On Hölder continuity of solutions to elliptic systems & variational integrals

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Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

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Variational problem - Hilbert's 19th problem

DATA:

- $\Omega \subset \mathbb{R}^d$ a given open bounded smooth domain
- $\boldsymbol{f} : \Omega \to \mathbb{R}^N$ a given smooth vector-valued function ($N \in \mathbb{N}$)
- $F : \mathbb{R}^N \times \mathbb{R}^{N\times d} \to \mathbb{R}$ being a smooth function fulfilling assumptions of uniform convexity, coercivity and growth condition, i.e., for some $p \in (1, \infty)$ and all $(u, \eta) \in \mathbb{R}^N \times \mathbb{R}^{N\times d}$ and all $\kappa \in \mathbb{R}^{N\times d}$

$$\boxed{-C_2 + C_1|\eta|^p \le F(u, \eta) \le C_2(1 + |\eta|^p)}$$

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GOAL: Minimize the functional

$$J(u) := \int_\Omega F(u(x), \nabla u(x)) - f(x) \cdot u(x)\, dx$$

over the space $W_0^{1,p}(\Omega; \mathbb{R}^N)$.

Variational problem - Hilbert's 19th problem II

Theorem

There exists a minimizer u to J. Moreover, if F does not depend on u then the minimizer is unique and it fulfills

$$\boxed{(1+|\nabla u|)^{\frac{p}{2}} \in W^{1,2}_{loc}(\Omega)}$$

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QUESTION: How smooth is the minimizer?

Hilbert: Set $p = 2$ and let F be independent of u. Is the minimizer analytic?

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- **Šverák & Yan (2002)**: "NO, they can be even unbounded"

Some answers for F depending on u

Consider the simplest case:

$$F(u,\eta) := A^{\alpha,\beta}(u)\eta_i^\alpha\eta_i^\beta \qquad |\partial_u A||u| \leq C$$

- **Frehse (1973)**: Construction of a discontinuous solution to the Euler-Lagrange equation even in $d = 2$ (but not minimizer!)

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- **Giaquinta & Giusti (1982)**: For $A^{\alpha\beta}(u) = a(u)\delta^{\alpha\beta}$ such that

$$2a(u) + a_u \cdot u \geq \alpha_0 > 0 \qquad \text{(one-sided condition)}$$

  the minimizer is **Hölder continuous** and consequently smooth. Moreover, if (one-sided condition) does not hold then the minimizer may not be continuous.
- **Giaquinta & Giusti (1982)**:For general A the theory is valid if $|A_u| \ll 1$

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provided that $p > d - 2$.

Notation

- Einstein summation convention is used
- $D_j := \frac{\partial}{\partial x_j}$
- $F_{\eta_j^\nu}(u, \eta) := \frac{\partial F(u,\eta)}{\partial \eta_j^\nu}$
- $F_{u^\nu}(u, \eta) := \frac{\partial F(u,\eta)}{\partial u^\eta}$

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Minimizers & Euler-Lagrange equations

- usually one derives the Euler-Lagrange equation and studies a solution of them
- usually one does not take care so much of the origin of the problem
- BUT not all solutions must be minimizers (F depending on u or F being non-convex)
- EVEN in case that the solution is a minimizer, we may hope that much better understanding of what is going on will come from the minimization property

Key consequences of minimizing property

Consider u being the minimizer of $J(u)$, i.e., $J(u) \leq J(v)$ for all $v \in W_0^{1,p}(\Omega; \mathbb{R}^N)$. The goal is to find a proper comparison function v giving optimal information

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- Euler-Lagrange equation: set $v(x) := u(x) + t\varphi(x)$ and let $t \to 0$

$$\boxed{-D_j(F_{\eta_j^\nu}(u, \nabla u)) + F_{u^\nu}(u, \nabla u) = f^\nu \qquad \nu = 1, \ldots, N} \tag{E-L}$$

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- Reverse Hölder inequality, Gehring lemma, Giaquinta & Giusti: set $v(x) := \theta(x)u(x) + (1-\theta(x))\bar{u}_R$

$$\boxed{\int_{B_R} \frac{|\nabla u|^{p+\varepsilon}}{R^d} \leq C\left(1 + \int_{B_{2R}} \frac{|\nabla u|^p}{R^d}\right)^{\frac{p+\varepsilon}{p}}} \implies u \in W_0^{1,p+\varepsilon}(\Omega;\mathbb{R}^N)$$

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- **Noether's (1918)** equation: set $v(x) := u(x + t\psi(x))$ and let $t \to 0$

$$\boxed{-D_i\left(F_{\eta_i^\nu}(u,\nabla u)D_k u^\nu\right) + D_k F(u,\nabla u) = f^\nu D_k u^\nu \qquad k = 1,\ldots,d} \tag{N-E}$$

Use of Noether's equation - testing by ∇u - Pohozaev like problem

Assume that $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ is a bounded solution to

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A: **If Ω is a star-shaped, regular domain and $p > \frac{2d}{d-2}$, then $u \equiv 0$.**

Proof of Pohozaev

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Multiply (P) by u^ν and integrate

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$$\int_{\partial\Omega} D_j u^\nu D_k u^\nu x_k n_j - \frac{1}{2}|\nabla u|^2 x_k n_k + \int_\Omega \frac{1}{2}|\nabla u|^2 D_k x_k - D_j u^\nu D_k u^\nu D_j x_k = \frac{1}{p}\int_\Omega |u|^p D_k x_k$$

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Thus if

$$\frac{d-2}{2} > \frac{d}{p} \Leftrightarrow p > \frac{2d}{d-2}$$

and the boundary integral is nonnegative then $u \equiv 0$. □

Use of Noether-harmonic mappings

Assume the simplest case, i.e., $u \in W^{1,2}(\Omega; \mathbb{R}^N)$ and $|u(x)| = 1$ for almost all $x \in \Omega$ fulfils

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The final (in)equality - monotonicity formula

$$\boxed{0 \leq 2 \int_{\partial B_R} \frac{|\nabla u \cdot x|^2}{|x|^d} = \frac{d}{dR} \int_{B_R} \frac{|\nabla u|^2}{R^{d-2}}}.$$

Use of monotonicity formula

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- neglect the positive term and integrate over $R \in (R_1.R_2)$

$$\int_{B_{R_1}} \frac{|\nabla u|^2}{R_1^{d-2}} \leq \int_{B_{R_2}} \frac{|\nabla u|^2}{R_2^{d-2}} \implies u \in BMO$$

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the same procedure should give BMO for general minimizers provided that the term $F_{\eta_i^\nu} D_k u^\eta x_i x_k$ has a sign $\implies$ minimizers are always in BMO provided that F satisfies **"splitting condition"**

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- DO NOT neglect the positive term and integrate over $R \in (0, r)$

$$\boxed{2\int_{B_r} \frac{|\nabla u \cdot x|^2}{|x|^d} \leq \int_{B_r} \frac{|\nabla u|^2}{r^{d-2}}}$$

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- Fill the hole, i.e., add $C\int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}}$

$$\int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}} \le \frac{C}{C+1}\int_{B_{2r}} \frac{|\nabla u|^2}{|x|^{d-2}} \Longrightarrow \int_{B_r} \frac{|\nabla u|^2}{r^{d-2+2\alpha}} \le C$$

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Let us choose the prototype case:

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$$\begin{aligned}\int_{B_R} \frac{|\nabla u|^2}{R^{d-2}} &\leq \int_{B_{2R}\setminus B_R} \frac{|u - \bar{u}_R||\nabla u \cdot x|}{R^d}\\ &\leq \varepsilon \int_{B_{2R}} \frac{|\nabla u|^2}{R^{d-2}} + C(\varepsilon)\int_{B_{2R}\setminus B_R} \frac{|\nabla u \cdot x|^2}{|x|^d}\end{aligned}$$

All together

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What we really needed - F independent of u:

- $$\varepsilon(1+|\eta|)^{p-2}|\eta \cdot x|^2 \leq F_{\eta_i^\nu \eta_j^\nu} x_i x_j$$
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- **One-sided condition appears**

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- Use one-sided condition for left hand side and use the "good" procedure for the right hand side

$$\boxed{\int \varepsilon |\nabla u|^2 \tau_R \le C \int |u - \bar{u}_R||D_k u D_k \tau_R| + |\bar{u}_R||D_k u D_k \tau_R|}$$

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- **Surprise:**

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- $|F_{u^\nu}(u,\nabla u)c^\nu| \sim |F_{u^\nu}(u,\nabla u)u^\nu|$

$$\boxed{|F_{u^\nu}(u,\eta)| \le C(1+|u^\nu|)^{-1}g(u^\nu)|\eta|^p}, \qquad \text{with } g(s) \to 0 \text{ as } s \to \infty.$$

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- Conditions for inhomogeneous hole-filling - one-sided condition

$$\boxed{F_{\eta_i^\nu}(u,\eta)\eta_i^\nu + F_{u^\nu}(u,\eta)u^\nu \ge \varepsilon|\eta|^p - K}$$

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The result

Theorem (Bulíček, Frehse, Steinhauer)

- *Let F satisfies the* **growth conditions** *and the* **conditions for Noether**. *Then any minimizer belongs to BMO.*
- *Moreover, if F satisfies* **conditions for Caccioppolli** *and* **one-sided condition**, *then any* **bounded** *minimizer is Hölder continuous.*
- *Even more, if there exists a constant C such that for $x_0 \in \Omega$ and all $R \in (0,1)$*

$$|\bar{u}_{B_R(x_0)}| \leq C(1 + |\ln R|)^{\min(\frac{1}{2}, \frac{1}{p'})}$$

  then minimizer is Hölder continuous in a neighborhood of x_0.
- *Moreover, if $p = 2$ then any minimizer is Hölder continuous.*
- *In addition, if $|F_u(u,\eta)||u| \to 0$ as $|u| \to \infty$ then minimizer is Hölder continuous.*
- *If $F(u, \lambda\eta) = \lambda^p F(u,\eta)$ then any bounded (or globally in BMO) minimizer on $\mathbb{R}^d$ is constant.*

Applicability of Theorem

Define

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Possible settings of F are

$$F(x,u,\eta) := (\sum_m Q_m(u,x,\eta,\eta))^{\frac{p}{2}} \qquad \text{(convex, not diagonal)},$$

$$F(x,u,\eta) := \prod_m (Q_m(u,x,\eta,\eta))^{\frac{p_m}{2}} \qquad \text{(not convex)}$$

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Generally

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is possible, while in the Uhlenbeck setting we require

$$\boxed{F(x,u,\eta) := \tilde{F}(x,u,|\nabla u|)} \text{ or more generally } \boxed{F(x,u,\eta) := \tilde{F}(x,u,|Q(u,x,\eta,\eta)|)}$$