Diffeomorphic approximation of $W^{1,1}$ planar Sobolev homeomorphisms

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2.5.2014, Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

Ball-Evans Problem

Problem [Ball-Evans]: $\Omega \subset \mathbf{R}^n$ domain, $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ homeomorphism. Can we find f_k piecewise affine (or diffeomorphisms) such that $f_k \to f$ in $W^{1,p}$?

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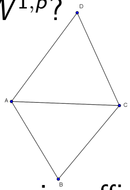

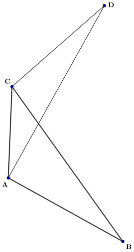

It is not easy: triangulization or mollification destroy injectivity

$\exists f_k$ smooth $\overset{\text{easy}}{\Rightarrow} \; \overset{\text{Pratelli\&Mora-Corral}}{\Leftarrow}$ $\exists f_k$ piecewise affine

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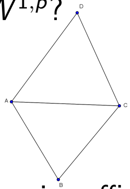

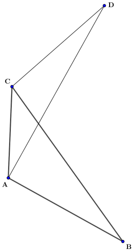

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- Regularity for models in Nonlinear Elasticity
  Ball models $\min \int W(Du)$ where $E(u) \to \infty$ as $J_u \to 0$

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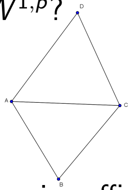

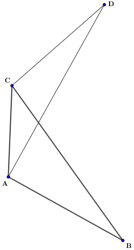

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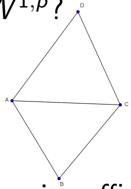

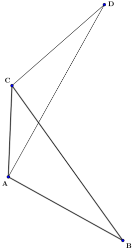

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- Regularity for models in Nonlinear Elasticity
  Ball models $\min \int W(Du)$ where $E(u) \to \infty$ as $J_u \to 0$
- Numerics - finite elements method
- Easier proofs of known (and new) statements

Known results

C. Mora-Corral: f smooth up to one point

Theorem (Iwaniec, Kovalev, Onninen)

Let $n = 2$ and $1 < p < \infty$. Given a homeomorphism $f \in W^{1,p}(\Omega, \mathbf{R}^2)$ there are diffeomorphisms f_k with $f_k \to f$ in $W^{1,p}$, $f_k \rightrightarrows f$ and $f_k - f \in W_0^{1,p}$

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Open problem: Can you find f_k with $f_k \to f$ in $W^{1,p}$ and $f_k^{-1} \to f^{-1}$ in $W^{1,p}$

Theorem (Daneri, Pratelli)

Let $n = 2$ and $1 \le p < \infty$. Given a bi-Lipschitz f there are diffeomorphims f_k with $f_k \to f$ in $W^{1,p}$ and $f_k^{-1} \to f^{-1}$ in $W^{1,p}$.

Idea of the proof - Iwaniec, Kovalev, Onninnen

$n = 2$, $1 < p < \infty$, $f \in W^{1,p}$ homeomorphism - $\exists\ f_k$

Theorem (Rado-Choquet-Knee; Allesandrini-Sigalloti)

Let $n = 2$ and $1 < p < \infty$. $f\ A \to Q$ homeomorphism onto convex Q. There is $g : A \to Q$, $f = g$ on ∂A such that g minimizes $\int_A |Dg|^p$. This g is a homeomorphism and it is smooth inside A.

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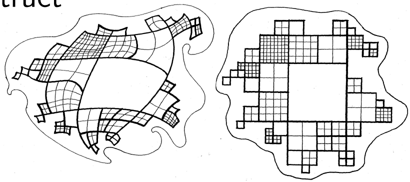
Cover $f(\Omega)$ by cubes ($\leq 1/k$) - construct f_k, smooth (technical difficulty)



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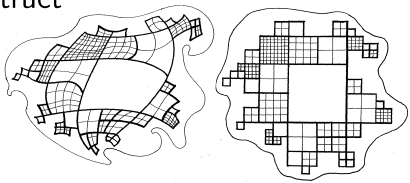

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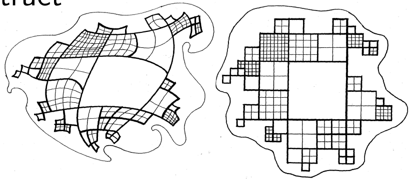

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$\Rightarrow f_k \to f$ in $W^{1,p}$

Allesandrini-Sigalloti - **works only for** $n = 2$ **and** $1 < p < \infty$

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$n = 2$, $1 \leq p < \infty$, f biLipschitz - $\exists$ f_k approximate f and f^{-1}

Theorem (Daneri, Pratelli)

Let $n = 2$ and $f\ \partial Q \to \mathbf{R}^2$ be L-biLipschitz and piecewise affine. There is piecewise affine and CL^4 biLipschitz $g : Q \to \mathbf{R}^2$ with $f = g$ on ∂Q.

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f is differentiable a.e. and a.e. point x is a Lebesque point of Df: $\int_{Q(x,2r)} |Df - Df(x)|^p < \varepsilon |Q|$ - Good centers
Cover Ω by cubes ($\leq 1/k$) - Good have measure $> |\Omega| - \delta$

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Approximate on the grid by piecewise linear mapping.

On good cubes use natural affine approximation - Lebesque squares - close norm
On bad cubes use Lemma - measure $< \delta \Rightarrow \int \leq \delta C L^{4p}$ small

New results

Theorem (H., Pratelli)

Let $n = 2$. Given a homeomorphism $f \in W^{1,1}(\Omega, \mathbf{R}^2)$ there are diffeomorphisms f_k with $f_k \to f$ in $W^{1,1}$.
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Theorem (Extension 1)

Let $\varphi : \partial Q_0 \to \mathbf{R}^2$ be a piecewise linear and one to one. Then there is a piecewise affine homeomorphism $h : Q_0 \to \mathbf{R}^2$ such that $h = \varphi$ on ∂Q_0 and $\int_{Q_0} |Dh(x)|\, dx \leq C \int_{\partial Q_0} |D\varphi(t)|\, dt$.

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Theorem (Extension 2)

Let $\varphi : \partial Q_0 \to \mathbf{R}^2$ be a piecewise linear and one to one with $\int_{\partial Q_0} \left| D\varphi(t) - \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} \tau \right| \, dt < \delta$. Then there is a piecewise affine homeomorphism $g : Q_0 \to \mathbf{R}^2$ such that $g = \varphi$ on ∂Q_0 and $\int_Q \left| Dg(x) - \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} \right| \, dx \leq C\delta$.

Idea of the proof - $W^{1,1}$ in the plane

1) Fine (Whitney type) grid on Ω: Good and Bad squares
$Q(c,r)$ Good $=$ f diff. at c on $5Q$, $\int_Q |Df - Df(c)|$ small

2) Adjust so that $(\operatorname{diam} Q)\int_{\partial Q} |Df| \leq C \int_{\frac{5}{4}Q} |Df|$
and $(\operatorname{diam} Q)\int_{\partial Q} |Df - Df(c)| \leq C \int_{\frac{5}{4}Q} |Df - Df(c)|$

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3) Approximate f on the grid by piecewise linear

4) On Bad and Zero ($J_f(c) = |Df(c)| = 0$) - Extension 1
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On Good natural affine approximation on 2 triangles

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$$\sum_{Q\in\mathcal{B}} \int_Q |Df - Dh| \leq \sum_{Q\in\mathcal{B}} \int_Q |Df| + (\text{diam } Q) \int_{\partial Q} |Df|$$

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$$\sum_{Q\in\mathcal{N}} \int_Q |Df - Dg| \leq \varepsilon + \sum_{Q\in\mathcal{N}} \int_Q |Dg - Df(c)|$$
$$\leq \varepsilon + \sum_{Q\in\mathcal{N}} \delta|Q| \leq 2\varepsilon$$

5) Mora-Corral and Pratelli $\Rightarrow$ approximation by smooth

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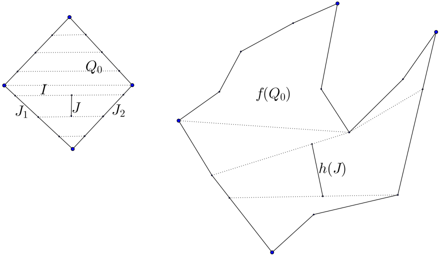

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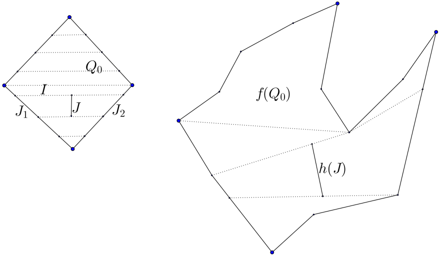

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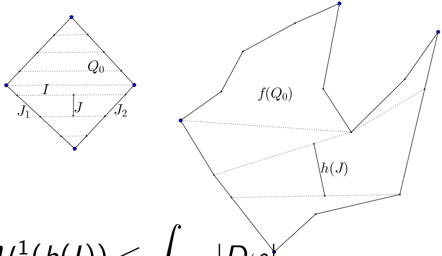

$$\int_I |D_x h| \leq \mathcal{H}^1(h(I)) \leq \int_{\partial Q_0} |D\varphi|$$

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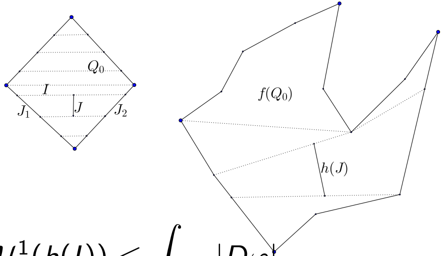

$$\int_I |D_x h| \leq \mathcal{H}^1(h(I)) \leq \int_{\partial Q_0} |D\varphi|$$

$$\int_J |D_y h| \leq \mathcal{H}^1(h(J)) \leq \max(\mathcal{H}^1(h(J_1)), \mathcal{H}^1(h(J_2)))$$

$$\leq \int_{J_1} |D\varphi| + \int_{J_2} |D\varphi|$$

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- $n = 2, p = 2$, $f \in W^{1,2}$, $f^{-1} \in W^{1,2}$ - Can we approximate? Are the minimizers of $\int |Df|^2 + \frac{|Df|^2}{J_f} (= \int |Df|^2 + \int |Df^{-1}|^2)$ smooth?

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