Homogenization of a non-Newtonian flow through a porous medium

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Homogenization in general

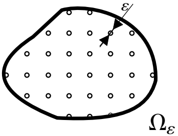
- equation with rapidly oscillating variables

$$-\operatorname{div} A\left(\frac{x}{\varepsilon}, u(x), \nabla u(x)\right) = g(x) \text{ in } \Omega$$

  A periodic in first varible

- equations on a domain with a shrinking microstructure

$$-\operatorname{div}(A(x, u(x), \nabla u(x)) = g(x) \text{ in } \Omega_\varepsilon$$



The aim is to establish equation without the dependence on the microstructure whose solution is a good approximation of the solution of initial problem.

Nondimensional form of stationary Stokes system

$$
\begin{aligned}
-\frac{1}{Re}\operatorname{div}\left(\eta(D\mathbf{u})D\mathbf{u}\right) + Eu\nabla p &= \frac{1}{Fr}\mathbf{f} \text{ in } \Omega \\
\operatorname{div}\mathbf{u} &= 0 \text{ in } \Omega \\
\mathbf{u} &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

macroscopic characteristics of the porous media Ω:

Re Reynolds number

Eu Euler number

Fr Froude number

Generalization of the power law

$\eta(D\mathbf{u}) = \frac{\varphi'(|D\mathbf{u}|)}{|D\mathbf{u}|}$
$\varphi : [0, \infty) \to [0, \infty)$ is N-function if $\exists \varphi'$ such that

1. φ' is (right)continuous, non-decreasing,
2. $\varphi'(0) = 0$,
3. $\varphi'(t) > 0$ for $t > 0$.

Δ_2-condition $\exists c > 0 \ \forall t > 0 : \varphi(2t) \leq c\varphi(t)$
examples:
$\varphi(t) = \frac{t^p}{p} \ p > 0, \ \varphi(t) = \frac{t^2}{\log(t+e)}, \ \varphi(t) = (t+1)\log(t+1) - t$

Sobolev-Orlicz spaces

Let $\Omega \subset \mathbb{R}^d$ be open.
Orlicz space

$$L^\varphi(\Omega) = \{u \in L^1_{loc}(\Omega), \int_\Omega \varphi(|u|) < \infty\}$$
$$\|u\|_\varphi = \inf \left\{ \lambda > 0; \int_\Omega \varphi\left(\frac{|u|}{\lambda}\right) \leq 1 \right\}$$

Sobolev-Orlicz space

$$W^{1,\varphi}(\Omega) = \{u \in L^\varphi(\Omega) \colon \nabla u \in L^\varphi(\Omega)\}$$
$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi$$
$$W^{1,\varphi}_{0(,\mathrm{div})}(\Omega) = \overline{C^\infty_{0(,\mathrm{div})}(\Omega)}^{\|\cdot\|_{1,\varphi}}$$

System of interest

ε ratio of the microscopic length and the characteristic length of the porous medium

$Re \sim \varepsilon^{-\gamma}$,$Eu \sim \varepsilon^{\delta}$,$Fr \sim \varepsilon^{-\beta}$

$$-\varepsilon^{\gamma}\,\mathrm{div}\left(\varphi'(|D\mathbf{u}_\varepsilon|)\frac{D\mathbf{u}_\varepsilon}{|D\mathbf{u}_\varepsilon|}\right)+\varepsilon^{\delta}\nabla p_\varepsilon=\varepsilon^{\beta}\mathbf{f} \text{ in } \Omega_\varepsilon$$

without loss of generality $\beta = 0,\ \delta = 0$

$$\begin{aligned}-\varepsilon^{\gamma}\,\mathrm{div}\left(\varphi'(|D\mathbf{u}_\varepsilon|)\frac{D\mathbf{u}_\varepsilon}{|D\mathbf{u}_\varepsilon|}\right)+\nabla p_\varepsilon&=\mathbf{f} \text{ in } \Omega_\varepsilon\\ \mathrm{div}\,\mathbf{u}_\varepsilon&=0 \text{ in } \Omega_\varepsilon\\ \mathbf{u}_\varepsilon&=0 \text{ on } \partial\Omega_\varepsilon\end{aligned} \qquad (\mathrm{GS}_\varepsilon)$$

The resulting system

Theorem

Let $\gamma = 1$. *The functions* $\mathbf{u}_0, p_0$ *satisfy for almost all* $x \in \Omega$

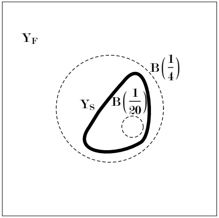
$$
\begin{aligned}
-\operatorname{div}_y \left(\varphi'(|D_y \mathbf{u}_0(x,y)|) \frac{D_y \mathbf{u}_0(x,y)}{|D_y \mathbf{u}_0(x,y)|} \right) + \nabla_y \pi(x,y) \\
= \mathbf{f}(x) - \nabla_x p_0(x) \text{ in } Y \\
\operatorname{div}_y \mathbf{u}_0 = 0 \text{ in } Y \\
\pi \ Y - \text{periodic.}
\end{aligned}
$$

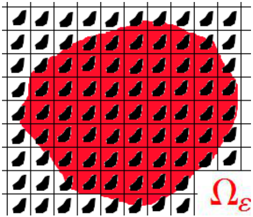
Known results

Tartar '80	derivation of Darcy law via homogenization of Stokes problem
Nguetseng '89	introduction and using of 2-s convergence
Allaire '92	obstacle size involves the form of the homogenized system
Bourgeat, Mikelić '96	homogenization of stationary p-NS
Nnang, Tachago '13	2-s convergence in Orlicz setting

Geometry of a porous medium

$Y = (0,1)^d, \ d = 2,3$
Y_S solid part of Y
Y_F fluid part of Y
$B(\frac{1}{20}) \subset Y_S \subset B(\frac{1}{4})$





a periodic repetition
$Y_k^\varepsilon = \varepsilon(Y + k), \ k \in \mathbb{Z}^d$ (Y scaled to ε-length)
$E_S = \cup_{k \in \mathbb{Z}^d} Y_S^\varepsilon$
$\Omega_\varepsilon = \Omega \setminus E_S$ fluid part of a porous medium $\Omega \in C^{0,1}(\mathbb{R}^d)$

Korn's type inequalitites

Lemma (Korn I)

$\exists c_1, c_2 > 0 \; \forall \mathbf{v} \in W_0^{1,\varphi}(\Omega)$:

$$\int_\Omega \varphi(|\nabla \mathbf{v}|) \leq c_1 \int_\Omega \varphi(|D\mathbf{v}|)$$

$$\|\nabla \mathbf{v}\|_{\varphi,\Omega} \leq c_2 \|D\mathbf{v}\|_{\varphi,\Omega}$$

Lemma (Korn II)

$\exists c_1, c_2, c_3, c_4 > 0 \; \forall \mathbf{v} \in W_0^{1,\varphi}(\Omega_\varepsilon)$:

$$\int_{\Omega_\varepsilon} \varphi(|\mathbf{v}|) \leq c_1 \int_\Omega \varphi(\varepsilon|\nabla \mathbf{v}|) \leq c_2 \int_\Omega \varphi(\varepsilon|D\mathbf{v}|)$$

$$\|\mathbf{v}\|_{\varphi,\Omega_\varepsilon} \leq c_3\varepsilon\|\nabla \mathbf{v}\|_{\varphi,\Omega_\varepsilon} \leq c_4\varepsilon\|D\mathbf{v}\|_{\varphi,\Omega_\varepsilon}$$

Two-scale convergence I

Definition

A sequence $\{v_{\varepsilon_k}\} \subset L^{\varphi}(\Omega)$ is said to converge weakly two-scale in $L^{\varphi}(\Omega)$ to $v \in L^{\varphi}(\Omega \times Y)$ ($v_{\varepsilon_k} \xrightarrow{2-s} v$) if for any $\sigma \in L^{\varphi^*}(\Omega; C_{per}(Y))$

$$\lim_{\varepsilon_k \to 0} \int_{\Omega} v_{\varepsilon_k}(x)\sigma\left(x, \frac{x}{\varepsilon_k}\right)\mathrm{d}x = \int_{\Omega}\int_{Y} v(x,y)\sigma(x,y)\mathrm{d}y\mathrm{d}x.$$

Theorem

From any bounded sequence in $L^{\varphi}(\Omega)$ one can extract a subsequence which converges weakly two-scale in $L^{\varphi}(\Omega)$.

Two-scale convergence II

Lemma

Let $\{v_{\varepsilon_k}\}, \{\varepsilon_k \nabla v_{\varepsilon_k}\}$ *be bounded in* $L^{\varphi}(\Omega) \Rightarrow \exists v \in L^{\varphi}(\Omega; W^{1,\varphi}_{per}(Y))$, $\{v_{\varepsilon_{k_l}}\} : v_{\varepsilon_{k_l}} \xrightarrow{2-s} v, \ \varepsilon_{k_l} \nabla_x v_{\varepsilon_{k_l}} \xrightarrow{2-s} \nabla_y v.$

Lemma

Let $w \in C^{\infty}_{C}(\Omega; C^{\infty}_{per}(Y))$. *Then*

$$\lim_{\varepsilon_k \to 0} \int_{\Omega} \Phi\left(\varepsilon_k \nabla_x w\left(x, \frac{x}{\varepsilon_k}\right)\right) \mathrm{d}x = \int_{\Omega}\int_{Y} \Phi(\nabla_y w(x,y)) \mathrm{d}y \mathrm{d}x.$$

Lemma

Let $\Phi \in C^1(\mathbb{R}^{d \times d})$ *be convex,* $0 \le \Phi(\xi) \le \varphi(|\xi|)$. *Let* $\xi_{\varepsilon_k} \xrightarrow{2-s} \xi$. *Then*

$$\liminf_{\varepsilon_k \to 0} \int_{\Omega} \Phi(\xi_{\varepsilon_k}) \mathrm{d}x \ge \int_{\Omega}\int_{Y} \Phi(\xi(x,y)) \mathrm{d}y \mathrm{d}x.$$

Weak solution

Let $\mathbf{f} \in L^{\varphi^*}(\Omega)$. $\mathbf{u}_\varepsilon \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$ is a weak solution of $(\mathrm{GS}_\varepsilon)$ if $\forall \mathbf{v} \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega_\varepsilon)$

$$\varepsilon^\gamma \int_{\Omega_\varepsilon} \varphi'(|D\mathbf{u}_\varepsilon|) \frac{D\mathbf{u}_\varepsilon}{|D\mathbf{u}_\varepsilon|} D\mathbf{v} = \int_{\Omega_\varepsilon} \mathbf{f}\mathbf{v}$$

$$\exists c_1, c_2 > 0 \ \forall \varepsilon : \int_{\Omega_\varepsilon} \varphi(|D\mathbf{u}_\varepsilon|) \leq c_1 \int_{\Omega_\varepsilon} \varphi^*(|\varepsilon^{1-\gamma}\mathbf{f}|)$$

$$\int_{\Omega_\varepsilon} \varphi^*\left(\varphi'(|D\mathbf{u}_\varepsilon|)\right) \leq c_2 \int_{\Omega_\varepsilon} \varphi^*(|\varepsilon^{1-\gamma}\mathbf{f}|)$$

Extensions I

- $\mathbf{u}_\varepsilon = 0$ on $\partial\Omega_\varepsilon \Rightarrow \mathbf{u}_\varepsilon$ extended by zero in $\Omega \setminus \Omega_\varepsilon$ is bounded uniformly with respect to ε in $W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$
- Extension of p_ε is not obvious. We want this extension to be bounded in $L^{\varphi^*}(\Omega)$!

Extensions I

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- Extension of p_ε is not obvious. We want this extension to be bounded in $L^{\varphi^*}(\Omega)$!
  Extension solved by L. Tartar who introduced and applied the restriction operator.

Restriction operator

Lemma

There exists a restriction operator $R_\varepsilon : W_0^{1,\varphi}(\Omega) \to W_0^{1,\varphi}(\Omega_\varepsilon)$ *with properties:*

$$R_\varepsilon \text{ is linear}$$

$$R_\varepsilon(\mathbf{w}) = \mathbf{w} \qquad \text{for } \mathbf{w} \in W_0^{1,\varphi}(\Omega_\varepsilon) \text{ extended by } 0 \text{ on } \Omega \setminus \Omega_\varepsilon$$

$$\operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \Rightarrow \operatorname{div} R_\varepsilon(\mathbf{w}) = 0 \text{ in } \Omega_\varepsilon$$

$$\|R_\varepsilon(\mathbf{w})\|_{\varphi;\Omega_\varepsilon} \le c\,(\|\mathbf{w}\|_{\varphi;\Omega} + \varepsilon\|\nabla \mathbf{w}\|_{\varphi;\Omega})$$

$$\|\nabla R_\varepsilon(\mathbf{w})\|_{\varphi;\Omega_\varepsilon} \le c\left(\frac{1}{\varepsilon}\|\mathbf{w}\|_{\varphi;\Omega} + \|\nabla \mathbf{w}\|_{\varphi;\Omega}\right)$$

Extensions II

pressure reconstruction

$$\langle \nabla p_\varepsilon, \mathbf{v} \rangle_{(W_0^{1,\varphi}(\Omega_\varepsilon))^*, W_0^{1,\varphi}(\Omega_\varepsilon)} = \varepsilon^\gamma \int_{\Omega_\varepsilon} \varphi'(|D\mathbf{u}_\varepsilon|) \frac{D\mathbf{u}_\varepsilon}{|D\mathbf{u}_\varepsilon|} D\mathbf{v} + \int_{\Omega_\varepsilon} \mathbf{f}\mathbf{v}.$$

define $G_\varepsilon \in (W_0^{1,\varphi}(\Omega))^*$

$$\langle G_\varepsilon, \mathbf{v} \rangle_{(W_0^{1,\varphi}(\Omega))^*, W_0^{1,\varphi}(\Omega)} = \langle \nabla p_\varepsilon, R_\varepsilon \mathbf{v} \rangle_{(W_0^{1,\varphi}(\Omega_\varepsilon))^*, W_0^{1,\varphi}(\Omega_\varepsilon)} .$$

$\operatorname{div} \mathbf{v} = 0$ in $\Omega \Rightarrow \operatorname{div} R_\varepsilon \mathbf{v} = 0$ in $\Omega_\varepsilon \Rightarrow \langle G_\varepsilon, \mathbf{v} \rangle = 0$
$\Rightarrow G_\varepsilon = \nabla \overline{p_\varepsilon}, \ \overline{p_\varepsilon} \in L^{\varphi^*}(\Omega)$

Pressure estimates

$$\langle \nabla \overline{p_\varepsilon}, \mathbf{v} \rangle_{(W_0^{1,\varphi}(\Omega))^*, W_0^{1,\varphi}(\Omega)}$$
$$\leq 2\varepsilon^\gamma \|\varphi'(|D\mathbf{u}_\varepsilon|)\|_{\varphi^*,\Omega_\varepsilon} \|DR_\varepsilon \mathbf{v}\|_{\varphi,\Omega_\varepsilon} + \|\mathbf{f}\|_{\varphi^*,\Omega_\varepsilon} \|R_\varepsilon \mathbf{v}\|_{\varphi,\Omega_\varepsilon}$$
$$\leq c(\varepsilon^{\gamma-1} + 1)\left(\|\mathbf{v}\|_{\varphi,\Omega} + \varepsilon \|\nabla \mathbf{v}\|_{\varphi,\Omega}\right)$$

$$\Rightarrow \|\nabla \overline{p_\varepsilon}\|_{(W_0^{1,\varphi}(\Omega))^*} \leq c \text{ if } \gamma \geq 1$$

$$\|q\|_{\varphi^*,\Omega} \leq c \|\nabla q\|_{(W_0^{1,\varphi}(\Omega))^*}$$
$$\Rightarrow \|\overline{p_\varepsilon}\|_{\varphi^*,\Omega} \leq c$$

Weak convergencies

$$\left.\begin{array}{l}\|D\mathbf{u}_\varepsilon\|_{\varphi,\Omega} \le c \xRightarrow{\text{Korn I}} \|\nabla\mathbf{u}_\varepsilon\|_{\varphi,\Omega} \le c \\ \|D\mathbf{u}_\varepsilon\|_{\varphi,\Omega} \le c \xRightarrow{\text{Korn II}} \varepsilon^{-1}\|\mathbf{u}_\varepsilon\|_{\varphi,\Omega} \le c\end{array}\right\} \Rightarrow \exists \mathbf{u}_0, \{\mathbf{u}_{\varepsilon_k}\}$$

- $\varepsilon^{-1}\mathbf{u}_\varepsilon \xrightharpoonup{2-s} \mathbf{u}_0$
- $\nabla_x \mathbf{u}_\varepsilon \xrightharpoonup{2-s} \nabla_y \mathbf{u}_0$
- $D_x \mathbf{u}_\varepsilon \xrightharpoonup{2-s} D_y \mathbf{u}_0$

$$\left.\begin{array}{r}\|\nabla\overline{p_\varepsilon}\|_{(W_0^{1,\varphi}(\Omega))^*} \le c \\ \|\overline{p_\varepsilon}\|_{\varphi^*,\Omega} \le c\end{array}\right\} \Rightarrow \exists p_0, \{\overline{p_{\varepsilon_k}}\}$$

- $\nabla\overline{p_{\varepsilon_k}} \rightharpoonup \nabla p_0$ in $(W_0^{1,\varphi}(\Omega))^*$
- $\overline{p_{\varepsilon_k}} \rightharpoonup p_0$ in $L^{\varphi^*}(\Omega)$

Strong convergence of the pressure I

Proposition

Let X be a reflexive Banach space. Let $\{f_n\} \subset X^*$ be such that $f_n \xrightharpoonup{n\to\infty} f$ in X^* and for any $\{w_n\} \subset X$ such that $w_n \xrightharpoonup{n\to\infty} w$ the following convergence holds true

$$\langle f_n, w_n \rangle \xrightarrow{n\to\infty} \langle f, w \rangle \,.$$

Then $f_n \xrightarrow{n\to\infty} f$ in X^*.

Strong convergence of the pressure II

Need

$$\langle \nabla \overline{p_{\varepsilon_k}}, \mathbf{w}_{\varepsilon_k} \rangle \xrightarrow{\varepsilon_k \to 0} \langle \nabla p_0, \mathbf{w} \rangle$$

for any $\mathbf{w}_{\varepsilon_k} \xrightarrow[\rightharpoonup]{\varepsilon_k \to 0} \mathbf{w}$ in $W_0^{1,\varphi}(\Omega)$

$$| \langle \nabla \overline{p_{\varepsilon_k}}, \mathbf{w}_{\varepsilon_k} \rangle - \langle \nabla p_0, \mathbf{w} \rangle | \leq | \langle \nabla \overline{p_{\varepsilon_k}}, \mathbf{w}_{\varepsilon_k} - \mathbf{w} \rangle | + | \langle \nabla \overline{p_{\varepsilon_k}} - \nabla p_0, \mathbf{w} \rangle |$$
$$=: I_{\varepsilon_k} + II_{\varepsilon_k}$$

$$I_{\varepsilon_k} \leq c \|\mathbf{w}_{\varepsilon_k} - \mathbf{w}\|_{\varphi,\Omega} + \varepsilon_k \|\nabla(\mathbf{w}_{\varepsilon_k} - \mathbf{w})\|_{\varphi,\Omega}$$

$$I_{\varepsilon_k} \xrightarrow{\varepsilon_k \to 0} 0, \;\; II_{\varepsilon_k} \xrightarrow{\varepsilon_k \to 0} 0$$

- $\nabla \overline{p_{\varepsilon_k}} \to \nabla p_0$ in $(W_0^{1,\varphi}(\Omega))^*$
- $\overline{p_{\varepsilon_k}} \to p_0$ in $L^{\varphi^*}(\Omega)$

Sketch of the proof I

$$\mathbf{v}_{\varepsilon_k} := \varepsilon_k^{-1}\mathbf{u}_{\varepsilon_k}$$
$$\boldsymbol{\Psi}_{\varepsilon_k}(x) := \Psi\left(x, \frac{x}{\varepsilon_k}\right)$$

$$\begin{aligned}
\int_\Omega \varphi(|\varepsilon_k D\boldsymbol{\Psi}_{\varepsilon_k})|) - \int_\Omega \varphi(|\varepsilon_k D\mathbf{v}_{\varepsilon_k}|) \\
&\geq \int_\Omega \varphi'(|\varepsilon_k D\mathbf{v}_{\varepsilon_k}|)\frac{D\mathbf{v}_{\varepsilon_k}}{|D\mathbf{v}_{\varepsilon_k}|}\varepsilon_k D(\boldsymbol{\Psi}_{\varepsilon_k} - \mathbf{v}_{\varepsilon_k}) \\
&= \int_\Omega \mathbf{f}(\boldsymbol{\Psi}_{\varepsilon_k} - \mathbf{v}_{\varepsilon_k}) + \int_\Omega \overline{p_{\varepsilon_k}}\,\mathrm{div}(\boldsymbol{\Psi}_{\varepsilon_k} - \mathbf{v}_{\varepsilon_k})
\end{aligned}$$

Sketch of the proof II

$$\int_\Omega \int_Y \varphi(|D_y \boldsymbol{\Psi}(x,y))|) - \int_\Omega \varphi(|D_y \mathbf{u}_0(x,y)|)$$
$$\geq \int_\Omega \int_Y \mathbf{f}(\boldsymbol{\Psi}(x,y) - \mathbf{u}_0(x,y)) + \int_\Omega \int_Y p_0(x,y)\, \mathrm{div}_x\, \boldsymbol{\Psi}(x,y)$$

$\lambda > 0,\ w \in L^\varphi(\Omega),\ \mathbf{z} \in C^1_{per}(Y), \mathrm{div}_y\, \mathbf{z} = 0$ in Y
Minty's trick with $\boldsymbol{\Psi} = \mathbf{u}_0 + \lambda w \mathbf{z}$

$$\left\langle \int_Y \varphi'(|D_y \mathbf{u}_0)|) \frac{D_y \mathbf{u}_0}{|D_y \mathbf{u}_0|} D_y \mathbf{z} - \int_Y (\mathbf{f} - \nabla p_0)\, z, w \right\rangle_{L^{\varphi^*}(\Omega), L^\varphi(\Omega)} = 0$$

i.e. for a.e. $x \in \Omega$

$$\int_Y \varphi'(|D_y \mathbf{u}_0)|) \frac{D_y \mathbf{u}_0}{|D_y \mathbf{u}_0|} D_y \mathbf{z} - \int_Y (\mathbf{f} - \nabla p_0)\, z = 0$$

Natural questions

Q_1 What can we expect in the stationary Navier-Stokes case?

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Q_1 What can we expect in the stationary Navier-Stokes case?
Q_2 How to proceed in the evolutionary case?
Q_3 How to treat the case $\gamma \neq 1$?

Thank you for attention!