Conditional regularity for p-parabolic systems with critical right hand side

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System of equations

We consider the system of nonlinear parabolic equations

$$\mathbf{u}_t - \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \mathbf{B}(\,\cdot\,, \mathbf{u}, \nabla \mathbf{u}),$$

for a vector $\mathbf{u} = (u^1, \dots u^m)$ of functions $\mathbf{u}\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and $p > 2$

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Growth condition

We assume that the functions $\mathbf{B} \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfy following growth condition

$$|\mathbf{B}(x, \mathbf{u}, \nabla \mathbf{u})| \leq \Lambda |\nabla \mathbf{u}|^p.$$

Example (p-harmonic flow)

The p-harmonic heat flow with values in a Riemannian manifold $\mathcal{N}$

$$\mathbf{u}_t - \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = |\nabla \mathbf{u}|^{p-2} A(\mathbf{u})(\nabla \mathbf{u}, \nabla \mathbf{u}),$$

where $A(\mathbf{u})(x)$ is the second fundamental form of the manifold at $\mathbf{u}(x)$ for any $x \in \Omega$.

Example ($\mathcal{N} = \mathbb{S}^d$)

$$\mathbf{u}_t - \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = |\nabla \mathbf{u}|^{p}\mathbf{u}$$

Case $p = 2$

Eells, Sampson ('64)	$m = 2$, $\mathrm{curv}\,\mathcal{N} \leq 0$
Struwe ('85, '88); Struwe, Chen ('89)	weak solutions, $\mathcal{N}$ – arbitrary
Coron, Ghidaglia ('89)	blow-up for $\mathcal{N} = \mathbb{S}^n$

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Case $p > 2$

Hungerbühler ('96)	$\mathcal{N}$ – homogeneous space
Hungerbühler ('97)	conformal case $p = m$
Misawa ('98)	$\mathrm{curv}\,\mathcal{N} \leq 0$
Misawa ('02)	partial regularity under small image condition

Proposition (L^q estimates)

Assume

- $\mathbf{u} \in L^p((0, T], W^{2,p}(\Omega))$ *is a weak solution;*
- $p < q < \infty$*;*
- $\|\mathbf{u}\|_{L^\infty((0,T],BMO(\Omega))} < \varepsilon$ *for a constant* $\varepsilon = \varepsilon(\Omega, n, p, q, \Lambda)$

Then $\nabla \mathbf{u} \in L^q_{loc}((0, T] \times \Omega)$.

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Proposition (L^∞ estimates)

Assume

- $\mathbf{u} \in L^p((0,T], W^{2,p}(\Omega))$ *is a weak solution;*
- $\vartheta_0 > \frac{p+m}{2}$;

Then

$$\|\nabla \mathbf{u}\|_{L^\infty(Q_{R/2})} \leq C \left(1 + \int_{Q_R} |\nabla u|^{2\vartheta_0} \right)^{\frac{1}{\vartheta_0 - \frac{p+n}{2}}}$$

whenever the right hand side is finite.

Corollary

Assume

- $\mathbf{u} \in L^p((0,T], W^{2,p}(\Omega))$;
- $\|\mathbf{u}\|_{L^\infty((0,T],BMO(\Omega))} < \varepsilon$ *for a constant* $\varepsilon = \varepsilon(\Omega, n, p, \Lambda)$;

Then $\mathbf{u} \in C^{1,\alpha}_{loc}((0,T] \times \Omega)$

see DiBenedetto's book.

$$-\mathrm{div}\,(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}) = B(\,\cdot\,, \mathbf{u}, \nabla \mathbf{u})$$

Theorem (T.Riviére, P.Strzelecki)

Let

- $\mathbf{u} \in W^{2,p}(\Omega, \mathbb{R}^n) \cap BMO(\Omega, \mathbb{R}^n)$ *be a weak solution of the above equation;*
- $\max\left\{ \|\mathbf{u}\|_{BMO(B_R)}, \left(R^p ⨍_{B_R} |\nabla \mathbf{u}|^p \, dx \right)^{1/p} \right\} < \varepsilon_1$ *for a constant* $\varepsilon_1 = \varepsilon_1(m, n, p, \Lambda)$*;*

Then $|\nabla \mathbf{u}| \in L^\infty(B_{R/4})$ *and*

$$\operatorname*{ess\,max}_{B_{R/4}} |\nabla \mathbf{u}| \leq \frac{C(m,n,p,\Lambda)}{R} \left(R^p ⨍_{B_R} |\nabla \mathbf{u}|^p \, dx \right)^{1/p}.$$

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Theorem

Let

- $\Omega \subset \mathbb{R}^m$, $\mathcal{N}$ *be a d-dimensional compact Riemannian manifold isometrically embedded in* $\mathbb{R}^n$*;*
- $2 < p \leq \frac{n}{2}$*;*
- $\mathbf{u} \in W^{2,p}(\Omega, \mathcal{N})$ *be a p-harmonic map;*

Then $\mathbf{u}$ *is locally Lipschitz on* $V \subset \Omega,\ V = \mathrm{Int}\, V$ *s.t.* $\mathcal{H}^{n-2p}(\Omega \setminus V) = 0$*.*
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Main tool: interpolation inequality

Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \geq 0$, $s \geq 2$, $u \in W^{2,1}_{\mathrm{loc}} \cap BMO$. Then

$$\int \psi^{s+2} |\nabla \mathbf{u}|^{s+2}\, dx \leq C s^2 \|\mathbf{u}\|^2_{BMO} \Big\{ \int \psi^{s+2} |\nabla \mathbf{u}|^{s-2} |\nabla^2 \mathbf{u}|^2\, dx + \|\nabla \psi\|^2_{L^\infty} \int \psi^s |\nabla \mathbf{u}|^s\, dx \Big\}.$$

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Parabolic case

$$\int_0^T\int_{\mathbb{R}^n}\psi^{s+2}|\nabla \mathbf{u}|^{s+2} \leq Cs^2\|\mathbf{u}\|^2_{L^\infty(BMO)} \cdot\left\{\int_0^T\int_{\mathbb{R}^n}\psi^{s+2}|\nabla \mathbf{u}|^{s-2}|\nabla^2\mathbf{u}|^2 + \|\nabla\psi\|^2_{L^\infty(L^\infty)}\int_0^T\int_{\mathbb{R}^n}\psi^s|\nabla \mathbf{u}|^s\right\}$$

For $\mathbf{u} \in L^\infty([0,T], W^{2,1}_{\mathrm{loc}}(\mathbb{R}^n)) \cap L^\infty([0,T], BMO(\mathbb{R}^n))$.

Sketch of the proof: Caccioppoli inequality

$$\mathbf{u}_t - \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = B(\,\cdot\,, \mathbf{u}, \nabla \mathbf{u})$$

- Differentiate both sides with respect to x_j;
- Test with $\varphi_{ij} = \zeta^2 |\nabla \mathbf{u}|^{2\alpha} u^i_{x_j}$, for $\alpha \geq 0$, $\zeta \in C_c^\infty((0,T)\times\Omega)$, $\zeta \geq 0$.

We obtain for $\mathbf{w} = |\nabla \mathbf{u}|^2$

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$$\frac{1}{2\alpha+2}\operatorname*{ess\,sup}_{t\in[0,T]}\int_\Omega \zeta(t,\cdot)^2 \mathbf{w}^{\alpha+1} + \frac{p-2+\alpha}{8}\int_0^T\int_\Omega \zeta^2 \mathbf{w}^{\frac{p}{2}+\alpha-2}|\nabla \mathbf{w}|^2$$
$$+\frac{1}{2}\int_0^T\int_\Omega \zeta^2 \mathbf{w}^{\frac{p}{2}+\alpha-1}|\nabla^2 \mathbf{u}|^2 \leq \left(\frac{2(p-1)^2}{p-2+2\alpha}+\frac{1}{2}\right)\int_0^T\int_\Omega |\nabla\zeta|^2 \mathbf{w}^{\frac{p}{2}+\alpha}$$
$$+\frac{1}{\alpha+1}\int_0^T\int_\Omega \zeta\zeta_t \mathbf{w}^{\alpha+1} + C_1(p+\alpha)\int_0^T\int_\Omega \zeta^2 \mathbf{w}^{\frac{p}{2}+\alpha+1},$$

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Sketch of the proof: smallness condition

To estimate the nasty term we assume

$$C_1 C_2 (p+2\alpha)^3 \|\mathbf{u}\|^2_{L^\infty(BMO)} = \frac{1}{2} - \delta < \frac{1}{2}$$

And by interpolation inequality

$$C_1(p+\alpha)\int_0^T\int_\Omega \zeta^2 \mathbf{w}^{\frac{p}{2}+\alpha+1} \le \left(\frac{1}{2}-\delta\right)\|\nabla\psi\|^2_{L^\infty(L^\infty)}\int_0^T\int_\Omega \psi^{p+2\alpha}\mathbf{w}^{\frac{p}{2}+\alpha} + \left(\frac{1}{2}-\delta\right)\int_0^T\int_\Omega \zeta^2 \mathbf{w}^{\frac{p}{2}+\alpha-1}|\nabla^2\mathbf{u}|^2,$$

where $\zeta^2 = \psi^{p+2\alpha+2}$ is a proper choice of a cut-off function.

Sketch of the proof: Parabolic embedding

$$\int_0^T \int_{B_r} \mathbf{w}^{\frac{p+2\alpha}{2}+\frac{2}{m}(\alpha+1)} \leq C(m) \operatorname*{ess\,sup}_{t\in[0,T]} \left(\int_{B_r} \mathbf{w}(t,\cdot)^{\alpha+1} \right)^{\frac{2}{m}} \cdot \left(\int_0^T \int_{B_r} |\nabla \mathbf{w}^{\frac{p+2\alpha}{4}}|^2 + r^{-2} \int_0^T \int_{B_r} |\mathbf{w}|^{\frac{p+2\alpha}{2}} \right)$$

Sketch of the proof: Parabolic embedding

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Apply

- Caccioppoli inequality
- Nasty term estimate

Sketch of the proof: Parabolic embedding

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Apply

- Caccioppoli inequality
- Nasty term estimate

Finally

$$\left(\iint_{Q_{R/2}} \mathbf{w}^{\frac{p+2}{2}\alpha+\frac{2}{m}(\alpha+1)} \right)^{\frac{1}{\kappa}} \leq C(m,n,R,p)(1+\alpha)^3 \left(\iint_{Q_R} \mathbf{w}^{\frac{p+2\alpha}{2}} + \left(\iint_{Q_R} \mathbf{w}^{\frac{p+2\alpha}{2}} \right)^{\frac{2\alpha+2}{2\alpha+p}} \right)$$

Sketch of the proof: L^∞ estimates

By

- parabolic embedding
- Caccioppoli inequality

We obtain

$$\iint_{Q_{\nu+1}} \mathbf{w}^{\frac{p+2\alpha_\nu}{2}+\frac{2}{m}(\alpha_\nu+1)} \leq C8^{\nu(1+2/m)} \left(1+\iint_{Q_\nu} \mathbf{w}^{\frac{p+2\alpha_\nu+2}{2}}\right)^{1+\frac{2}{m}},$$

Thus

$$\iint_{Q_{\nu+1}} \mathbf{w}^{\vartheta_{\nu+1}} \leq C8^{\nu\mu} \left(1+\iint_{Q_\nu} \mathbf{w}^{\vartheta_\nu}\right)^{\mu}$$

Notation

$$I_\nu = \iint_{Q_\nu} \mathbf{w}^{\vartheta_\nu}$$

Sketch of the proof: L^∞ estimates

Which is precisely

$$I_{\nu+1} \leq C^\nu \left(1 + I_\nu^\mu\right)$$

$$I_\nu \leq C^{b_\nu} \left(1 + I_0^{\mu^\nu}\right)$$

$$\|\nabla \mathbf{u}\|_{L^\infty(Q_{R/2})} = \sup_{\nu \in \mathbb{N}} I_\nu^{1/\vartheta_\nu} \leq C^B (1 + I_0^A)$$

□

Thank you for your attention!