B)} \frac{|\langle \mathcal{A}\nabla(u-h), \nabla\xi\rangle|}{\|\nabla h\|_{q'}}$$

Use $W_0^{1,\infty}$-Lipschitz truncation for ξ and weak-type estimates.

Almost p-harmonic

Definition

$u \in W^{1,\max\{p-1,1\}}(B)$ is almost harmonic on B if for suitable small $\delta > 0$

$$\left| \fint_B |\nabla u|^{p-2} \nabla u \nabla \xi \, dx \right| \leq \delta \fint_B |\nabla u|^{p-1} \, dx \|\nabla \xi\|_\infty \qquad \text{for } \xi \in C_0^\infty(B).$$

Duzaar, Mingione, ...

Theorem (Diening, Stroffolini, Verde)

Then the p-harmonic h with $u - h = 0$ on ∂B satisfies

$$\left(\fint_B |\nabla (u-h)|^{\theta p} \, dx \right)^{\frac{1}{\theta}} \leq \varepsilon(\delta, \theta) \fint_B |\nabla u|^p \, dx \qquad \theta \in (0,1)$$

Problem: Cannot go below power p.

Parabolic problems

Assume $u \in L^p(W^{1,p})$ and $H \in L^{p'}(L^{p'})$ and

$$\partial_t u = \operatorname{div} H \qquad \text{in } \mathcal{D}'(I \times \Omega)$$

Parabolic Poincaré-inequality

On parabolic cube $Q_r := (-\alpha r^2, \alpha r^2) \times B_r$

$$\fint_{Q_r} \left| \frac{u - u_{Q_r}}{r} \right| dx\,dt \leq c \fint_{Q_r} |\nabla u|\,dx\,dt + c\alpha \fint_{Q_r} |H|\,dx\,dt,$$

To match ∇u with H we need $\alpha = \lambda^{2-p}$.

Parabolic Lipschitz truncation [Kinnunen, Lewis '02, D. R. W. '10]

Roughly $u_\lambda \in L^\infty(W^{1,\infty}) \cap L^\infty(W^{-1,\infty})$

- Solenoidal version is possible! [Breit, Diening, Schwarzacher '13]

Parabolic Lipschitz truncation

Theorem (Diening, Schwarzacher, Stroffolini, Verde)

Let $\partial_t w = \operatorname{div} G$ *on* Q *with* $w = 0$ *on* $\partial_p Q$*. Then there exists* w_λ^α *with*

1. $w_\lambda^\alpha = 0$ *on* $\partial_p Q$.
2. $\|\nabla w_\lambda^\alpha\|_\infty \leq c\,\lambda$.
3. w_λ^α *is* λ*-Lipschitz with respect to metric* $\left(\frac{|t-s|}{\alpha}\right)^{\frac{1}{2}} + |x-y|$
4. $\langle \partial_t w_\lambda^\alpha, w - w_\lambda^\alpha \rangle \leq c\,\frac{\lambda^2}{\alpha} |\mathtt{Bad}_\lambda^\alpha|$.

where $\mathtt{Bad}_\lambda^\alpha := \{\mathcal{M}^\alpha(\nabla u) + \alpha \mathcal{M}^\alpha(G) > \lambda\}$.

p-caloric approximation lemma

Extension of: Duzaar, Mingione, Bögelein, Scheven, ...

Theorem (Diening, Schwarzacher, Stroffolini, Verde)

Let $\partial_t u = \operatorname{div} H$ and let u be p-caloric in the sense that for $\xi \in C_0^\infty(Q)$

$$\left| \fint_Q -u\partial_t \xi + A(\nabla u)\nabla \xi \, dz \right| \leq \delta \left(\fint_Q |\nabla u|^p dz + \fint_Q |H|^{p'} dz + \|\nabla \xi\|_\infty^{p'} \right)$$

Then the p-caloric h with $h = u$ on $\partial_p Q$ satisfies for $\theta \in (0,1)$

$$\left(\fint_Q |\nabla u - \nabla h|^{p\theta} \, dz \right)^{\frac{1}{\theta}} \leq \varepsilon(\delta, \theta) \left(\fint_Q |\nabla u|^p dz + \fint_Q |H|^{p'} dz \right).$$

Use parabolic Lipschitz truncation perserving zero boundary values.

Conclusion

- Lipschitz truncation is useful in many PDE problems

  (Existence; almost harmonic; almost caloric)

- Close relation to Calderón-Zygmund decomposition