On the existence of weak solution to the coupled fluid-structure interaction problem for non-Newtonian shear-dependent fluid

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Telč, 2014

Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

Introduction

The motion of several rigid bodies in viscous fluids

Desjardins, Esteban 99, 2000 *the existence of local in time solutions for incompressible newtonian fluids - up to the first collision in* R^2 *or "up to the blow up of the velocity gradient in a certain Sobolev norm " in the case* $\Omega \subset R^3$

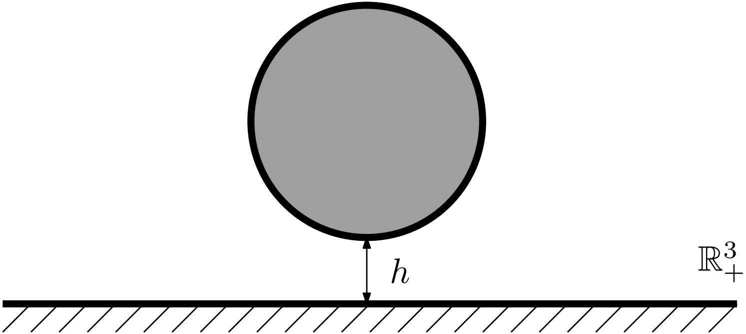
Conca, Starovoitov, Tucsnak 2000

Gunzburger, Lee, Seregin, 2000
the existence results "up to the first collisions"

San Martin, Starovoitov, Tucsnak, 2002
2D case , possible collisions must be "smooth" it means with zero relative velocities

Starovoitov, 2005

- *nonuniqueness of a solution to the problem on motion of a rigid body in a viscous incompressible fluid*
- *If bodies are of the class* $C^{1,1}$ *and velocity field belong to* $L^\infty(0,T,L^2(\Omega)) \cap L^1(0,T,K_p(S,\Omega))$ *then if* $p \geq n+1$ *then there are no collisions*



- **Hesla, 2005**
- **Hillairet, 2006**
  2D - absence of collisions in viscous fluids
- **Hillairet, Takahashi** 2009 extend result of no collision(falling body in the half space)
- **Varet, Hillairet** 2013 -existence of weak solution with slip (local)

Elastic body or boundary

- *Desjardins, Esteban, Grandmont, Le Tallec* (2001)
- *Boulakia* (2007)
- *Grandmont* (2008)
- *S. Canic, B. Muha* 2013
- *D. Lengeler, M. Růžička*- Koiter Shell
- B. da Veiga, 2004
- Guidoboni, Guidorzi, Padula,2011
- Guidorzi, Padula, Plotnikov, 2008
- Surulescu, 2007
- Zauskova, Filo,2010
- D. Coutand, S. Shkoller - elastic bodies(2007, 2010,..)

Equations of motion

Two dimensional non-Newtonian flow

$$
\begin{aligned}
\rho \partial_t \mathbf{v} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} - \operatorname{div} [2\mu(|e(\mathbf{v})|) e(\mathbf{v})] + \nabla \pi &= 0 \qquad (1) \\
\operatorname{div} \mathbf{v} &= 0
\end{aligned}
$$

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the symmetric deformation tensor

$$e(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

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typical example

$$\mu(|e(\mathbf{v})|) = \mu(1 + |e(\mathbf{v})|^2)^{\frac{p-2}{2}} \quad p > 1, \qquad (2)$$

$p > 2$

Assumption on shear - dependent model

Assumptions : A potential $\mathcal{U} \in C^2(R^{2\times 2})$ of shear stress tensor τ: $1 < p < \infty,\ C_1,\ C_2 > 0$ we have for all $\eta,\ \xi \in R^{2\times 2}_{sym}$ and $i,\ j,\ k,\ l \in \{1,2\}$,

$$\frac{\partial \mathcal{U}(\eta)}{\partial \eta_{ij}} = \tau_{ij}(\eta) \tag{3}$$

$$\mathcal{U}(\mathbf{0}) = \frac{\partial \mathcal{U}(\mathbf{0})}{\partial \eta_{ij}} = 0 \tag{4}$$

$$\frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{mn} \partial \eta_{rs}} \xi_{mn} \xi_{rs} \geq C_1\ (1 + |\eta|)^{p-2} |\xi|^2 \tag{5}$$

$$\left| \frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{ij} \partial \eta_{kl}} \right| \leq C_2 (1 + |\eta|)^{p-2}. \tag{6}$$

Description of domain

$$\Omega(\eta(t)) \equiv \{(x_1, x_2);\ 0 < x_1 < L,\ 0 < x_2 < R_0(x_1) + \eta(x_1, t)\},$$
$$0 < t < T$$

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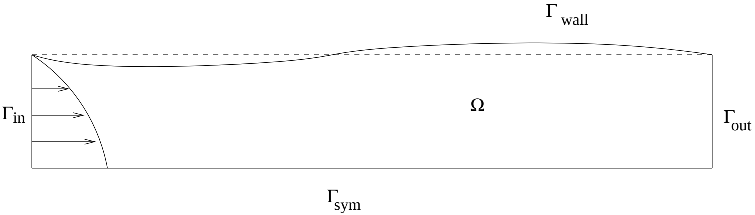
$\eta(x_1, t)$ the domain deformation.

Coupling the fluid and the geometry of the computational domain



Dirichlet boundary condition on the deformable part of the boundary $\Gamma_w(t)$

$$\mathbf{v}(x_1, R_0(x_1) + \eta(x_1,t), t) = \left(0, \frac{\partial \eta(x_1,t)}{\partial t}\right), \qquad (7)$$

$$\Gamma_w(t) = \{(x_1,x_2);\ x_2 = R_0(x_1) + \eta(x_1,t),\ x_1 \in (0,L)\}.$$

The generalized string equation

$$\tilde{E}\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right] = \qquad (8)$$
$$g\left(-\mathbf{T}_f^{ref} - P_w^{ref}\mathbf{I}\right)\mathbf{n}^{ref}\cdot\mathbf{e}_2 \quad \text{on } \Gamma_w^0.$$

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$[(\mathbf{T}_f^{ref} - P_w^{ref}\mathbf{I})\mathbf{n}^{ref}](x^{ref}) = [(\mathbf{T}_f - P_w\mathbf{I})\mathbf{n}](x),\ x\in\Gamma_w(t),\ x^{ref}\in\Gamma_w^0,$

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$\mathbf{T}_f = -\pi\mathbf{I} + 2\mu(|e(\mathbf{v})|)e(\mathbf{v}),$

$\mathbf{n}$ is the unit outward normal on $\Gamma_w(t)$, $\mathbf{n}|\mathbf{n}| = (-\partial_{x_1}(R_0+\eta), 1)^T$

and $\Gamma_w^0 = \Gamma_w(0)$.

The generalized string equation

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$\mathbf{n}$ is the unit outward normal on $\Gamma_w(t)$, $\mathbf{n}|\mathbf{n}| = (-\partial_{x_1}(R_0+\eta), 1)^T$

and $\Gamma_w^0 = \Gamma_w(0)$. $g = \frac{(R_0+\eta)\sqrt{1+(\partial_{x_1}(R_0+\eta))^2}}{R_0\sqrt{1+(\partial_{x_1}R_0)^2}}$

Boundary and initial conditions for (8)

$$\eta(0,t) = \eta(L,t) = 0 \quad \text{and} \quad \eta(x_1,0) = \frac{\partial \eta}{\partial t}(x_1,0) = 0,$$
$$\eta_{x_1}(0,t) = \eta_{x_1}(L,t) = 0, \tag{9}$$

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constants in (8)

$$\tilde{E} = \rho_w \hbar, \qquad a = \frac{|\sigma_z|}{\left(1 + \left(\frac{\partial R_0}{\partial x_1}\right)^2\right)^2}, \quad b = \frac{\mathcal{E}}{(R_0 + \eta) R_0}, \qquad c > 0,$$

$\mathcal{E}$ - the Young modulus,
$\hbar$ - the wall thickness,
ρ_w - the density of the vessel wall tissue,
$c = \gamma/(\rho_w \hbar)$,
γ positive constant.
$|\sigma_z| = G\kappa$ - the longitudinal stress,
$\kappa = 1$ - the Timoshenko's shear correction factor
G - the shear modulus
$G = \mathcal{E}/2(1+\sigma)$ with $\sigma = 1/2$ for incompressible materials.

Linearization of (8)

$$E\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \Big[-\mathbf{T}_f\mathbf{n}|\mathbf{n}|\cdot\mathbf{e}_2 - P_w\Big](x_1, R_0(x_1)+\eta(x_1,t),t), \tag{10}$$

$x_1 \in (0, L)$.

$E = \tilde{E}\sqrt{1+(\partial_{x_1}R_0)^2}$.

We assume that E is bounded.

Initial and boundary conditions(for 1)

inflow part of the boundary Γ_{in}

$$v_2(0, x_2, t) = 0, \tag{11}$$

$$\left(2\mu(|e(\mathbf{v})|)\frac{\partial v_1}{\partial x_1} - \pi + P_{in} - \frac{\rho}{2}|v_1|^2\right)(0, x_2, t) = 0 \tag{12}$$

for any $0 < x_2 < R_0(0)$, $0 < t < T$ and for a given function $P_{in} = P_{in}(x_2, t)$.

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for any $0 < x_2 < R_0(0)$, $0 < t < T$ and for a given function $P_{in} = P_{in}(x_2, t)$.

outflow part of the boundary Γ_{out}

$$v_2(L, x_2, t) = 0\ , \qquad (13)$$

$$\left(2\mu(|e(\mathbf{v})|)\frac{\partial v_1}{\partial x_1} - \pi + P_{out} - \frac{\rho}{2}|v_1|^2\right)(L, x_2, t) = 0 \qquad (14)$$

for any $0 < x_2 < R_0(L)$, $0 < t < T$ and for a given function $P_{out} = P_{out}(x_2, t)$.

Initial and boundary conditions(for 1)

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for any $0 < x_2 < R_0(L)$, $0 < t < T$ and for a given function $P_{out} = P_{out}(x_2, t)$.

Γ_c, the flow symmetry condition

$$v_2(x_1, 0, t) = 0 \ , \quad \mu(|e(\mathbf{v})|)\frac{\partial v_1}{\partial x_2}(x_1, 0, t) = 0 \tag{15}$$

for any $0 < x_1 < L$, $0 < t < T$.
The initial conditions

$$\mathbf{v}(x_1, x_2, 0) = \mathbf{0} \quad \text{for any } 0 < x_1 < L, \ 0 < x_2 < R_0(x_1). \tag{16}$$

Weak formulation

Definition 1.1 [Weak formulation] We say that $(\mathbf{v}, \eta)$ is a weak solution of (1)–(16) on $[0, T)$ if the following conditions hold

- $\mathbf{v} \in L^p(0, T; W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0, T; L^2(\Omega(\eta(t))))$,
- $\eta \in W^{1,\infty}(0, T; L^2(0, L)) \cap H^1(0, T; H_0^2(0, L))$,
- $\operatorname{div} \mathbf{v} = 0$ a.e. on $\Omega(\eta(t))$,
- $\mathbf{v}\big|_{\Gamma_w(t)} = (0, \eta_t)$ for a.e. $x \in \Gamma_w(t),\ t \in (0, T)$,
  $v_2\big|_{\Gamma_{in} \cup \Gamma_{out} \cup \Gamma_c} = 0$,

$$\int_0^T \int_{\Omega(\eta(t))} \left\{ -\rho \mathbf{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\mathbf{v})|) e(\mathbf{v}) e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i} \boldsymbol{\varphi}_j \right\}$$

$$+ \int_0^T \int_0^{R_0(L)} \left(P_{out} - \frac{\rho}{2} |v_1|^2 \right) \varphi_1(L, x_2, t) \, dx_2 \, dt \qquad (17)$$

$$- \int_0^T \int_0^{R_0(0)} \left(P_{in} - \frac{\rho}{2} |v_1|^2 \right) \varphi_1(0, x_2, t) \, dx_2 \; dt$$

$$+ \int_0^T \int_0^L P_w \varphi_2(x_1, R_0(x_1) + \eta(x_1, t), t) - a \frac{\partial^2 R_0}{\partial x_1^2} \xi \, dx_1 \, dt$$

$$+ \int_0^T \int_0^L -\frac{\partial \eta}{\partial t} \frac{\partial \xi}{\partial t} + c \, \frac{\partial^3 \eta}{\partial x_1^2 \partial t} \frac{\partial^2 \xi}{\partial x_1^2} + a \frac{\partial \eta}{\partial x_1} \frac{\partial \xi}{\partial x_1} + b\eta \, \xi \, dx_1 \, dt \; = \; 0$$

for every test functions

$$
\begin{aligned}
&\boldsymbol{\varphi}(x_1, x_2, t) \in H^1(0, T; W^{1,p}(\Omega(\eta(t)))) \text{ such that} \\
&\operatorname{div} \boldsymbol{\varphi} = 0 \text{ a.e on } \Omega(\eta(t)), \\
&\varphi_2\big|_{\Gamma_w(t)} \in H^1(0, T; H_0^2(\Gamma_w(t))),\ \varphi_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = \varphi_1\big|_{\Gamma_w(t)} = 0 \quad \text{and} \\
&\xi(x_1, t) = E\rho\, \varphi_2(x_1, R_0(x_1) + \eta(x_1, t), t).
\end{aligned} \tag{18}
$$

Main result: existence of a weak solution

Theorem (Main result: existence of a weak solution)

Let $p \geq 2$. Assume that the boundary data fulfill $P_{in} \in L^{p'}(0,T;L^2(0,R_0(0)))$, $P_{out} \in L^{p'}(0,T;L^2(0,R_0(L)))$, $P_w \in L^{p'}(0,T;L^2(0,L))$, $\frac{1}{p}+\frac{1}{p'}=1$. Furthermore, assume that the properties (3)–(6) for the viscous stress tensor hold. Then there exists a weak solution $(\mathbf{v},\eta)$ of the problem (1)-(16) such that

- $\mathbf{v} \in L^p(0,T; W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0,T; L^2(\Omega(\eta(t))))$,

  $\eta \in W^{1,\infty}(0,T; L^2(0,L)) \cap H^1(0,T; H_0^2(0,L))$,
- $\mathbf{v}\big|_{\Gamma_w(t)} = (0, \eta_t)$ for a.e. $x \in \Gamma_w(t),\ t \in (0,T)$,
  $v_2\big|_{\Gamma_{in} \cup \Gamma_{out} \cup \Gamma_c} = 0$,
- $\mathbf{v}$ satisfies the condition $\operatorname{div} \mathbf{v} = 0$ a.e on $\Omega(\eta(t))$ and (17) holds.

- approximation of the solenoidal spaces on a moving domain by the artificial compressibility approach: ε - approximation (26)
- splitting of the boundary conditions (7)–(8) by introducing the semi-pervious boundary: κ - approximation (23), (24)
- assuming a given, sufficiently smooth free boundary deformation $\delta(x_1, t)$ and actual radius $h(t) := R_0 + \delta(t)$ we transform the weak formulation on a time dependent domain $\Omega(h(t)) := \Omega(\delta(t))$ to a fixed reference domain $D = (0, L) \times (0, 1)$: h - approximation
- limiting process for $\varepsilon \to 0$, $\kappa \to \infty$ respectively.
- fixed point procedure for the domain deformation $\eta(x_1, t)$.

Formulation of the (κ, ε, h) - problem

We approximate the deformable boundary Γ_w by a given function $h = R_0 + \delta$, $\delta \in H^1(0, T; H_0^2(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L))$, $R_0(x_1) \in C^2[0, L]$ satisfying for all $x_1 \in [0, L]$

$$0 < \alpha \leq h(x_1, t) \leq \alpha^{-1}, \quad \left| \frac{\partial h(x_1, t)}{\partial x_1} \right| + \int_0^T \left| \frac{\partial h(x_1, t)}{\partial t} \right|^2 dt \leq K$$

$h(0, t) = R_0(0), \quad h(L, t) = R_0(L).$

We look for a solution $(\mathbf{v}, \pi, \eta)$ of the following problem

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \operatorname{div}[2\mu(|e(\mathbf{v})|)e(\mathbf{v})] - \nabla\pi \quad \text{in} \quad \Omega(h(t)), \quad (20)$$

and for all $x_1 \in (0, L)$, see (10), $0 < t < T$

$$-E\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1, t) = \quad (21)$$

$$\left[\mu(|e(\mathbf{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial h}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - \pi + P_w\right](\bar{x}, t),$$

$$\mathbf{v}(\bar{x}, t) = \left(0, \frac{\partial\eta}{\partial t}(x_1, t)\right), \quad (22)$$

$\bar{x} = (x_1, h(x_1, t))$.

The boundary condition (7)-(8), cf. (21)-(22), is splitted in the following way:

$$\left[\mu(|e(\mathbf{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right)\frac{\partial h}{\partial x_1}+2\frac{\partial v_2}{\partial x_2}\right\}-\pi+P_w\right](\bar{x},t) \quad (23)$$

$$-\frac{\rho}{2}v_2\Big(v_2(\bar{x},t)-\frac{\partial h}{\partial t}(x_1,t)\Big)=\rho\kappa\Big[\frac{\partial \eta}{\partial t}(x_1,t)-v_2(\bar{x},t)\Big]$$

and

$$-E\Big[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta+c\frac{\partial^5\eta}{\partial t\partial x_1^4}-a\frac{\partial^2 R_0}{\partial x_1^2}\Big](x_1,t) \qquad (24)$$

$$=\kappa\Big[\frac{\partial\eta}{\partial t}(x_1,t)-v_2(\bar{x},t)\Big] \qquad (25)$$

with $\kappa \gg 1$.

Artificial compressibility

$$\varepsilon \left(\frac{\partial \pi_\varepsilon}{\partial t} - \Delta \pi_\varepsilon \right) + \operatorname{div} \mathbf{v}_\varepsilon = 0 \quad \text{in } \Omega(h(t)), \acute{t} \in (0, T), \quad (26)$$

$$\frac{\partial \pi_\varepsilon}{\partial \mathbf{n}} = 0, \quad \text{on} \quad \partial\Omega(h(t)), t \in (0, T), \pi_\varepsilon(0) = 0 \text{ in } \Omega(h(0)), \ \varepsilon > 0.$$

we will reformulate it to a fixed rectangular domain. Set

$$\begin{aligned} \mathbf{u}(y_1, y_2, t) &\stackrel{\text{def}}{=} \mathbf{v}(y_1, h(y_1, t)y_2, t) \\ q(y_1, y_2, t) &\stackrel{\text{def}}{=} \rho^{-1}\pi(y_1, h(y_1, t)y_2, t) \\ \sigma(y_1, t) &\stackrel{\text{def}}{=} \frac{\partial \eta}{\partial t}(y_1, t) \end{aligned} \tag{27}$$

for $y \in D = \{(y_1, y_2);\ 0 < y_1 < L,\ 0 < y_2 < 1\}$, $0 < t < T$.

Definition of the following spaces

$$
\begin{aligned}
\mathbf{V} &\equiv \{\mathbf{w} \in W^{1,p}(D) : \; w_1 = 0 \text{ on } S_w \text{ and} \\
w_2 &= 0 \text{ on } S_{in} \cup S_{out} \cup S_c,
\end{aligned}
$$

(28)

$$
\mathrm{S}_w = \{(y_1, 1) : 0 < y_1 < L\}, \qquad S_{in} = \{(0, y_2) : 0 < y_2 < 1\}, (29)
$$

$$
\mathrm{S}_{out} = \{(L, y_2) : 0 < y_2 < 1\}, \qquad S_c = \{(y_1, 0) : 0 < y_1 < L\}.
$$

$$\mathrm{div}_h\mathbf{u} \stackrel{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2},$$

$$a_1(q,\phi) = \int_D \left\{ \left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_1} + \left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_2}\right\} dy, \qquad (30)$$

viscous term

$$((\mathbf{u}, \boldsymbol{\psi})) = \int_D h\tau_{ij}((\mathbf{u}))_{ij}(\boldsymbol{\psi})dy, \tag{31}$$

$$\tau_{ij}((\mathbf{u})) = 2\rho^{-1}\mu(|(\mathbf{u})|)_{ij}(\mathbf{u}), \qquad {}_{ij}(\mathbf{u}) = \frac{1}{2}(\hat{\partial}_i(u_j) + \hat{\partial}_j(u_i)),$$

$$\hat{\partial}_1 = \left(\frac{\partial}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_2}\right), \quad \hat{\partial}_2 = \frac{1}{h}\frac{\partial}{\partial y_2},$$

convective term

$$b(\mathbf{u}, , \boldsymbol{\psi}) = \int_D \left(hu_1\left(\frac{\partial}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_2}\right) + u_2\frac{\partial}{\partial y_2}\right) \cdot \boldsymbol{\psi} + \frac{h}{2} \cdot \boldsymbol{\psi}\, \mathrm{div}_h \mathbf{u}\, dy$$

$$-\frac{1}{2}\int_0^1 R_0 u_1 z_1 \psi_1 \left(L, y_2\right) dy_2 + \frac{1}{2}\int_0^1 R_0 u_1 z_1 \psi_1 \left(0, y_2\right) dy_2$$

$$-\frac{1}{2}\int_0^L u_2 z_2 \psi_2 \left(y_1, 1\right) dy_1.$$

Definition 2.1

[Weak solution of (κ, ε, k) - approximate problem]
Let $\mathbf{u} \in L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;L^2(D))$,
$q \in L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ and
$\sigma \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^2_0(0,L))$. A triple $\mathbf{w} = (\mathbf{u}, q, \sigma)$ is called a weak solution of the regularized problem (1)–(16) if the following equation holds

$$
\begin{aligned}
&-\int_0^T \left\langle \frac{\partial (h\mathbf{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \;= \qquad (33)\\
&\int_0^T \int_D \left(-\frac{\partial h}{\partial t} \frac{\partial (y_2 \mathbf{u})}{\partial y_2} \cdot \boldsymbol{\psi} + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\psi}) - h\, q\, \mathrm{div}_h \boldsymbol{\psi} \right) dy + ((\mathbf{u}, \boldsymbol{\psi}))dt \\
&+ \int_0^T \int_0^1 h(L,t) q_{out} \psi_1 \; (L, y_2, t) - h(0,t) q_{in} \psi_1 \; (0, y_2, t) \; dy_2 dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_0^L\left(q_w+\frac{1}{2}u_2\frac{\partial h}{\partial t}+\kappa\left(u_2-\sigma\right)\right)\psi_2\ (y_1,1,t)\,dy_1dt\\
&+\varepsilon\int_0^T\left\langle\frac{\partial(hq)}{\partial t},\phi\right\rangle dt \qquad (34)\\
&+\int_0^T\int_D\left(-\varepsilon\frac{\partial h}{\partial t}\frac{\partial(y_2q)}{\partial y_2}\phi+\varepsilon a_1(q,\phi)+\ h\,\mathrm{div}_h\mathbf{u}\ \phi\right)dy\,dt\\
&+\frac{\varepsilon}{2}\ \int_0^T\int_0^L\frac{\partial h}{\partial t}(y_1,t)q\phi(y_1,1,t)\,dy_1dt+
\end{aligned}
$$

$$
+\int_0^T\int_0^L\left(\frac{\partial\sigma}{\partial t}\xi + c\,\frac{\partial^2\sigma}{\partial y_1^2}\frac{\partial^2\xi}{\partial y_1^2} + a\frac{\partial}{\partial y_1}\int_0^t\sigma(y_1,s)ds\frac{\partial\xi}{\partial y_1}\right.
$$

$$
\left. + b\int_0^t\sigma(y_1,s)ds\,\xi - a\frac{\partial^2 R_0}{\partial y_1^2}\xi + \frac{\kappa}{E}(\sigma - u_2)\,\xi\right)(y_1,t)\,dy_1
$$

for every
$(\boldsymbol{\psi},\phi,\xi)\in H_0^1(0,T;\mathbf{V})\times L^2(0,T;H^1(D))\times L^2(0,T;H_0^2(0,L))$.

Existence of stationary solution

Approximate time derivatives first order backward finite differences

$$\frac{\partial(h\mathbf{u})}{\partial t} \approx \frac{h^i\mathbf{u}^i - h^{i-1}\mathbf{u}^{i-1}}{\Delta t}, \frac{\partial(hq)}{\partial t} \approx \frac{h^i q^i - h^{i-1}q^{i-1}}{\Delta t}, \frac{\partial \sigma}{\partial t} \approx \frac{\sigma^i - \sigma^{i-1}}{\Delta t},$$

- Existence of stationary problem
- Existence of unsteady problem

Problem with $\varepsilon = 0, \ \kappa = \infty$

First a priori estimate

$$\max_{0\le t\le T}\int_D h(t)\left(|\mathbf{u}_\kappa|^2+\varepsilon|q_\kappa|^2\right)(t)dy+\frac{E}{2}\int_0^L|\sigma_\kappa(t)|^2dy_1$$
$$+\int_0^T\int_D \delta|\nabla\mathbf{u}_\kappa|^p+\frac{2\alpha\varepsilon}{2+K^2}|\nabla q_\kappa|^2dy+E\int_0^L c\left|\frac{\partial^2\sigma_\kappa}{\partial y_1^2}\right|^2 dy_1\,dt$$
$$+\int_0^L\frac{aE}{2}\left|\int_0^t\frac{\partial\sigma_\kappa(s)}{\partial y_1}ds\right|^2+\frac{bE}{2}\left|\int_0^t\sigma_\kappa(s)s\right|^2dy_1$$
$$+\int_0^T\int_0^L 2\kappa\left|\sigma_\kappa-u_{2\kappa}\right|^2dy_1\,dt\le\tilde M\int_0^T\|q_{\partial D}\|^{p'}_{L^2(\partial D)}+c_1\left\|\frac{\partial^2R_0}{\partial y_1^2}\right\|^2_{L^2}$$

where $c_1=c_1(p,E,a,c),\ \tilde M=\tilde M(p,K,\alpha)$

Limiting process $\kappa = \varepsilon^{-1} \to \infty$

- the weak convergence of

$$(\mathbf{u}_\kappa, \sqrt{\varepsilon} q_\kappa, \sigma_\kappa) \rightharpoonup (\mathbf{u}, \tilde{q}, \sigma) \quad (36)$$
$$\text{in} \quad L^p(0,T;\mathbf{V}) \times L^2(0,T;H^1(D)) \times L^2(0,T;H^2(0,L))$$

- $$\mathrm{div}_h \mathbf{u} = 0 \qquad a.e. \quad \text{on} \quad (0,T) \times D.$$
- $\mathrm{div}_h \psi = 0$ a.e. on D
- to obtain the strong convergence $(\mathbf{u}_\kappa, \sigma_\kappa) \to (\mathbf{u}, \sigma)$ use the equicontinuity in time

$$\int_0^{T-\tau} \int_D |(h\mathbf{u}_\kappa)(t+\tau) - (h\mathbf{u}_\kappa)(t)|^2 \, dy \, dt$$
$$+ \int_0^{T-\tau} \varepsilon |(hq_\kappa)(t+\tau) - (hq_\kappa)(t)|^2 \, dy \, dt$$
$$+ \int_0^{T-\tau} \int_0^L |(h\sigma_\kappa)(t+\tau) - (h\sigma_\kappa)(t)|^2 dy_1 \, dt \le C(K,\alpha)\tau, \quad (37)$$

let us consider test functions $\boldsymbol{\psi} \in L^p(0,T;X), \boldsymbol{\psi}(T) = 0$

$$X = \{\boldsymbol{\psi}(t) \in \mathbf{V}_{div}\,;\ \boldsymbol{\psi}_2(t)\big|_{S_w} \in H_0^2(0,L), \}, \tag{38}$$
$$\mathbf{V}_{div} := \{f \in \mathbf{V},\ \mathrm{div}_h f = 0 \ a.e. \text{ on } D\},\ \text{cf. } (28)$$

- With this choice of test functions the boundary terms with κ are canceled.
- $\kappa \to \infty$ in Def. 2

$$
\begin{aligned}
\int_0^T \int_D \left\{ h\mathbf{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial (y_2 \mathbf{u})}{\partial y_2} \cdot \boldsymbol{\psi} \right\} dy = \\
\int_0^T \Bigg\{ \; ((\mathbf{u}, \boldsymbol{\psi}))_h \; + \; b_h(\mathbf{u}, \mathbf{u}, \boldsymbol{\psi}) \\
+ \int_0^1 h(L) q_{out}(y_2, t) \psi_1 (L, y_2, t) - h(0) q_{in}(y_2, t) \psi_1 (0, y_2, t) \\
+ \int_0^L \left(q_w + \frac{1}{2} \frac{\partial h}{\partial t} u_2 \right) \psi_2 (y_1, 1, t) \, dy_1 \\
+ \int_0^L -\sigma \frac{\partial \xi}{\partial t} + c \, \frac{\partial^2 \sigma}{\partial y_1^2} \frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1} \int_0^t \sigma(y_1, s) ds \frac{\partial \xi}{\partial y_1} \\
- a \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \int_0^t \sigma(y_1, s) ds \, \xi(y_1, t) \, dy_1 \; \Bigg\} \, dt
\end{aligned}
$$

Theorem (Existence of weak solution for $\varepsilon = 0$, $\kappa = \infty$)

Assume that $h \in H^1(0, T; H_0^2(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L))$ *satisfies (19). Let the boundary data fulfill* $q_{in}, q_{out} \in L^{p'}(0, T; L^2(0, 1))$, $q_w \in L^{p'}(0, T; L^2(0, L))$. *Furthermore, assume that the properties (3)–(6) for the viscous stress tensor hold. Then there exists a weak solution* $(\mathbf{v}, \eta)$ *of the problem (1)–(16), such that*

i) $(\mathbf{u},\eta)\in[L^p(0,T;\mathbf{V})\times H^1(0,T;H_0^2(0,L))]\cap[L^\infty(0,T;L^2(D))\times W^{1,\infty}(0,T;L^2(0,L))]$, where $\mathbf{u}$ is defined in (27),

ii) the time derivative $\bar{\partial}_t(h\mathbf{u})\in L^{p'}(0,T;X^*)$ for $p>2$ and $\bar{\partial}_t(h\mathbf{u})\in L^{p'}(0,T;X^*)\oplus L^{4/3}((0,T)\times D)$ for $p=2$,

$$\int_0^T\int_D\left\{h\mathbf{u}\cdot\frac{\partial\boldsymbol{\psi}}{\partial t}+\frac{\partial h}{\partial t}\frac{\partial(y_2\mathbf{u})}{\partial y_2}\cdot\boldsymbol{\psi}\right\}dy\,dt=-\int_0^T\left\langle\bar{\partial}_t(h\mathbf{u}),\boldsymbol{\psi}\right\rangle dt,$$

where $\bar{\partial}_t(h\mathbf{u})=\frac{\partial(h\mathbf{u})}{\partial t}-\frac{1}{h}\frac{\partial h}{\partial t}\frac{\partial(y_2h\mathbf{u})}{\partial y_2}=h\partial_t^y\mathbf{u}$ and $\boldsymbol{\psi}\in\mathcal{M}\cap H_0^1(0,T;X)$,

$$\begin{aligned}\mathcal{M}\ =\ &\{\omega\in L^p(0,T;X)\text{ for }p>2;\\ &\ \omega\in L^p(0,T;X)\cap L^4((0,T)\times D)\text{ for }p=2\}.\end{aligned}\tag{40}$$

iii) $\mathbf{v}$ satisfies the condition $\operatorname{div}\mathbf{v}=0$ a.e on $\Omega(h(t))$, $v_2(x_1,h(x_1,t),t)=\partial_t\eta(x_1,t)$ for a.e. $x_1\in(0,L),\ t\in(0,T)$

and the following integral identity holds

$$\int_0^T \int_{\Omega(h(t))} \Big\{ -\rho \mathbf{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\mathbf{v})|)e(\mathbf{v})e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i} \boldsymbol{\varphi}_j \Big\} dx$$

$$
\begin{aligned}
&+ \int_0^T \int_0^{R_0(L)} \left(P_{out} - \frac{\rho}{2}\left|v_1\right|^2\right) \varphi_1(L, x_2, t)\, dx_2\, dt \\
&- \int_0^T \int_0^{R_0(0)} \left(P_{in} - \frac{\rho}{2}\left|v_1\right|^2\right) \varphi_1(0, x_2, t)\, dx_2\, dt \\
&+ \int_0^T \int_0^{L} \left(P_w - \frac{\rho}{2} v_2 \left(v_2 - \frac{\partial h}{\partial t}\right)\right) \varphi_2(x_1, h(x_1, t), t)\, dx_1\, dt \\
&+ \int_0^T \int_0^{L} -\frac{\partial \eta}{\partial t}\frac{\partial \xi}{\partial t} + c\, \frac{\partial^3 \eta}{\partial x_1^2 \partial t}\frac{\partial^2 \xi}{\partial x_1^2} + a \frac{\partial \eta}{\partial x_1}\frac{\partial \xi}{\partial x_1} dx_1\, dt \\
&+ \int_0^T \int_0^{L} -a \frac{\partial^2 R_0}{\partial x_1^2}\xi + b\eta\, \xi\, dx_1\, dt \;=\; 0
\end{aligned}
$$

for every test functions

$$
\begin{aligned}
&\boldsymbol{\varphi}(x_1, x_2, t) = \boldsymbol{\psi}\left(x_1, \frac{x_2}{h(x_1, t)}, t\right) \quad \text{such that} \\
&\boldsymbol{\psi} \in H^1_0(0, T; \mathbf{V}), \ \ \psi_2\big|_{S_w} \in H^1_0(0, T; H^2_0(0, L)), \\
&\operatorname{div} \boldsymbol{\varphi} = 0 \quad \textit{a.e.} \text{ on } \Omega(h(t)), \\
&\textit{and} \quad \xi(x_1, t) = E\rho\, \varphi_2(x_1, h(x_1, t), t).
\end{aligned}
$$

Note that the structure equation is fulfilled in a slightly modified sense,

$$E\rho\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta + c\frac{\partial^5\eta}{\partial t\partial x_1^4} - a\frac{\partial^2 R_0}{\partial x_1^2}\right](x_1,t) = \left[-(\mathbf{T}_f + P_w\mathbf{I})\mathbf{n}|\mathbf{n}|\cdot\mathbf{e}_2 + \frac{\rho}{2}\partial_t\eta(\partial_t\eta - \partial_t h)\right](x_1,h(x_1,t),t)$$

a.e. on $(0,T)\times(0,L)$, compare (10).

Fixed point iterations

We have proved the existence of weak solution of the original problem in a domain given by a known deformation function δ, i.e. $h = R_0 + \delta$, $\delta \in H^1(0, T; H_0^2(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L))$, $R_0(x_1) \in C^2[0, L]$.

Aim: Proof of the existence of the weak solution of (17), which implies, that the domain deforms according to the function $\eta(x_1, t)$, i.e. $h = R_0 + \eta$.
This will be realized with the use of the Schauder fixed point theorem.

- The compactness argument based on the equicontinuity in time we obtain that bounded sequence sequence $(\mathbf{v}^{(k)}, \eta^{(k)})$ defined on $\Omega(\delta^{(k)})$ for some sequence $\delta^{(k)} \to \delta$ converges to the limit $(\mathbf{v}, \eta)$ defined on $\Omega(\delta)$.
- The Schauder fixed point argument implies, that the weak solution η is associated with the time dependent domain $\Omega(\eta)$.
- Finally we obtain the main result: existence of weak solution for a fully coupled fluid structure interaction problem (1)–(16).

$Y = H^1(0, T; L^2(0, L))$. For each test function $\psi \in L^p(0, T; X)$, $\psi(T) = 0$, recalling (38), and for any $h = R_0 + \delta \in Y$ we construct solutions $\{\mathbf{u}, \eta\}_{k=1}^{\infty}$ of the following problem defined on the reference domain D, $\sigma = \partial_t \eta$

$$
\begin{aligned}
&-\int_0^T \left\langle \bar{\partial}_t(h\mathbf{u}), \boldsymbol{\psi} \right\rangle \, dt \qquad (41)\\
&= \int_0^T \Big\{ \, ((\mathbf{u}, \boldsymbol{\psi}))_h \, + \, b_h(\mathbf{u}, \mathbf{u}, \boldsymbol{\psi}) \\
&\quad + \int_0^1 h(L) q_{out}(y_2, t) \psi_1 \, (L, y_2, t) dy_2 \\
&\quad - \int_0^1 -h(0) q_{in}(y_2, t) \psi_1 \, (0, y_2, t) \, dy_2 \\
&\quad + \int_0^L \Big(q_w + \frac{1}{2} \frac{\partial h}{\partial t} \sigma \Big) \psi_2 \, (y_1, 1, t) \, dy_1 \\
&\quad + \langle \partial_t \sigma, \xi \rangle + \int_0^L c \, \frac{\partial^2 \sigma}{\partial y_1^2} \frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1} \int_0^t \sigma(y_1, s) ds \frac{\partial \xi}{\partial y_1} \\
&\qquad\qquad - a \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \int_0^t \sigma(y_1, s) ds \, \xi(y_1, t) \, dy_1 \, \Big\} \, dt \; .
\end{aligned}
$$

$$\mathcal{F} : B_{\alpha,K} \to Y;$$

$$\mathcal{F}(\delta) = \eta, \delta = h - R_0, \text{where}$$

$$B_{\alpha,K} = \Big\{(\delta \in Y;\ \|\delta\|_Y \leq C_{\alpha,K},\ 0 < \alpha \leq R_0(y_1) + \delta(y_1,t) \leq \alpha^{-1};$$

$$\left|\frac{\partial\delta(y_1,t)}{\partial y_1}\right| \leq K, \quad \delta(y_1,0) = 0,\ \forall y_1 \in [0,L],\ \forall t \in [0,T],$$

$$\int_0^T \left|\frac{\partial\delta y_1,t)}{\partial t}\right|^2 dt \leq K,\ \forall y_1 \in [0,L]\Big\},$$

- $\mathcal{F}(B_{\alpha,K}) \subset B_{\alpha,K}$.
- $\mathcal{F}(\delta) = \eta$ is relatively compact in Y. on the equicontinuity in time
- $\mathcal{F}$ is continuous with respect to the strong topology in Y. We have to prove that for any convergent subsequences $\delta^k \in (B_{\alpha,K}), \delta^{(k)} \to \delta$ in Y

$$\mathcal{F}(\delta^{(k)}) = \eta^{(k)} \to \mathcal{F}(\delta) = \eta.$$

Limiting proces

we can construct sufficiently smooth test functions $\tilde{\boldsymbol{\psi}}(y,t) = \tilde{\boldsymbol{\varphi}}(x,t)$, which are independent on k and divergence free in $\Omega(h)$ (i.e. $\mathrm{div}_h \tilde{\boldsymbol{\psi}} = 0$). They are also well defined on infinitely many approximate domains $\Omega(h)$ and dense in the space of admissible test functions $L^p(0,T;X)$, cf. (38). Such a test functions $\tilde{\boldsymbol{\varphi}}$ can be constructed on $(0,T) \times B_M$ as algebraic sum,

$$\tilde{\varphi} = \varphi_0 + \varphi_1,$$

φ_0 is a smooth function with compact support in $\Omega(h)$,

$\mathrm{div}\varphi_0 = 0$ on $\Omega(h)$ and φ_0 is extended by 0 to $(0, T) \times B_M$.

$\xi \in H^1(0, T; H_0^2(0, L))$ we define $\varphi_1 \overset{\text{def}}{=} (0, \xi(x_1)/E)$ on $B_M \backslash B_\alpha$,
$B_\alpha = (0, L) \times (0, \alpha) \in R^2$, the constant E comes from (18)

$\mathrm{div}\boldsymbol{\varphi}_1 = 0$ on $B_M \backslash B_\alpha$

$\boldsymbol{\varphi}_1$ such that

$\int_{\partial B_\alpha} \boldsymbol{\varphi}_1 \cdot n = \int_0^\alpha \boldsymbol{\varphi}_1^1(L, x_2, t) - \boldsymbol{\varphi}_1^1(0, x_2, t) dx_1 + \int_0^L \frac{\xi}{E}(x_1, t) dx_2 = 0$
can be extended into B_α by a divergence-free extension,

$\tilde{\boldsymbol{\psi}}(y, t) = \tilde{\boldsymbol{\psi}}(x_1, \frac{x_2}{h^{(k)}(x_1,t)}, t) = \tilde{\boldsymbol{\varphi}}(x, t),\ x \in \Omega(h^{(k)}),\ y \in D,$

the set of admissible test functions $\boldsymbol{\psi}^{(k)}$ by transformation of $\tilde{\boldsymbol{\varphi}}$ from $\Omega(h^{(k)})$ into D

test functions $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}}$, which are independent on k and smooth enough.

$$\boldsymbol{\psi}^{(k)} : D \to R^2; \quad \mathrm{div}_{h^{(k)}} \boldsymbol{\psi}^{(k)} = 0, \quad E\psi_2^{(k)}(y_1, 1, t) = \xi(y_1, t), \quad \text{and}$$

$$\left.\begin{array}{l} \boldsymbol{\psi}^{(k)} \to \tilde{\boldsymbol{\psi}}, \\ \hat{e}(\boldsymbol{\psi}^{(k)}) \to \hat{e}(\tilde{\boldsymbol{\psi}}) \end{array}\right\} \quad \text{uniformly on } (0, T) \times D. \tag{42}$$

Summary:

For all $p \geq 2$ there exists at least one weak solution to the original *fluid-structure interaction* problem (1) – (16) such that

i) $\mathbf{v} \in L^p(0,T;W^{1,p}(\Omega(\eta(t)))) \cap L^\infty(0,T;L^2(\Omega(\eta(t))))$,
$\eta \in W^{1,\infty}(0,T;L^2(0,L)) \cap H^1(0,T;H_0^2(0,L))$,

ii) $\operatorname{div}\mathbf{v} = 0$ a.e. on $\Omega(\eta(t))$,

iii) $\mathbf{v}\big|_{\Gamma_w(t)} = (0,\eta_t)$ for a.e. $x \in \Gamma_w(t),\ t \in (0,T)$,
$v_2\big|_{\Gamma_{in}\cup\Gamma_{out}\cup\Gamma_c} = 0$,

and the following integral identity holds

$$
\begin{aligned}
&\int_0^T \int_{\Omega(\eta(t))} \left\{ -\rho \mathbf{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + 2\mu(|e(\mathbf{v})|) e(\mathbf{v}) e(\boldsymbol{\varphi}) + \rho \sum_{i,j=1}^{2} v_i \frac{\partial v_j}{\partial x_i} \boldsymbol{\varphi}_j \right\} dx \\
&+ \int_0^T \int_0^{R_0(L)} \left(P_{out} - \frac{\rho}{2} |v_1|^2 \right) \varphi_1(L, x_2, t)\, dx_2\, dt \\
&- \int_0^T \int_0^{R_0(0)} \left(P_{in} - \frac{\rho}{2} |v_1|^2 \right) \varphi_1(0, x_2, t)\, dx_2\, dt \\
&+ \int_0^T \int_0^L P_w \varphi_2(x_1, R_0(x_1) + \eta(x_1, t), t)\, dx_1\, dt \\
&+ \int_0^T \int_0^L - \frac{\partial \eta}{\partial t} \frac{\partial \xi}{\partial t} + c\, \frac{\partial^3 \eta}{\partial x_1^2 \partial t} \frac{\partial^2 \xi}{\partial x_1^2} + a \frac{\partial \eta}{\partial x_1} \frac{\partial \xi}{\partial x_1}\ dx_1\, dt \\
&+ \int_0^T \int_0^L -a \frac{\partial^2 R_0}{\partial x_1^2} \xi + b\eta\, \xi\ dx_1\, dt \quad = \quad 0
\end{aligned}
\qquad (4
$$

for every test functions $\boldsymbol{\varphi}$ with the property (18).