Regularity of evolutionary symmetric p-Laplacian

Jan Burczak

Institute of Mathematics, Polish Academy of Sciences, Warsaw

Telč, 02.05.2013

Focus

Local regularity for the symmetric p-Laplace system

$$u_{,t} - \operatorname{div}\left[\left(\mu + |\mathbb{D}u|^2\right)^{\frac{p-2}{2}} \mathbb{D}u\right] = 0$$

with $p \geq 2$ and its $\mathcal{A}$-generalizations.

Motivation

The physical one: power-law Stokes, nonlinear Kelvin-Voigt.
The analytical one: full-gradient-case resolved ($C^{1,\alpha}_{loc}$).

Focus

Local regularity for the symmetric p-Laplace system

$$u_{,t} - \operatorname{div}\left[(\mu + |\mathbb{D}u|^2)^{\frac{p-2}{2}} \mathbb{D}u\right] = 0$$

with $p \geq 2$ and its $\mathcal{A}$-generalizations.

Motivation

The physical one: power-law Stokes, nonlinear Kelvin-Voigt.
The analytical one: full-gradient-case resolved ($C^{1,\alpha}_{loc}$).

Trouble

Essentially different pointwise structure.
For instance lack of *the semilinear subsolution property* of the full-gradient case

$$w_{,t} - \left(A_{jk}(\mu + w^2)^{\frac{p-2}{2}} w_{,x_k}\right)_{,x_j} \leq 0$$

$w = |\nabla u|^2$, coefficients

$$A_{jk} := \delta_{jk} + (p-2)\frac{(\nabla u)^i_j (\nabla u)^i_k}{\mu + |\nabla u|^2}$$

bounded.

p-Navier-Stokes

- short time MR theory for general BVPs (Prüss, Bothe),
- $C^{1,\alpha}$ regularity, planar case, periodic or Dirichlet b.c. (Kaplický, Málek, Stará)
- partial $C^{1,\alpha}$ regularity for $p \in (12/5; 10/3)$, $d = 3$ (Ladyzhenskaya, Seregin),
- partial $C^{1,\alpha}$ regularity for modified $p(x)$-Navier-Stokes (Acerbi, Mingione, Seregin),
- strong solutions, $p \geq 2$, Dirichlet b. c. (Málek, Nečas, Růžička)

p-Navier-Stokes

- short time MR theory for general BVPs (Prüss, Bothe),
- $C^{1,\alpha}$ regularity, planar case, periodic or Dirichlet b.c. (Kaplický, Málek, Stará)
- partial $C^{1,\alpha}$ regularity for $p \in (12/5; 10/3)$, $d = 3$ (Ladyzhenskaya, Seregin),
- partial $C^{1,\alpha}$ regularity for modified $p(x)$-Navier-Stokes (Acerbi, Mingione, Seregin),
- strong solutions, $p \geq 2$, Dirichlet b. c. (Málek, Nečas, Růžička)

symmetric p-Laplace and around

- $W^{2,p}$ regularity for stationary symmetric p-Laplace with p close 2/small data (Crispo, DaVeiga, Grisanti),
- C^α/Morrey-space estimates for stationary problems (Buliček, Frehse, Steinhauer)

We want local (interior) results!

- Systems with general structure $\mathcal{A}(z, u, \mathbb{D}u)$.
  - Partial $C^{1,\alpha}_{loc}$ regularity.

- Systems with main part without lower-order terms $\mathcal{A}(\mathbb{D}u)$ (joint work with Petr Kaplický).
  - Local second-order estimates in space.
  - Local second-order estimates in time (iteration in Nikolskii-Bochner spaces).
  - Local regularity by stationary estimates.

$$u_{,t} - \mathrm{div} \mathcal{A}(z, u, \mathbb{D}u) = 0 \quad (1)$$

General structure

Tensor $\mathcal{A}(z, \eta, \cdot)$

is *strongly elliptic*	$\mathcal{A}(z, \eta, Q^s) : Q^s \geq \lambda \lvert Q^s \rvert^p$
is *weakly symmetrizing*	$\mathcal{A}(z, \eta, Q^s) : P^s \geq A(z, \eta, Q^s) P^s$
has $p-1$ growth	$\lvert \mathcal{A}(z, \eta, Q) \rvert \leq C(1 + \lvert Q \rvert^{p-1})$
	$\lvert \mathcal{A}(z, \eta, Q) - \mathcal{A}(\tilde{z}, \tilde{\eta}, Q) \rvert \leq C\theta(z, \tilde{z}, \eta, \tilde{\eta})(1 + \lvert Q \rvert^{p-1})$

where $\beta \in (0, 1)$, $\theta = \min \left[1, K(|\eta| + |\tilde{\eta}|) \left(d_2(z - \tilde{z}) + |\eta - \tilde{\eta}|^{\beta} \right) \right]$ and $K : \mathbb{R}_+ \to [1, \infty)$ is non-decreasing.

Moreover $\mathcal{A}$ is differentiable with respect to the matrix argument and $\partial \mathcal{A}(z, \eta, \cdot)$

is *Legendre-Hadamard elliptic*	$\partial \mathcal{A}(z, \eta, Q) P^s : P^s \geq \lambda (1 + \lvert Q \rvert^2)^{\frac{p-2}{2}} \lvert P^s \rvert^2$
is *strongly symmetrizing*	$(\partial A(z, \eta, Q))^{ij}_{kl} = (\partial A(z, \eta, Q))^{kl}_{ij} = (\partial_Q A(z, \eta, Q))^{ji}_{lk}$
grows in a general way	$\lvert \eta \rvert + \lvert Q \rvert \leq M \implies \lvert \partial \mathcal{A}(z, \eta, Q) \rvert \leq C(M)$
is continuous	$\lvert \eta \rvert + \lvert Q \rvert + \lvert \eta - \tilde{\eta} \rvert + \lvert Q - \tilde{Q} \rvert \leq M \implies$

$$\left| \partial \mathcal{A}(z, \eta, Q) - \partial \mathcal{A}(\tilde{z}, \tilde{\eta}, \tilde{Q}) \right| \leq C(M)\, \omega(M, d_2^p(z - \tilde{z}) + |\eta - \tilde{\eta}|^p + |Q - \tilde{Q}|^p)$$

the local modulus of continuity ω satisfies: $\omega(\cdot, s)$, $\omega(t, \cdot)$ are nondecreasing, $\omega(t, 0) = 0$ and $\omega(t, \cdot)$ is continuous at zero, $\omega^p(t, \cdot)$ is concave.

Theorem (Partial regularity)

Weak solutions to (1) with the general structure, $p \geq 2$ have a.e. locally Hölder continuous gradients.

More precisely, there is an open set $\tilde{Q}$ of full Lebesgue measure containing

$$\left\{ z \in Q : \liminf_{\varrho \to 0} \fint_{Q_\varrho(z)} |\mathbb{D}u - (\mathbb{D}u)|^p = 0 \ \wedge \ \limsup_{\varrho \to 0} |(u)_{z,\varrho}| + |(\nabla u)_{z,\varrho}| < +\infty \right\}$$

for which

$$\nabla u \in C^{\beta, \frac{\beta}{2}}(\tilde{Q}), \quad u \in C^{1, \frac{1}{2}}(\tilde{Q})$$

Proof Along the lines of Duzaar, Mingione, Steffen.

1. (*symmetric caloric approximation lemma*) Fix ε. Function f solves δ_ε-approximatively a linear Legendre-Hadamard parabolic system $\Rightarrow$ f is ε-close in an appropriate $L^2 - L^p$ sense to the exact solution of a linear parabolic system.
2. Linearization and Caccioppoli inequality $\Rightarrow$ an appropriately rescaled weak solution to (1) solves D-approximatively a linear Legendre-Hadamard parabolic system. Here
$$D = \omega\left(M+1, \tilde{E}_{z_0,l}(\varrho)\right) + \tilde{E}^{\frac{1}{2}}_{z_0,l}(\varrho) + \delta_0/2$$
3. Take z_0 such that
$$\liminf_{\varrho\to 0} \fint_{Q_\varrho(z_0)} |\mathbb{D}u - (\mathbb{D}u)|^p = 0, \quad \limsup_{\varrho\to 0} |(u)_{z_0,\varrho}| + |(\nabla u)_{z_0,\varrho}| < +\infty$$
then D is small (qualitatively) in a cylinder Q_{ρ_0} $\Rightarrow$ control of excess energies on $Q_{\sigma\rho_0}$, $\sigma \in (0,1)$ via
(i) the symmetric caloric approximation lemma,
(ii) the Campanato theory for the linear Legendre-Hadamard parabolic systems.
Iteratively, we can control quantitatively the excess energies in a neighborhood of z_0.

$$u_{,t} - \mathrm{div}\mathcal{A}(\mathbb{D}u) = 0 \qquad (2)$$

Structure

For any $P, Q \in Sym^{d\times d}$

$$\begin{aligned}(\mathcal{A}(P) - \mathcal{A}(Q)) : (P - Q) &\geq c\varphi''(|P| + |Q|)|P - Q|^2 \\ |\mathcal{A}(P) - \mathcal{A}(Q)| &\leq C\varphi''(|P| + |Q|)|P - Q|\end{aligned}$$

φ is a $\mathcal{N}$-function with good φ' property: $\varphi''(t)t \sim \varphi'(t)$

$$u_{,t} - \mathrm{div} \mathcal{A}(\mathbb{D}u) = 0 \tag{2}$$

Structure

For any $P, Q \in Sym^{d\times d}$

$$\begin{aligned}(\mathcal{A}(P) - \mathcal{A}(Q)) : (P - Q) &\geq c\varphi''(|P| + |Q|)|P - Q|^2 \\ |\mathcal{A}(P) - \mathcal{A}(Q)| &\leq C\varphi''(|P| + |Q|)|P - Q|\end{aligned}$$

φ is a $\mathcal{N}$-function with good φ' property: $\varphi''(t)t \sim \varphi'(t)$

The "square root" tensor $\mathcal{V}$

$$\bar{\varphi}'(t) := \sqrt{t\varphi'(t)}, \qquad \mathcal{V} := \partial_Q \bar{\varphi}(|Q|)$$

$$|\mathcal{V}(P) - V(Q)|^2 \sim (\mathcal{A}(P) - \mathcal{A}(Q)) : (P - Q)$$

One can justify testing (2) with $-\mathrm{div}((\nabla u)\eta^2)$ and get

Theorem (Strong solutions in space)

A local weak solution u to (2) enjoys

$$\nabla u \in L^\infty_{loc}(L^2_{loc}),\ \nabla \mathcal{V}(\mathbb{D}u) \in L^2_{loc}(L^2_{loc}),\ \nabla^2 u \in L^2_{loc}(L^2_{loc})$$

with the following estimate

$$\operatorname*{ess\,sup}_{\tau \in I_{r^2}} \int_{B_r} |\nabla u|^2(\tau) + \int_{Q_r} |\nabla \mathcal{V}(\mathbb{D}u)|^2 + \varphi''(0) \int_{Q_r} |\nabla^2 u|^2 \leq$$

$$\left(1 + \frac{1}{\varphi''(0)}\right) \frac{C(G(\varphi'))}{(R-r)^2} \left[\sup_{\tau \in I_{R^2}} \int_{B_R} |u|^2 (\tau) + \int_{Q_R} \varphi(|\mathbb{D}u|) \right]$$

for any $r < R$ and concentric parabolic cylinders Q_r, Q_R.

One can justify testing (2) with $-\mathrm{div}((\nabla u)\eta^2)$ and get

Theorem (Strong solutions in space)

A local weak solution u to (2) enjoys

$$\nabla u \in L^\infty_{loc}(L^2_{loc}),\ \nabla \mathcal{V}(\mathbb{D}u) \in L^2_{loc}(L^2_{loc}),\ \nabla^2 u \in L^2_{loc}(L^2_{loc})$$

with the following estimate

$$\operatorname*{ess\,sup}_{\tau \in I_{r^2}} \int_{B_r} |\nabla u|^2(\tau) + \int_{Q_r} |\nabla \mathcal{V}(\mathbb{D}u)|^2 + \varphi''(0) \int_{Q_r} |\nabla^2 u|^2 \leq$$
$$\left(1 + \frac{1}{\varphi''(0)}\right) \frac{C(G(\varphi'))}{(R-r)^2} \left[\sup_{\tau \in I_{R^2}} \int_{B_R} |u|^2(\tau) + \int_{Q_R} \varphi(|\mathbb{D}u|) \right]$$

for any $r < R$ and concentric parabolic cylinders Q_r, Q_R.

<u>Proof</u> A modification of the elliptic technique by Diening, Ettwein (do not need a covering argument).

Trouble Even formally, closing estimate by testing (2) with $(u_{,t}\, \eta)_{,t}$ is troublesome

$$\sup_{\tau \in I} \int_\Omega |u_{,t}\, \eta|^2(\tau) + \int_{\Omega_I} \varphi''(|\mathbb{D}u|)|\mathbb{D}u_{,t}|^2 \eta^2 \leq$$

$$C(\eta) \int_{\Omega_I} (|u_{,t}|^2 + \varphi''(|\mathbb{D}u|)|u_{,t}|^2)\eta \quad (3)$$

$$\varphi''(t) \sim 1 + t^{p-2}, \quad p \geq 2$$

Trouble Even formally, closing estimate by testing (2) with $(u_{,t}\,\eta)_{,t}$ is troublesome

$$\sup_{\tau\in I}\int_\Omega |u_{,t}\,\eta|^2(\tau) + \int_{\Omega_I} \varphi''(|\mathbb{D}u|)|\mathbb{D}u_{,t}|^2\eta^2 \leq C(\eta)\int_{\Omega_I}(|u_{,t}|^2 + \varphi''(|\mathbb{D}u|)|u_{,t}|^2)\eta \quad (3)$$

$$\varphi''(t) \sim 1 + t^{p-2}, \quad p \geq 2$$

Ways out

- Split $\varphi''(|\mathbb{D}u|)|u_{,t}|^2\eta$ and use space regularity + parabolic embedding.
  Limitations: one needs a priori a lot of smoothness, $p \leq 2 + \frac{4}{d}$.

Trouble Even formally, closing estimate by testing (2) with $(u_{,t}\,\eta)_{,t}$ is troublesome

$$\sup_{\tau\in I}\int_\Omega |u_{,t}\,\eta|^2(\tau) + \int_{\Omega_I}\varphi''(|\mathbb{D}u|)|\mathbb{D}u_{,t}|^2\eta^2 \le C(\eta)\int_{\Omega_I}(|u_{,t}|^2 + \varphi''(|\mathbb{D}u|)|u_{,t}|^2)\eta \quad (3)$$

$$\varphi''(t) \sim 1 + t^{p-2}, \quad p \ge 2$$

Ways out

- Split $\varphi''(|\mathbb{D}u|)|u_{,t}|^2\eta$ and use space regularity + parabolic embedding.
  Limitations: one needs a priori a lot of smoothness, $p \le 2 + \frac{4}{d}$.
- Be less greedy. Interpolation of weak regularity data

$$u_{,t} \in (L^p(I; W_0^{1,p}(\Omega)))^* \quad \text{and} \quad u \in L^p(I; W_0^{1,p}(\Omega))$$

gives a fractional time regularity. Next we try to iterate.

Theorem (Strong solutions in time)

Assume $\varphi''(t) \sim 1 + t^{p-2}$, u is a local weak solution to (2)

- *Case $p \in (2, 2 + \frac{2}{\sqrt{d+1}})$. Locally we have u in*

$$N^{1+\gamma,2}(L^p),\ W^{2,2}(W^{-1,2}),\ W^{1,\infty}(L^2),\ W^{1,2}(W^{1,2}),\ N^{\frac{2}{p},p}(W^{1,p})$$

for a positive γ. Tensor $\mathcal{V}(\mathbb{D}u)$ is in

$$W^{1,2}(L^2)$$

- *Case $p \geq \frac{2}{\sqrt{d+1}}$. For any $\alpha < \frac{2p}{(p-2)(d(p-2)+p)}$ u is in*

$$N^{\alpha,\infty}(L^2),\ N^{1+\alpha,p'}(W^{-1,p'}),\ N^{\alpha,2}(W^{1,2}),\ N^{\frac{2\alpha}{p},p}(W^{1,p})$$

and $\mathcal{V}(\mathbb{D}u)$ belongs to

$$N^{\alpha,2}(L^2)$$

$+$ *Estimates.*

Theorem (Strong solutions in time)

Assume $\varphi''(t) \sim 1 + t^{p-2}$, *$u$ is a local weak solution to* (2)

- *Case* $p \in (2, 2 + \frac{2}{\sqrt{d+1}})$. *Locally we have u in*

$$N^{1+\gamma,2}(L^p),\ W^{2,2}(W^{-1,2}),\ W^{1,\infty}(L^2),\ W^{1,2}(W^{1,2}),\ N^{\frac{2}{p},p}(W^{1,p})$$

for a positive γ. *Tensor* $\mathcal{V}(\mathbb{D}u)$ *is in*

$$W^{1,2}(L^2)$$

- *Case* $p \geq \frac{2}{\sqrt{d+1}}$. *For any* $\alpha < \frac{2p}{(p-2)(d(p-2)+p)}$ *u is in*

$$N^{\alpha,\infty}(L^2),\ N^{1+\alpha,p'}(W^{-1,p'}),\ N^{\alpha,2}(W^{1,2}),\ N^{\frac{2\alpha}{p},p}(W^{1,p})$$

and $\mathcal{V}(\mathbb{D}u)$ *belongs to*

$$N^{\alpha,2}(L^2)$$

$+$ *Estimates.*

Comment Range for full derivatives better than parabolic embedding for $d > 3$.

<u>Idea of Proof</u> Buliček, Ettwein, Kaplický, Pražak.
Difference: here bad terms (localization) are worse.

<u>Single iteration step</u>

1. $u \in N^{\alpha_i,p}(L^p) \Rightarrow \mathcal{V}(\mathbb{D}u) \in N^{\alpha_i,2}(L^2)$ (second-order fractional estimate)
2. $\mathcal{V}(\mathbb{D}u) \in N^{\alpha_i,2}(L^2) \Rightarrow$
$$\begin{cases} u\eta \in N^{1+\alpha_i,p'}\left(W^{-1,p'}\right) & \text{(evolutionary estimate)} \\ u\eta \in N^{\frac{2\alpha_i}{p},p}(W^{1,p}) & \text{(structure)} \end{cases}$$
3. $\begin{cases} u\eta \in N^{1+\alpha_i,p'}(W^{-1,p'}) \\ u\eta \in N^{\frac{2\alpha_i}{p},p}(W^{1,p}) \end{cases} \Rightarrow u \in N^{\alpha_{i+1},p}(L^p)$ (interpolation and embedding)

$$\alpha_{i+1} = \alpha_i A + B, \qquad A := \frac{2}{p} + \frac{p-2}{d(p-2)+2p} \quad B := \frac{2}{d(p-2)+2p}$$

<u>Possible improvement</u> Use at every iteration step improved integrability of $\mathbb{D}u$.

Lemma (Regularity via the stationary estimates)

A local weak solution u to (2) *enjoys*

$$\nabla \mathcal{V}(\mathbb{D}u) \in L^{\infty}_{loc}(L^{2}_{loc}), \quad u \in L^{\infty}_{loc}(W^{2,2}_{loc})$$

with the following estimate

$$\operatorname*{ess\,sup}_{t\in I} \int_{B_r} |\nabla \mathcal{V}(\mathbb{D}u(t))|^2 + \varphi''(0)|\nabla \mathbb{D}u(t)|^2 \leq$$

$$\operatorname*{ess\,sup}_{t\in I} \left(1 + \frac{1}{\varphi''(0)}\right) \frac{C(G(\varphi'))}{(R-r)^2} \int_{B_R} \varphi(|\nabla u(t)|) + |u_{,t}(t)|^2$$

Lemma (Regularity via the stationary estimates)

A local weak solution u to (2) *enjoys*

$$\nabla\mathcal{V}(\mathbb{D}u) \in L^\infty_{loc}(L^2_{loc}), \quad u \in L^\infty_{loc}(W^{2,2}_{loc})$$

with the following estimate

$$\operatorname*{ess\,sup}_{t\in I} \int_{B_r} |\nabla\mathcal{V}(\mathbb{D}u(t))|^2 + \varphi''(0)|\nabla\mathbb{D}u(t)|^2 \leq$$
$$\operatorname*{ess\,sup}_{t\in I} \left(1 + \frac{1}{\varphi''(0)}\right) \frac{C(G(\varphi'))}{(R-r)^2} \int_{B_R} \varphi(|\nabla u(t)|) + |u_{,t}(t)|^2$$

Comment Planar case: $\mathbb{D}u \in L^\infty_{loc}(BMO)$. Next step: Hölder continuity of $\mathbb{D}u$.

- Systems with general structure $\mathcal{A}(z, u, \mathbb{D}u)$.
  - Partial $C^{1,\alpha}_{loc}$ regularity.
    $\rightarrow$ singular set estimates, Orlicz structure, Stokes
- Systems with main part without lower-order terms $\mathcal{A}(\mathbb{D}u)$.
  - Local second-order estimates in space.
  - Local second-order estimates in time (iteration in Nikolskii-Bochner spaces).
    $\rightarrow$ larger range of p's
  - Local regularity by stationary estimates.

- Systems with general structure $\mathcal{A}(z, u, \mathbb{D}u)$.
  - Partial $C^{1,\alpha}_{loc}$ regularity.

    $\rightarrow$ singular set estimates, Orlicz structure, Stokes

- Systems with main part without lower-order terms $\mathcal{A}(\mathbb{D}u)$.
  - Local second-order estimates in space.
  - Local second-order estimates in time (iteration in Nikolskii-Bochner spaces).

    $\rightarrow$ larger range of p's
  - Local regularity by stationary estimates.

<u>Even further research</u>

$\rightarrow$ low L^q non-linear Calderón-Zygmund theory

$\rightarrow$ full-range non-linear Calderón-Zygmund theory in the planar case

- $C^{1,\alpha}_{loc}$-regularity for evolutionary symmetric p-Laplace system, p close to 2
- $C^{1,\alpha}_{loc}$-regularity for (evolutionary) symmetric p-Laplace system

- $C^{1,\alpha}_{loc}$-regularity for evolutionary symmetric p-Laplace system, p close to 2
- $C^{1,\alpha}_{loc}$-regularity for (evolutionary) symmetric p-Laplace system

DĚKUJI