A quantitative modulus of continuity for the two-phase Stefan problem

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The Stefan problem

The Stefan problem (≈ 1890) is a simplified model to describe the behavior of a substance changing phase. When a change of phase takes place, a latent heat is either absorbed or released, while the temperature of the material changing its phase remains constant. The classical formulation is

$$\begin{cases} \partial_t u - \triangle u = 0 & \textit{in} \quad \{u > 0\} \cup \{u < 0\} \\ V_\nu = |\nabla u^+| - |\nabla u^-| & \textit{on} \quad \partial(\{u > 0\} \cup \{u < 0\}). \end{cases}$$

Here u^+ and u^- denotes respectively the limit taken from $\{u > 0\}$ and $\{u < 0\}$, respectively, and V_ν is the outward normal velocity of the free boundary with respect to $\{u > 0\}$.

The Stefan problem

It can be shown (see for example Kim-Požár, *CPDE '11*) that in rather general situations the problem can be reformulated as

$$\partial_t \big[u + \mathcal{L}_h H_0(u) \big] \ni \triangle u \qquad \text{in } \Omega_T = \Omega \times (0, T),$$

where $\mathcal{L}_h$ is a positive constant usually called as the latent heat. In practice, in the weak sense for $v \in u + \mathcal{L}_h H_0(u)$ we have

$$\int_{\Omega \times (t_1, t_2)} \big[- v \partial_t \varphi + \langle \nabla u, \nabla \varphi \rangle \big] \, dx \, dt + \int_\Omega v \varphi \, dx \Big|_{t=t_1}^{t_2} = 0$$

holds for $\varphi \in C_c^\infty(\Omega_T)$ and a.e. $0 < t_1 < t_2 < T$.

More general structures

More in general we can replace the Laplacian with $a : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ having linear growth:

$$\langle a(x,t,u,\xi), \xi \rangle \geq \Lambda^{-1}|\xi|^2, \qquad |a(x,t,u,\xi)| \leq \Lambda|\xi|$$

to take into account convective effects and also non-linear growth of the parabolic part, i.e.,

$$\partial_t \big[\beta(u) + \mathcal{L}_h H_a(\beta(u))\big] \ni \operatorname{div} a(x,t,u,\nabla u).$$

β is a C^1 diffeomorphism of $\mathbb{R}$ and contains thermal properties of the water. The jump here occurs when $\beta(u) = b \in \mathbb{R}$. Calling $v = \beta(u)$ and assuming $\mathcal{L}_h = 1$, we have

$$\partial_t \big[v + H_b(v)\big] \ni \operatorname{div} a(x,t,v,\nabla v)$$

Objective:

Find a quantitative modulus of continuity $\omega(r)$ in the two-phase Stefan problem such that

$$\operatorname*{osc}_{Q_r} u \lesssim \omega(r)\,, \qquad Q_r := B_r(x_0) \times (t_0 - r^2, t_0) \subset \Omega_T$$

for weak solutions to $\partial_t \big[\beta(u) + \mathcal{L}_h H_a(\beta(u))\big] \ni \operatorname{div} a(x, t, u, \nabla u)$ in Ω_T.

What can be found in the literature: Continuity in one-phase

One-phase Stefan problem: with $b = 0$, we just allow $u \geq 0$ (that is, ice is at temperature $0\,^\circ$C). Then

$$\operatorname*{osc}_{Q_R} u \lesssim \omega(r) \qquad \text{with}$$

$$\omega(r) = \left[\ln\left(\frac{1}{r}\right)\right]^{-\epsilon}, \qquad \text{if } n \geq 3 \qquad 0 < \epsilon < \frac{2n}{n-2};$$

$$\omega(r) = 2^{-[\ln(\frac{1}{r})]^{\gamma}}, \qquad \text{if } n = 2\,, \qquad 0 < \gamma < \frac{1}{2}.$$

[Caffarelli & Friedman, Indiana Univ. Math. J., '79]

Elliptic operator: **Laplacian**, Proof: heavy use of the positivity of u; maximum principle & representation formulae.

What can be found in the literature: Continuity in two-phase

Two-phase Stefan problem: with $b = 0$, we don't impose sign restrictions (i.e., ice can reach $-10\,^{\circ}$C, for instance).

Only qualitative continuity in [Caffarelli & Evans, ARMA, '83] and [DiBenedetto, AMPA, '82].

Implicit some kind of $\log\log$ continuity in the former, proved in the case of **Laplacian**, while the second handles nonlinear operators & lower order terms. CE proof: De Giorgi iteration + Green formula to reduce the supremum of u_+; DiB proof: only energy methods.

The modulus

$$\omega(r) = \left[\ln\ln\left(\frac{1}{r}\right)\right]^{-\sigma} \qquad \text{for some } \sigma > 0,$$

is stated as a Remark in [DiBenedetto & Friedman, Crelle's J., '84]; explicit proof (at the boundary) in [DiBenedetto, JDE '86].

Remark 3. 1. The same arguments prove that we have the modulus of continuity (3. 25) also for the weak solutions of the two-phase Stefan problem and certain extensions of the porous medium equations [2], [3]; such a modulus was not calculated in these papers.

From (3. 22)—(3. 24) it follows that ∇u is continuous with modulus of continuity

$$\left(\log\log\frac{A}{r}\right)^{-\sigma} \qquad (A>0,\ \sigma>0) \tag{3. 25}$$

Our first theorem

Theorem (Baroni, T6 & Urbano)

Let v be a weak solution to

$$\partial_t[v + H_b(v)] \ni \operatorname{div} a(x,t,v,\nabla v) \qquad \text{in } \Omega_T,$$

$a(x,t,u,\nabla v) \approx \nabla v$, $b \in \mathbb{R}$. Then

$$\omega(r) = const \cdot \left[\ln\left(\frac{1}{r}\right)\right]^{-\gamma}, \qquad \begin{cases} \gamma = \dfrac{2}{n+2} & \text{if } n \geq 3, \\ 0 < \gamma < \frac{1}{2} & \text{if } n = 2. \end{cases}$$

Open problem: Give an example of a solution in the case $a(x,t,v,\nabla v) = \nabla v$ for the above modulus of continuity (especially when $n \geq 3$).

Proof: a tool - weak supersolution

We consider weak supersolution:

$$(\star)\ \ w_t - \operatorname{div} a(x,t,w,\nabla w) \geq 0 \qquad \text{in } Q = B_R(x_0) \times (t_0 - R^2, t_0)$$

with $a(x,t,w,\nabla w) \approx \nabla w$.

Theorem (Weak Harnack inequality (Trudinger, CPAM '68))

Let $w > 0$ be a weak supersolution to $(\star)$, bounded in Q^*. Then

$$⨍_{Q^*} w\,dx\,dt \leq c \inf_{Q^-} w$$

$$⨍_{Q^* \cap \{t = \tau^*\}} w(\cdot,t)\,dx \leq c \inf_{Q^-} w$$

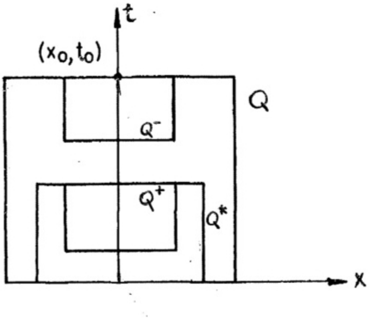

Figure 1.

Proof: a tool - weak supersolution - II

As corollary it follows:

Corollary (Decay of weak supersolution)

Let $w > 0$ be a weak supersolution to $(\star)$, bounded in Q. Then

$$\inf_{B/2\times\{t_0-R^2\}} w \geq k, \qquad k > 0$$

$$\Downarrow$$

$$\inf_{B/2\times(t_0-R^2,t_0)} w \geq \frac{k}{c} e^{-\frac{c}{R^2}(t-(t_0-R^2))}$$

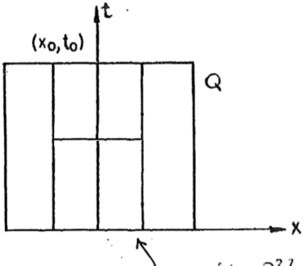

Figure 2.

Two exercises

Exercise

If v is a solution to

$$(\star) \qquad v_t - \triangle v = 0 \qquad \left(v_t - \operatorname{div} a(x,t,v,\nabla v) = 0\right),$$

then $w = \min\{v,k\}$, $k \in \mathbb{R}$ is a supersolution to $(\star)$.

Proof.

Formally, test $(\star)$ with $\varphi\chi_{\{v<k\}}$ and discard the negative term. □

Exercise

If v is a solution to

$$\partial_t[v + H_b(v)] \ni \triangle v \qquad \left(\partial_t[v + H_b(v)] \ni \operatorname{div} a(x,t,v,\nabla v)\right),$$

then $w = \min\{v,k\}$ with $k < b$ is a supersolution to $(\star)$.

Remark about formal computations

In this talk **I will proceed formally**. Rigorously one should mollify Heaviside jump:

$$H_{b,\varepsilon}(v) := (H_b * \theta_\varepsilon)(v), \qquad \operatorname{supp} H'_{b,\varepsilon} \subset (b-\varepsilon, b+\varepsilon),$$

and consider the approximating solutions v_ε. For these we prove

$$\operatorname*{osc}_{Q_R} v_\varepsilon \lesssim \omega(R) + \varepsilon$$

and then we use local uniform convergence, letting $\varepsilon \searrow 0$.

Towards the proof - reductions

Take $Q \equiv Q_R$ and $\omega(\cdot)$ modulus of continuity. We can suppose

$$\textit{Reduction } 1: \qquad \inf_Q v = 0 \qquad \Longrightarrow \qquad \operatorname*{osc}_Q v = \sup_Q v;$$

$$\textit{Reduction } 2: \qquad b \in [0, \sup_Q v] = [0, \operatorname*{osc}_Q v],$$

(if not, v is Hölder continuous by the standard regularity theory);

$$\textit{Reduction } 2': \qquad b \in \left[\frac{1}{2}\sup_Q v, \sup_Q v\right],$$

$$\text{(if not, consider instead } \tilde{v} := \sup_Q v - v\text{);}$$

$$\textit{Reduction } 3: \qquad \sup_Q v > \omega(R).$$

The alternatives

We fix two alternatives (recall that the jump $= b \geq \sup v/2$):

$$\text{(Alt. 1)} \qquad \left| Q^* \cap \left\{ v \geq \frac{\sup v}{4} \right\} \right| > \varepsilon_1 \left[\omega(R)\right]^{1+\frac{n}{2}} |Q^*|$$

OR

$$\text{(Alt. 2)} \qquad \left| Q^* \cap \left\{ v \geq \frac{\sup v}{4} \right\} \right| \leq \varepsilon_1 \left[\omega(R)\right]^{1+\frac{n}{2}} |Q^*|.$$

We consider just the case $n > 2$; ε_1 to be fixed.

The first alternative

Consider the case where the first alternative holds true:

$$\text{(Alt. 1)} \qquad \left| Q^* \cap \left\{ v \geq \frac{\sup v}{4} \right\} \right| > \varepsilon_1 \left[\omega(R)\right]^{1+\frac{n}{2}} |Q^*|.$$

We truncate $\hat{v} := \min\{v, \sup v/4\}$, which is a supersolution (recall that $b \geq \sup v/2$). We then simply have for such positive supersolution:

$$\begin{aligned} \inf_{Q^-} \hat{v} \geq \frac{1}{c} \fint_{Q^*} \hat{v}\, dx\, dt &\geq \frac{1}{c|Q^*|} \int_{Q^* \cap \{v \geq \sup v/4\}} \hat{v}\, dx\, dt \\ &\overset{\hat{v} = \sup v/4}{\geq} \frac{\varepsilon_1}{c} \left[\omega(R)\right]^{1+\frac{n}{2}} \frac{\operatorname{osc} v}{4} \\ &\overset{\sup v > \omega(R)}{\geq} \frac{\varepsilon_1}{4c} \left[\omega(R)\right]^{2+\frac{n}{2}}. \end{aligned}$$

The first alternative - part II

Using the decay of supersolutions, we moreover get

$$\inf_{Q^- \cup Q/2} v \geq \inf_{Q^- \cup Q/2} \hat{v} \geq \frac{\varepsilon_1}{c} \left[\omega(R)\right]^{2+\frac{n}{2}} .$$

In particular,

$$\begin{aligned} \operatorname*{osc}_{Q/2} v &\leq \sup_{Q} v - \frac{\varepsilon_1}{c} \left[\omega(R)\right]^{2+\frac{n}{2}} \\ &= \operatorname*{osc}_{Q} v - \frac{\varepsilon_1}{c} \left[\omega(R)\right]^{2+\frac{n}{2}} . \end{aligned}$$

The second alternative

Now we analyze the occurrence of the second alternative:

$$\left|Q^* \cap \left\{v \geq \sup v - \frac{\omega(R)}{4}\right\}\right| \leq \left|Q^* \cap \left\{v \geq \frac{\sup v}{4}\right\}\right|$$

$$\text{(Alt. 2)} \qquad \leq \varepsilon_1 \left[\omega(R)\right]^{1+\frac{n}{2}} |Q^*|.$$

This is the starting point of a De Giorgi-type iteration that proves that

$$\sup_{Q^*/2} v \leq \sup_{Q} v - \frac{\omega(R)}{8},$$

provided ε_1 is chosen appropriately.

The second alternative - The Caccioppoli

$$\sup_\tau \int_B \left[(v-k)_+^2\phi^2\right](\cdot,\tau)\,dx + \int_Q |\nabla(v-k)_+|^2\phi^2\,dx\,dt$$
$$\le c \int_Q (v-k)_+^2\left[|\nabla\phi|^2 + (\partial_t\phi^2)_+\right]dx\,dt$$
$$+ c \int_Q (b-k)_+\chi_{\{v\ge k\}}(\partial_t\phi^2)_+\,dx\,dt =: RHS.$$

Parabolic Sobolev inequality:

$$LHS := \int_Q \left[(v-k)_+^2\phi^2\right]^{1+\frac{2}{n}}\,dx\,dt$$
$$\le cR^2\left[\sup_\tau \int_B \left[(v-k)_+^2\phi^2\right](\cdot,\tau)\,dx\right]^{\frac{2}{n}} \int_Q \left|\nabla[(v-k)_+\phi]\right|^2 dx\,dt.$$

The second alternative - Concluded

A logarithmic Lemma now transforms the pointwise information into an information in measure, but in the future:

$$\frac{\left|Q/2 \cap \left\{v \geq \sup v - \varsigma(\nu)[\omega(R)]^{2+\frac{n}{2}}\right\}\right|}{|Q/2|} \leq \nu,$$

Now we can perform another De Giorgi iteration, with test function independent of time - and this makes the inhomogeneity of the Caccioppoli disappear - and this yields

$$\operatorname*{osc}_{Q/4} v \leq \operatorname{osc} v - \frac{\varsigma(\nu)}{2}\,[\omega(R)]^{2+\frac{n}{2}}\,,$$

just if we take ν (and hence ς) small enough, **independently of $\boldsymbol{\omega}$(R)**.

Conclusion

All in all, we proved

$$\operatorname*{osc}_{Q/4} v \le \operatorname*{osc}_{Q} v \le \omega(R)$$

OR

$$\operatorname*{osc}_{Q/4} v \le \operatorname*{osc}_{Q} v - \theta\,[\omega(R)]^{2+\frac{n}{2}}\,,$$

with θ a small constant depending on the data.
Using induction, for $R_j := 4^{-j}R$ and $Q_j := Q_{R_j}$, one estimates (assuming $\operatorname{osc}_{Q_i} v \le \omega(R_i)$ for $i \in \{0, \dots, j\}$)

$$\operatorname*{osc}_{Q_{j+1}} v \le \prod_{i=0}^{j} \left(1 - \theta[\omega(R_i)]^{1+\frac{n}{2}}\right) \omega(R).$$

Conclusion - part II

$$
\begin{aligned}
\prod_{i=0}^{j}\left(1-\theta[\omega(R_i)]^{1+\frac{n}{2}}\right) &= \exp\left(\sum_{i=0}^{j}\ln\left(1-\theta[\omega(R_i)]^{1+\frac{n}{2}}\right)\right)\\
&\leq \exp\left(-\theta\sum_{i=0}^{j}[\omega(R_i)]^{1+\frac{n}{2}}\right)\\
&\leq \exp\left(-\frac{\theta}{\ln 4}\int_{R_{j+1}}^{R}[\omega(\rho)]^{1+\frac{n}{2}}\frac{d\rho}{\rho}\right)\\
&\overset{???}{=} \exp\left(-\ln\left[\frac{\omega(R)}{\omega(R_{j+1})}\right]\right) = \frac{\omega(R_{j+1})}{\omega(R)}\,.
\end{aligned}
$$

The second alternative

Thus the question reduces to ask when

$$\frac{\theta}{\ln 4}\int_{R_{j+1}}^{R} [\omega(\rho)]^{1+\frac{n}{2}} \frac{d\rho}{\rho} \overset{???}{=} -\ln\left(\frac{\omega(R)}{\omega(R_{j+1})}\right)$$

holds?

Claim

$$\omega(r) = \left[\frac{\theta}{\ln 4}\left(\frac{n+2}{2}\right)\ln\left(\frac{1}{r}\right)\right]^{-\frac{1}{1+\frac{n}{2}}}$$

gives the needed equality; hence the induction works!

Proof.

Just a simple computation. □

The degenerate Stefan problem

We take here

$$\partial_t[v + H_b(v)] \ni \operatorname{div}\left[|\nabla v|^{p-2}\nabla v\right], \qquad p > 2;$$

not very much is known then. Several things are still valid, but the Caccioppoli estimate is problematic in two ways:

$$\begin{aligned}\sup_\tau \int_B \left[(v-k)_+^2\phi^p\right](\cdot,\tau)\,dx + \int_Q |\nabla(v-k)_+|^p\phi^p\,dx\,dt \\ \leq c\int_Q \left[(v-k)_+^p|\nabla\phi|^p + (v-k)_+^2(\partial_t\phi^p)_+\right]dx\,dt \\ + \int_Q (a-k)_+\chi_{\{v\geq k\}}(\partial_t\phi^p)_+\,dx\,dt.\end{aligned}$$

The degenerate Stefan problem - solution

We use the approach in [T6, Mingione & Nyström, JMPA '13]: one considers cylinders of the type

$$Q_R^{\lambda\omega(\cdot)}(x_0, t_0) = B_R(x_0) \times (t_0 - \lambda^{2-p}[\omega(R)]^{2-p} R^p, t_0)$$

where

$$\lambda \approx \frac{1}{\omega(R)} \operatorname*{osc}_{Q_R^{\lambda\omega(\cdot)}} v.$$

It turns out that these cylinders reveal to be the appropriate ones to treat, in the sharp way, $C^{\omega(\cdot)}$ property for the parabolic obstacle problem. Indeed

$$\text{Obstacle} \in C^{\omega(\cdot)} \quad \Longrightarrow \quad \text{solution} \in C^{\omega(\cdot)}, \quad \operatorname*{osc}_{Q_R^{\lambda\omega(\cdot)}} v \lesssim \lambda\omega(R).$$

The degenerate Stefan problem - solution - part II

Formally,

$$\lambda \approx \frac{1}{\omega(R)} \operatorname*{osc}_{Q_R^{\lambda\omega(\cdot)}} v \approx \frac{R}{\omega(R)} |\nabla v|.$$

Hence the p-Laplace operator rewrites as

$$v_t - \operatorname{div}\left[|\nabla v|^{p-2}\nabla v\right] \approx v_t - \left[\frac{\omega(R)\lambda}{R}\right]^{p-2} \triangle v = 0$$

and this "rescale to the heat equation" in Q_1 if considered in

$$B_R(x_0) \times (t_0 - \lambda^{2-p}[\omega(R)]^{2-p} R^p, t_0);$$

this allows to perform blow-up arguments. Note the two borderline cases.

The degenerate Stefan problem - solution - part III

Hence, to handle this problem, we have to consider two time scales (once fixed ω):

$$t_0 - [\omega(R)]^{(2-p)(2+\frac{n}{p})} R^p, \qquad t_0 - \widetilde{T}\big(\omega(R)\big) R^p, \qquad t_0,$$

but the same alternatives as before (with p in place of 2):

$$\left| Q^* \cap \left\{ v \geq \frac{\sup v}{4} \right\} \right| > \varepsilon_1 \left[\omega(R)\right]^{1+\frac{n}{p}} |Q^*|$$

OR

$$\left| Q^* \cap \left\{ v \geq \frac{\sup v}{4} \right\} \right| \leq \varepsilon_1 \left[\omega(R)\right]^{1+\frac{n}{p}} |Q^*|.$$

The degenerate Stefan problem - our second theorem

The result here is the following

Theorem (Baroni, T6 & Urbano)

Let v be a weak solution to

$$\partial_t[v + H_a(v)] \ni \operatorname{div}\left[|\nabla v|^{p-2}\nabla v\right] \qquad \text{in } \Omega_T, \quad 2 < p < n.$$

Then, with

$$Q_R^{\omega(\cdot)}(z_0) := B_R(x_0) \times (t_0 - [\omega(R)]^{(2-p)(2+\frac{n}{p})} R^p, t_0),$$

we have

$$\operatorname*{osc}_{Q_r^{\omega(\cdot)}(z_0)} v \leq \omega(r) \approx \left[\ln\left(\frac{1}{r}\right)\right]^{-\frac{p}{n+p}}.$$

Thank you for your attention!