High regularity results of solutions to modified p-Navier-Stokes equations

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joint work with P. Maremonti

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Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

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Introduction

We consider the following problem:

$$\nabla \cdot T(u,\pi) = F + (u \cdot \nabla)u\,, \quad \nabla \cdot u = 0 \ \text{ in } \mathbb{R}^n\,, \tag{1}$$

where $u = (u_1, \cdots, u_n)$ is a vector field, π is a scalar field,

$T(u,\pi)$ is a special tensor of the kind

$$T(u,\pi) = -\pi I + |\nabla u|^{p-2}\,\nabla u\,, \tag{2}$$

with I identity operator, $p \in (1,2)$.

The system is nonlinear and singular.
It was studied for the first time by J.L.Lions (1969), through the monotone operators theory.

Introduction

Tensor T is "close" to the well known stress tensor $\widetilde{T}$ of non-Newtonian fluids, in the singular case,

$$\widetilde{T}(u,\pi) = -\pi I + |\mathcal{D}u|^{p-2}\mathcal{D}u\,,$$

$\mathcal{D}u = \frac{\nabla u + (\nabla u)^T}{2}$, which gives rise to the so called p-Navier-Stokes problem

$$\nabla \cdot \big(|\mathcal{D}u|^{p-2}\,\mathcal{D}u\big) - \nabla\pi = F + (u\cdot\nabla)u\,, \quad \nabla\cdot u = 0 \ \text{ in } \mathbb{R}^n\,. \tag{3}$$

Hence, we call, on the contrary, system (1) *modified* p-Navier-Stokes system.

The difference between T and $\widetilde{T}$ makes our results not interesting from a fluid dynamic point of view. Nevertheless the results gain a special interest from the analytical point of view.

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Introduction

The literature concerning the “high regularity” of solutions to the singular p-(Navier-)Stokes problem is not satisfactory. We are just aware of the following results ($n=3$):

Naumann & Wolf (JMFM 2005)

$$\Omega \text{ bounded}, \;\; p\in(9/5,2), \;\; (u,\pi)\in W^{2,\frac{3p}{p+1}}_{\ell oc}(\Omega)\times L^{p'}_{\ell oc}(\Omega),$$

Berselli, Diening & Růžička (JMFM 2010)

$$\Omega \text{ space-periodic}, \;\; p\in(9/5,2), \;\; u\in W^{2,\frac{3p}{p+1}}(]0,1[^3),$$

Ebmeyer (MMAS 2006)

$$\Omega \text{ bounded, slip boundary conditions}, \;\; p\in(9/5,2), \;\; u\in W^{2,\frac{3p}{p+1}}(\Omega)\,.$$

No global regularity result for the pressure field π.

Introduction

The corresponding elliptic system is the well known p-Laplacian, whose study has a wide literature.

We quote in particular the results due to:

Acerbi & Fusco ('89), DiBenedetto ('93), Ebmeyer ('06), Iwaniec & Manfredi ('89), Liu & Barrett ('93), Málek & Rajagopal & Růžička ('95), Acerbi & Mingione & Seregin ('04), Tolksdorf ('05), Mingione (06);

high regularity with $D^2u \in L^q(\Omega)$:
Beirão da Veiga & Crispo ('12) and ('13),
Crispo & Maremonti ('14).

These results lead us to wonder if it is possible to deduce analogous results related to the p-(Navier-)Stokes problem.

Introduction

Our task is to understand if the couple of the vector field u and the scalar field π can give a regular solution of (1) at least with ∇u in place of $\mathcal{D}u$.

In other words, we investigate if, in spite of the presence of the scalar field π, the regularity of the field u can be compared with the one of solutions to the corresponding elliptic system.

The results I'm going to introduce are part of the papers

F.C. and P. Maremonti, *A high regularity result of solutions to modified p-Stokes equations*, submitted;

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“high regularity” for the modified p-Navier-Stokes system (1)

Definition 1 (High regular solution)

A pair (u, π) is a high regular solution of (1) if:

- *for some* $q \in (n, +\infty)$, $D^2 u, \nabla\pi \in L^q(\mathbb{R}^n)$, $\nabla u \in L^p(\mathbb{R}^n)$, $u \in L^{p^*}(\mathbb{R}^n)$, $\pi \in L^{p'}(\mathbb{R}^n)$,
- $\nabla \cdot u = 0$, *a. e. in* $\mathbb{R}^n$,
- $(|\nabla u|^{p-2}\nabla u, \nabla\varphi) = (\nabla\pi, \varphi) + ((u \cdot \nabla)u, \varphi) + (F, \varphi)$, *for all* $\varphi \in C_0^\infty(\mathbb{R}^n)$.

We set

$$M(r) := 1 - (2-p)H(r')(5 + H(r)),$$

and

$$\overline{M}(2) := 2p - 3 - (2-p)(1 + H(2)),$$

where $H(s)$ is the L^s-singular norm of Calderón-Zygmund type, r' coniugate exponent of r.

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Theorem 1:“high regularity” for the modified p-Stokes

Let

- $p \in (\frac{3}{2}, 2]$, $q_1 = \frac{np}{n+p}$, $q \in (n, +\infty)$;
- $F \in L^q(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$;
- $\overline{M}(2) > 0,\ M(q_1) > 0, M(q) > 0.$

Then there exists a solution (u, π) *in the sense of Definition 1 of the modified p-Stokes system, and*

$$\|D^2 u\|_q + \|D^2 u\|_{q_1} \le c\,(\|F\|_q + \|F\|_{q_1})^{\frac{1}{p-1}}\,. \tag{4}$$

$$\|\nabla\pi\|_q \le c\,\|F\|_q, \quad \|\nabla\pi\|_{q_1} \le c\,\|F\|_{q_1}\,. \tag{5}$$

Moreover, the solution (u, π) *is unique in the class of solutions with* $\nabla u \in L^p(\mathbb{R}^n)$.

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Approximating solutions

We develop our arguments starting from a nonsingular problem:

$$\nabla \cdot u = 0, \text{ in } \mathbb{R}^n, \quad \nabla \cdot \left[(\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right] - \nabla \pi = F, \text{ in } \mathbb{R}^n, \qquad (6)$$

where $\mu > 0$.

The idea is to work on this problem and to to obtain all the estimates uniformly with respect to the parameter μ.

Some comments on the statement of the theorem

- The choice of the exponent $q_1 = \frac{np}{n+p}$ has been made in order to get $\nabla u \in L^p(\mathbb{R}^n)$ by Sobolev embedding.

  $\nabla u \in L^p(\mathbb{R}^n)$ enables us to discuss uniqueness. Hence in particular to compare our solution with weak solutions.

  (Note that $q_1 > 1$ if and only if $p > \frac{n}{n-1}$, which excludes the value $n = 2$).

- The request on the constants $M(q_1) > 0$ and $M(q) > 0$ translates a condition of proximity of p to 2, which is a sufficient condition in order to get the following kind of estimate

$$\left\| \frac{D^2 u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \right\|_{L^r(\mathbb{R}^n)} \le c \, \|F\|_{L^r(\mathbb{R}^n)} \,, \text{ for all } \mu > 0, \qquad (7)$$

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  for exponents $r = q_1, q$.

Why does the theorem not work under the assumption $\widetilde{T}(u,\pi) = -\pi I + |\mathcal{D}u|^{p-2}\mathcal{D}u$? Hydrodynamic case!

The reason is connected with the fact that we employ the pointwise estimate

$$\frac{|D_{x_i}u_j(x)|^2}{\mu + |\nabla u(x)|^2} \leq 1\,,$$

that clearly does not hold, in general, with

$$\mu + |\mathcal{D}u(x)|^2 \text{ in place of } \mu + |\nabla u(x)|^2.$$

However, we could obtain the results for $\widetilde{T}(u,\pi)$ if we were able to show the following crucial kind of estimates:

$$\int_\Omega \frac{|D_{x_h}u_i D_{x_i}|\mathcal{D}u|^2|^r}{(\mu + |\mathcal{D}u|^2)^{\frac{4-p}{2}r}}\,dx \leq c\int_\Omega \frac{|\mathcal{D}u|^r|D^2u|^r}{(\mu + |\mathcal{D}u|^2)^{\frac{3-p}{2}r}}\,dx\,,$$

for $r = 2$ and bounded Ω, $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$,
and for $r \neq 2$ and $\Omega = \mathbb{R}^n$, $\nabla u \in L^p(\mathbb{R}^n)$ and $D^2u \in L^r(\mathbb{R}^n)$.

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Why does our proof work? What is new?

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When we study the p-(Navier)-Stokes equations, we usually study the existence and regularity of the kinetic field and then the ones of the pressure field.

So, when we employ the classical methods to exhibit a solution (see J.L. Lions), we aim at a twofold objective:

- the former is to drop the pressure field,
- the latter is to get a uniform bound of a suitable "energy norm".

The first objective is a consequence of the classical Helmohltz-Weyl orthogonality.
The second objective is a consequence of the coercive properties of the chief operator.

For instance, if we look for a weak solution, the Helmohltz-Weyl orthogonality is between the fields u and $\nabla\pi$, while the metric concerns $|\nabla u|_p$.

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If we look for the regularity, for example in the L^2 setting, then for the **classical** Stokes problem we need as staring point

- the Helmohltz-Weyl orthogonality between $P\Delta u$ and $\nabla\pi$.
- Then, the "energy metric" is $\|P\Delta u\|_2$, which implies, by a suitable estimate, the same bound on the second derivatives, and so on.

If we argue in a similar way (L^2-theory) to obtain the “high regularity” of solutions to the p-Stokes problem,

- then the Helmohltz-Weyl orthogonality between $P\Delta u$ and $\nabla\pi$ clearly continues to hold,
- but we should be able to evaluate the quantity

$$\big(\nabla\cdot\widetilde{T}(u,\pi), P\Delta u\big) \quad \text{or} \quad \big(\nabla\cdot T(u,\pi), P\Delta u\big),$$

or, in $\mathbb{R}^n$,

$$\big(\nabla\cdot\widetilde{T}(u,\pi), \Delta u\big) \quad \text{or} \quad \big(\nabla\cdot T(u,\pi), \Delta u\big),$$

which doesn't give an estimate for $\|P\Delta u\|_2$ or for $\|\Delta u\|_2$.

This is the impasse that we meet if we formally reproduce the approach inherited from the classical analytic theory of the Stokes problem.

Our approach

In our case we proceed in a different way.
The new idea is to investigate the existence and the regularity looking at the problem as a **perturbed elliptic problem**.

Since we work in $\mathbb{R}^n$, we are able to arrive at a representation formula of the pressure field of this kind

$$\pi := (2-p)\int_{\mathbb{R}^n} D_{y_i}\mathcal{E}(x-y)\,\frac{D_{y_i}u_s(y)D_{y_s}|\nabla u|^2}{(\mu+|\nabla u(y)|^2)^{\frac{4-p}{2}}}\,dy\,,$$

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This representation is crucial for our aims.

Setting $K[\nabla u] := \pi(x)$, where $\pi(x)$ is the pressure field already defined, then we study the problem

$$\nabla \cdot \left[(\mu + |\nabla u|)^{p-2} \nabla u\right] - \nabla K[\nabla u] = F, \text{ for all } \mu > 0. \tag{8}$$

Here the perturbation is the term $\nabla K[\nabla u]$, and for p close to 2 it can be small in suitable norms.

The advantage to handle system (8) is that now we can employ the methods of the elliptic problems, the eigenfunctions of the Laplace operator as special Galerkin basis (on a bounded domain), and all the estimates up to the second derivatives are made in L^q spaces and not in the spaces of Hydrodynamics (which intrinsically contain the Helmohltz-Weyl orthogonality).

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In particular, our estimates on second derivatives have, as a starting point, the estimate of the kind:

$$\Big\| \frac{D^2 u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \Big\|_{L^r(\mathbb{R}^n)} \leq c \, \|F\|_{L^r(\mathbb{R}^n)} \,, \text{ for all } \mu > 0, \tag{9}$$

for exponents $r = q_1, q$. Estimate (9) plays the same role as the estimate of $\|P \Delta u\|_r$ for the classical Stokes system.

A crucial tool for the above estimate is the following one

$$\Big\| \frac{D^2 u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \Big\|_{L^r(\mathbb{R}^n)} \leq c(r,p) \Big\| \frac{\Delta u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \Big\|_{L^r(\mathbb{R}^n)}$$

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$$\|D^2 u\|_{L^r(\mathbb{R}^n)} \leq c(r) \|\Delta u\|_{L^r(\mathbb{R}^n)} \, .$$

We gain, roughly speaking, a solution of the problem

$$\nabla \cdot T(u, \pi, \mu) = F\,, \quad \text{in } \mathbb{R}^n\,,\ \text{for all } \mu > 0\,, \tag{10}$$

with

$$\pi = (2-p)\int_{\mathbb{R}^n} D_{y_i}\mathcal{E}(x-y)\,\frac{D_{y_i}u_s D_{y_s}|\nabla u|^2}{(\mu + |\nabla u|^2)^{\frac{4-p}{2}}}\,dy\,. \tag{11}$$

So the divergence free equation

$$\nabla \cdot u = 0 \tag{12}$$

seems missing.

But our pressure field has the representation formula of a pressure field arisen from the equations of the pressure:

$$\Delta\pi = \nabla \cdot \left[\nabla \cdot \left((\mu + |\nabla u|)^{p-2}\nabla u\right)\right]\,,\ \textbf{joint with the condition } \nabla \cdot u = 0.$$

Therefore equation (12) is satisfied by our solution as a compatibility condition between the equations (10) and the representation formula (11) and finally by the condition $\nabla \cdot u \to 0$ for $|x| \to \infty$.

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Therefore equation (12) is satisfied by our solution as a compatibility condition between the equations (10) and the representation formula (11) and finally by the condition $\nabla \cdot u \to 0$ for $|x| \to \infty$.

Indeed, the compatibility between the equation (10) and (11) formally gives, for $\nabla \cdot u$, the equation

$$\Delta(\nabla \cdot u) + \frac{(2-p)}{2} \frac{\nabla(\nabla \cdot u) \cdot \nabla |\nabla u|^2}{(\mu + |\nabla u|^2)} = 0\,, \text{ in } \mathbb{R}^n,$$

to which we append the condition

$$\nabla \cdot u \ \to 0 \text{ as } |x| \to \infty.$$

Therefore the maximum principle ensures that $\nabla \cdot u = 0$.
So, the solution of (10) is divergence free.
Hence u is a solution to the modified p-Stokes problem (1).

Regularity under more general assumptions

In order to obtain a high regularity result for the modified p-Navier-Stokes system, it is crucial to improve the result obtained for the modified p-Stokes system, by making different assumptions on the integrability of the force term F.
This is the aim of the following theorem, whose proof relies on Theorem 1.

Theorem 2: “high regularity” for the modified p-Stokes

Let

- $p \in (\frac{3}{2}, 2]$, $q_1 = \frac{np}{n+p}$, $q \in (n, +\infty)$;
- $F \in L^q(\mathbb{R}^n) \cap (\widehat{W}^{1,p}(\mathbb{R}^n))'$;
- $\overline{M}(2) > 0,\ M(q_1) > 0, M(q) > 0.$

Then there exists a solution (u, π) *in the sense of Definition 1 of the modified p-Stokes system, and*

$$\|D^2 u\|_q \le c\,(\|F\|_q \|F\|_{-1,p'}^{\frac{(1-a)(2-p)}{p-1}})^{\frac{1}{1-a(2-p)}}\,, \tag{13}$$

$a = \frac{nq}{n(q-p)+pq}$ *. Moreover, the solution* (u, π) *is unique in the class of solutions with* $\nabla u \in L^p(\mathbb{R}^n)$.

Theorem 3: “high regularity” for the modified p-Navier-Stokes

Let

- $p \in (\frac{9}{5}, 2]$, $q_1 = \frac{3p}{3+p}$, $q \in (3, +\infty)$;
- $F \in L^q(\mathbb{R}^n) \cap (\widehat{W}^{1,p}(\mathbb{R}^3))'$;
- $\overline{M}(2) > 0,\ M(q_1) > 0, M(q) > 0$.

Then there exists a solution (u, π) in the sense of Definition 1 of the modified p-Navier- Stokes system.

The high regularity obtained in the above theorem is completely similar to the one given for the p-Stokes system, as it happens for the classical Navier-Stokes equations.
Likewise, no uniqueness result is at disposal.

The leading ideas behind this result are the following.

- We consider the "regularized" nonlinear system

$$\nabla\cdot\big(|\nabla v|^{p-2}\nabla v\big)-\nabla\pi_v=\chi^\rho J_\varepsilon\big((v\cdot\nabla)J_\varepsilon(v\chi^\rho)\big)+f\,,\quad \nabla\cdot v=0 \ \text{ in } \mathbb{R}^3\,. \tag{14}$$

Using the Galerkin method and the monotone operator theory, for $f\in(\widehat{W}^{1,p}(\mathbb{R}^3))'$ there exists a weak solution of (14) satisfying

$$\|\nabla v\|_p\le c\|f\|_{-1,p'}^{\frac{1}{p-1}}\,,\ \forall\varepsilon>0,\ \forall\rho>0\,. \tag{15}$$

Note that $(\chi^\rho J_\varepsilon((v\cdot\nabla)J_\varepsilon(v\chi^\rho)),v)=((v\cdot\nabla)J_\varepsilon(v\chi^\rho),J_\varepsilon(v\chi^\rho))=0\,.$

The p-Navier-Stokes equations

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- We consider the modified p-Stokes system

$$\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) - \nabla \pi_u = F\,, \quad \nabla \cdot u = 0 \ \text{ in } \mathbb{R}^3\,. \tag{16}$$

with

$$F := f + F_{\varepsilon,\rho} := f + \chi^{\rho} J_{\varepsilon}\big((v\cdot\nabla)J_{\varepsilon}(v\chi^{\rho})\big) \in L^q(\mathbb{R}^3) \cap \big(\widehat{W}^{1,p}(\mathbb{R}^3)\big)'\,, \tag{17}$$

for which our Theorem 2 holds. Hence there exists a high regular solution u.

- By uniqueness in the class of weak solutions, $u \equiv v$.

Then one needs to get estimates on $\|D^2 u\|_q$, uniformly in ε and ρ, in order to use weak sequential compactness results, and pass to the limit on $\varepsilon \to 0$ and $\rho \to \infty$.

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THANK YOU FOR YOUR ATTENTION