Update on nonlinear potential theory

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with Josek Málek at paseky (2005)

Part 1: Size bounds

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

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- **We have**

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The classical potential estimates

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

- **We have**

$$u(x) = \int G(x,y)\mu(y)$$

where

$$G(x,y) \approx \begin{cases} |x-y|^{2-n} & \text{se } n>2 \\ -\log|x-y| & \text{se } n=2 \end{cases}$$

Estimates via Riesz potentials

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

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$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

- while, after differentiation, we obtain

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

Local versions

- **In bounded domains one uses**

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho} \qquad\qquad \beta\in(0,n]$$

since

$$\begin{aligned}\mathbf{I}^{\mu}_{\beta}(x,R) &\lesssim \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-\beta}} \\ &= I_\beta(|\mu|\llcorner B(x,R))(x) \\ &\leq I_\beta(|\mu|)(x)\end{aligned}$$

for non-negative measures

What happens in the nonlinear case?

- **For instance for nonlinear equations with linear growth**

$$-\text{div } a(Du) = \mu$$

that is equations well posed in $W^{1,2}$ (p-growth and $p = 2$) that is

$$|\partial a(z)| \leq L \qquad \nu|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda \rangle$$

- **And degenerate ones like**

$$-\text{div } (|Du|^{p-2}Du) = \mu$$

- To be short, we shall concentrate on the case $p \geq 2$

Nonlinear potentials

- **The nonlinear Wolff potential is defined by**

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n]$$

- **The nonlinear Wolff potential** plays in nonlinear potential theory the same role the Riesz potential plays in the linear one

The first nonlinear potential estimate

Theorem (Kilpeläinen & Malý, Acta Math. 94)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B(x,R)} |u|^{p-1}\,dy\right)^{1/(p-1)}$$

holds

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For $p = 2$ we are back to the Riesz potential $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_{2}$ - the above estimate is non-trivial already in this situation

Corollary: optimal integrability

- **Indeed**

$$\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \qquad q \in (1,n)$$

and more in general estimates in rearrangement invariant function spaces

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and more in general estimates in rearrangement invariant function spaces

- **This property follows by another pointwise estimate**

$$\int_0^\infty \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} \lesssim I_\beta\left\{[I_\beta(|\mu|)]^{1/(p-1)}\right\}(x)$$

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- The quantity in the right-hand side is usually called Havin-Mazya potential
- More applications in this direction are proposed in Cianchi's paper (Ann. Pisa 2011)

Foundations of Nonlinear Potential Theory

NON-LINEAR POTENTIAL THEORY

V. G. Maz'ya and V. P. Khavin

Contents

A first gradient potential estimate

Theorem (Min., JEMS 2011)

When $p = 2$, if u solves

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{I}_1^{\mu}(x,R) + \fint_{B(x,R)} |Du|\, dy$$

holds

A first gradient potential estimate

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$$|Du(x)| \lesssim \mathbf{I}_1^{\mu}(x, R) + \fint_{B(x,R)} |Du| \, dy$$

holds

For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

New viewpoint - Let's twist!!!

- Consider

$$-\text{div}\, v = \mu$$

with

$$v = |Du|^{p-2}Du$$

Joint work with Tuomo Kuusi

Indeed

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2013)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{\mu}(x,R) + \left(\fint_{B(x,R)} |Du|\,dy\right)^{p-1}$$

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holds

The theorem still holds for general equations of the type $-\text{div}\,a(Du) = \mu$ Note that

$$\mathbf{I}_1^{\mu}(x,R) \lesssim [\mathbf{W}^{\mu}_{1/p,p}(x,R)]^{p-1}$$

Unified approach to gradient regularity

- **For the model case**

$$-\text{div}\ (|Du|^{p-2}Du) = \mu$$

all the known gradient integrability result now follow

- **Moreover, delicate and still open borderline cases (Lorentz and Orlicz regularity), immediately follow**

Potential characterisation of Lebesgue points

Theorem (Kuusi & Min., Bull. Math. Sci.)

If x is a point such that

$$\mathbf{I}_1^{\mu}(x,R) < \infty$$

for some $R > 0$ then x is a Lebesgue point of Du that is, the following limit

$$\lim_{\varrho\to 0} \fint_{B(x,\varrho)} Du(y)\,dy$$

exists

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exists

Part 2: Oscillation bounds

The general continuity criterion

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2013)

If u solves

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu$$

and

$$\lim_{R\to 0} \mathbf{I}_1^{\mu}(x,R) = 0 \text{ uniformly w.r.t. } x$$

then

$$Du \text{ is continuous}$$

A classical theorem of Stein

Theorem (Stein, Ann. Math. 1981)

$$Dv \in L(n,1) \Longrightarrow v \textit{ is continuous}$$

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We recall that

$$g \in L(n,1) \Longleftrightarrow \int_0^\infty |\{x \,:\, |g(x)| > \lambda\}|^{1/n} \, d\lambda < \infty$$

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It follows that

$$\triangle u = \mu \in L(n,1) \Longrightarrow Du \text{ is continuous}$$

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An example of $L(n,1)$ function is given by

$$\frac{1}{|x| \log^\beta(1/|x|)} \qquad \beta > 1$$

in the ball $B_{1/2}$

A nonlinear Stein theorem

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2013)

If u solves the p-Laplacean equation

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu \in L(n,1)$$

then

Du is continuous

A vectorial nonlinear Stein theorem

Theorem (Kuusi & Min., Calc. Var.)

If $u : \Omega \to \mathbb{R}^m$ solves the p-Laplacean system

$$-\operatorname{div}(|Du|^{p-2}Du) = F$$

and

$$F : \Omega \to \mathbb{R}^m \quad \textit{with} \quad F \in L(n,1)$$

then

Du is continuous

The special role of $L(n,1)$ - local and global results

- Duzaar & Min. (Ann. IHP 2010) prove $\|Du\|_{L^\infty_{\text{loc}}} < \infty$ when $F \in L(n,1)_{\text{loc}}$ for general equations $-\text{div}\, a(Du) = F$ and Uhlenbeck systems, for $n \geq 3$; interior regularity is obtained
- Cianchi & Maz'ya (Comm. PDE 2011) prove $\|Du\|_{L^\infty} < \infty$ when $F \in L(n,1)$ for $-\triangle_p u = F$ with zero boundary values, and up to the boundary, still in the case $n \geq 3$
- Kuusi & Min. (ARMA 2013) prove $Du \in C^0$ for $n \geq 2$ for general equations with coefficients $-\text{div}\, a(x, Du) = F$
- Cianchi & Maz'ya (ARMA 2014) $\|Du\|_{L^\infty} < \infty$ when $F \in L(n,1)$, for the system $-\triangle_p u = F$ with zero boundary values, and up to the boundary, for Uhlenbeck systems, for $n \geq 3$
- Kuusi & Min. (to appear in Calc. Var. & Math. Ann.) prove $Du \in C^0$ for Uhlenbeck systems with coefficients, for $n \geq 2$ and also in the parabolic case

Part 3: Interlude on fully nonlinear

A fully nonlinear Stein theorem

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 2014)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2u) = f \in L(n,1)$$

then

Du is continuous

A fully nonlinear Stein theorem

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 2014)

If u solves the uniformly elliptic fully nonlinear equation

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Previous results of Caffarelli (Ann. Math. 1989) assert that

$$f \in L^{n+\varepsilon} \Longrightarrow Du \in C^{0,\alpha}$$

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Notice that

$$L^{n+\varepsilon} \subset L(n,1) \qquad \varepsilon > 0$$

Modified potentials

Key to the proof, a new potential estimate

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$$\mathbf{I}_1^f(x,r) := \int_0^r \fint_{B_\varrho(x)} |f(y)|\, dy\, d\varrho$$

Key to the proof, a new potential estimate

$$\begin{aligned} \mathbf{I}_1^f(x,r) &:= \int_0^r \frac{1}{\varrho^{n-1}} \int_{B_\varrho(x)} |f(y)|\, dy\, \frac{d\varrho}{\varrho} \\ &= \int_0^r \fint_{B_\varrho(x)} |f(y)|\, dy\, d\varrho \\ &\leq \int_0^r \left(\fint_{B_\varrho(x)} |f(y)|^p\, dy \right)^{1/p} d\varrho =: \mathbf{II}_1^f(x,r)\,. \end{aligned}$$

Modified potentials

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 2014)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2u) = f \in L(n,1)$$

then

$$|Du(x)| \leq c\,\mathbf{I}_1^f(x,r) + c\left(\fint_{B_r(x)} |Du|^q\,dy\right)^{1/q}$$

for $p \geq n - \varepsilon$ and $q > n$

Consequences

- It holds, with $n - \varepsilon < p$ that

$$\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \Longrightarrow Du \in \mathrm{BMO}$$

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- In particular

$$f \in \mathcal{M}^n \equiv L(n, \infty) \Longrightarrow Du \in \mathrm{BMO}$$

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- In particular

$$f \in \mathcal{M}^n \equiv L(n, \infty) \Longrightarrow Du \in \mathrm{BMO}$$

- Moreover

$$\lim_{r \to 0} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy = 0 \Longrightarrow Du \in \mathrm{VMO}$$

Comparisons

Borderline case of a theorem of Caffarelli, who proved

$$\sup_{B_r(x)} r^{n(1-\alpha)-n} \int_{B_r(x)} |f|^n \, dy < \infty \Longrightarrow Du \in C^{0,\alpha}$$

Comparisons

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$$\sup_{B_r(x)} r^{n(1-\alpha)-n} \int_{B_r(x)} |f|^n \, dy < \infty \Longrightarrow Du \in C^{0,\alpha}$$

Corollary (Teixeira, ARMA 2014)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2u) = f \in M^n \equiv L(n, \infty)$$

then u is Log-Lipschitz, that is

$$|u(x) - u(y)| \leq -|x-y| \log\left(\frac{1}{|x-y|}\right)$$

Part 4: Evolution

Parabolics

- **The model case is here given by**

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu\,, \qquad \text{in } \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}$$

more in general we consider

$$u_t - \operatorname{div} a(Du) = \mu\,.$$

- The basic reference for existence and a priori estimates in the setting of SOLA is the work of Boccado, Dall'Aglio, Galloüet and Orsina, JFA, 1997

Degenerate equations - basic results

Theorem (Boccardo, Dall'Aglio, Gallouët & Orsina, JFA, 1997)

$$|Du| \in L^q(\Omega \times (-T,0)), \quad 1 \leq q < p - 1 + \frac{1}{N-1}$$

$$N = n + 2 \qquad \text{is the parabolic dimension}$$

Consider the caloric Riesz potential

$$\mathbf{I}_1^{\mu}(x,t;r) := \int_0^r \frac{|\mu|(Q_\varrho(x,t))}{\varrho^{N-1}}\,\frac{d\varrho}{\varrho}\,, \quad N := n+2\,,$$

The heat equation

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then for solutions to

$$u_t - \triangle u = \mu$$

we have

$$|Du(x,t)| \leq c\mathbf{I}_1^{\mu}(x,t;r) + c\fint_{Q_r(x,t)} |Du|\,dz$$

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$$|Du(x,t)| \leq c\mathbf{I}_1^{\mu}(x,t;r) + c \fint_{Q_r(x,t)} |Du|\, dz$$

we recall that

$$Q_r(x,t) := B(x,r) \times (t-r^2,t)$$

Inhomogeneous a priori estimates

Theorem (DiBenedetto & Friedman, Crelle J. 85)

$$\sup_{Q_{r/2}(x_0,t_0)} |Du| \leq c(n,p) \fint_{Q_r(x_0,t_0)} (|Du|+1)^{p-1}\, dz$$

- **The basic analysis is the following: consider intrinsic cylinders**

  $$Q_\varrho^\lambda(x,t) = B(x,\varrho) \times (t - \lambda^{2-p}\varrho^2, t)$$

  where it happens that

  $$|Du| \approx \lambda \qquad \text{in } Q_\varrho^\lambda(x,t)$$

  then the equation behaves as

  $$u_t - \lambda^{p-2} \triangle u = 0$$

  that is, scaling back in the same cylinder, as the heat equation
- **On intrinsic cylinders estimates "ellipticize"; in particular, they become homogeneous**

DiBenedetto's intrinsic estimate

- The homogenizing effect of intrinsic geometry

Theorem (DiBenedetto & Friedman, Crelle J. 85)

There exists a universal constant $c \geq 1$ *such that*

$$c \left(\fint_{Q_r^\lambda(x,t)} |Du|^{p-1} \, dz \right)^{1/(p-1)} \leq \lambda$$

then

$$|Du(x,t)| \leq \lambda$$

Intrinsic Riesz potentials

- **Define the intrinsic Riesz potential** such that

$$\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) := \int_0^r \frac{|\mu|(Q^{\lambda}_{\varrho}(x,t))}{\varrho^{N-1}}\,\frac{d\varrho}{\varrho}$$

with

$$Q^{\lambda}_{\varrho}(x,t) = B(x,\varrho)\times(t-\lambda^{2-p}\varrho^2,t)$$

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with

$$Q^{\lambda}_{\varrho}(x,t) = B(x,\varrho) \times (t - \lambda^{2-p}\varrho^2, t)$$

- **Note that**

$$\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) = \mathbf{I}^{\mu}_{1}(x,t;r) \qquad \text{when } p = 2 \text{ or when } \lambda = 1$$

The parabolic Riesz gradient bound

Theorem (Kuusi & Min., JEMS, ARMA 2014)

There exists a universal constant $c \geq 1$ *such that*

$$c\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) + c\left(\fint_{Q_r^{\lambda}(x,t)} |Du|^{p-1}\, dz\right)^{1/p-1} \leq \lambda$$

then

$$|Du(x,t)| \leq \lambda$$

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then

$$|Du(x,t)| \leq \lambda$$

- **When $\mu \equiv 0$ this reduces to the sup estimate of DiBenedetto & Friedman (Crelles J. 84)**

Sharpness

- Consider the equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \delta,$$

where δ denotes the Dircac unit mass charging the origin

- The so called Barenblatt (fundamental solution) is

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-\frac{n}{\theta}}\left(c_b - \theta^{\frac{1}{1-p}}\left(\frac{p-2}{p}\right)\left(\frac{|x|}{t^{1/\theta}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} & t > 0 \\ 0 & t \leq 0\,. \end{cases}$$

for $\theta = n(p-2)+p$ and a suitable constant c_b such that

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x,t)\,dx = 1 \qquad \forall\ t > 0$$

Sharpness

- A direct computation shows the following upper optimal upper bound
$$|D\mathcal{B}_p(x,t)| \leq ct^{-(n+1)/\theta}$$
- The intrinsic estimate above **exactly reproduces this upper bound**
- This decay estimate is indeed reproduced for all those solutions **that are initially compactly supported**

Intrinsic bounds imply explicit bounds

- The previous bound always implies a priori estimates on standard parabolic cylinders

Theorem (Kuusi & Min., JEMS, ARMA 2014)

$$|Du(x,t)| \lesssim \mathbf{I}_1^{\mu}(x,t;r) + \fint_{Q_r(x,t)} (|Du|+1)^{p-1}\, dz$$

holds for every standard parabolic cylinder Q_r

Bact to classical potentials

Corollary (Kuusi & Min., JEMS, ARMA 2014)

Assume that u solves

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \qquad \textit{in } \mathbb{R}^{n+1}\,.$$

Then

$$|Du(x_0,t_0)| \lesssim \int_{\{t<t_0\}} \frac{d|\mu|(x,t)}{d_{\mathrm{par}}((x,t),(x_0,t_0))^{N-1}}$$

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recall that

$$d_{\rm par}((x,t),(x_0,t_0)) := \max\left\{|x-x_0|, \sqrt{|t-t_0|}\right\}$$

Gradient continuity via potentials

Theorem (Kuusi & Min., ARMA 2014)

Assume that

$$\lim_{r\to 0} \mathbf{I}_1^{\mu}(x,t;r) = 0 \qquad \textit{uniformly w.r.t. } (x,t)$$

then

$$Du \textit{ is continuous in } Q_T$$

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then

$$Du \textit{ is continuous in } Q_T$$

Theorem (Kuusi & Min., ARMA 2014)

Assume that

$$|\mu|(Q_\varrho) \lesssim \varrho^{N-1+\delta}$$

holds, then thtere exists α, depending on δ, such that

$$Du \in C^{0,\alpha} \qquad \textit{locally in } \ Q_T$$

A nonlinear parabolic Stein theorem

Theorem (Kuusi & Min., ARMA 2014)

Assume that

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \in L(N,1)$$

that is

$$\int_0^\infty |\{|\mu| > \lambda\}|^{1/N}\, d\lambda < \infty$$

then Du is continuous in Q_T

DiBenedetto proved that Du is continuous when $\mu \in L^{N+\varepsilon}$

A nonlinear, vectorial parabolic Stein theorem

Theorem (Kuusi & Min., Math. Ann., to appear)

Assume that u is a vector valued solutions to the parabolic p-system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu \in L(N,1)$$

that is

$$\int_0^\infty |\{|\mu| > \lambda\}|^{1/N}\, d\lambda < \infty$$

then

- *Du is continuous in Q_T*
- *The condition relaxes in $\mu \in L(n,1) = L(N-2,1)$ in the case μ is time independent*

Thanks for the attention, with a work of Serena Nono