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BMO Estimates for p-Parabolic Systems

The p-Parabolic System

We consider local solution of

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\operatorname{div}(g)$$

Aim: Transfer properties from g to $|\nabla u|^{p-2}\nabla u$. I.e. find spaces X $\||\nabla u|^{p-2}\nabla u\|_X \leq c\|g\|_X$+l.o.t.

Theorem

$g \in L^{p'}(I \times B)$ implies $|\nabla u|^{p-2}\nabla u \in L^{p'}(I \times B)$.

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Linear Calderón-Zygmund theory

Let u solve $-\Delta u = -\mathrm{div} g$.

Then $g \mapsto \nabla u$ is singular integral operator.
$\|\nabla u\|_X \leq c\|g\|_X +$l.o.t.
for $X = L^q, 1 < q < \infty$ and $X = C^\alpha, 0 < \alpha < 1$.
$g \in L^\infty \not\Longrightarrow \nabla u \in L^\infty$. However

Theorem

$g \in \mathrm{BMO}$ *implies* $\nabla u \in \mathrm{BMO}$

$g \in \mathrm{BMO}(\Omega)$ if $\|g\|_{\mathrm{BMO}(\Omega)} := \sup_{B \subset \Omega} ⨍_B |g - \langle g \rangle_B| < \infty$.

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Non-linear Calderón-Zygmund theory

For $u \in W^{1,p}$ with

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\mathrm{div}(g)$$

we have:

Theorem (Non-linear Calderón-Zygmund theory)

If $g \in X$, then $|\nabla u|^{p-2}\nabla u \in X$.

Case: $X = L^s$ with $s \geq p'$ [Iwaniec '83, 92'].

Case: $X = \mathrm{BMO}, \mathrm{VMO}, C^{0,\alpha}$ [DiBenedetto, Manfredi '96 for $p \geq 2$; Diening, Kaplický, Schwarzacher '12 for all $1 < p < \infty$].

p-Caloric Functions

Maximal regularity for

$$\partial_t h - \mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0.$$

Theorem (DiBenedetto, Friedmann '85)

Let h be p-caloric, then $h \in C^{1,\alpha}_{par}$.

Observe, h is p-Caloric $\not\Longrightarrow$ ah p-Caloric: No scaling!
Assume $|\nabla h| = \lambda \Longrightarrow \partial_t h(t,x) - \lambda^{p-2}\Delta h(t,x) = 0.$
Take $\tau = \lambda^{p-2}t \Longrightarrow \partial_\tau h(\tau,x) - \Delta h(\tau,x) = 0.$
Definition: $Q_r^\lambda(t,x) := (t, t-\lambda^{2-p}r^2) \times B_r(x)$ is called (sub)-intrinsic, if

$$\fint_{Q_r^\lambda(t,x)} |\nabla h|^p (\leq) = \lambda^p.$$

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Quantitative estimates for p-Caloric Functions

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Theorem (DiBenedetto, Friedmann '85, Acerbi, Mingione '07)

If $p > \frac{2n}{n+2}$, and Q_r^λ sub intrinsic, then
$\sup_{y,z \in Q_{r/2}^\lambda} |\nabla h(y) - \nabla h(z)| \leq c r^\alpha \lambda.$

Theorem (Schwarzacher '13)

If Q_r^λ is intrinsic, then
$\sup_{y,z \in Q_{r/2}^\lambda} |\nabla h(y) - \nabla h(z)|^p \leq c r^\alpha ⨍_{Q_r^\lambda} |\nabla h - \langle \nabla h \rangle_{Q_r^\lambda}|^p.$

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Parabolic Non Linear CalderónZygmund Theory

$$\partial_t u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\mathrm{div}(g)$$

What spaces hold $\||\nabla u|^{p-2}\nabla u\|_X \leq c\|g\|_X +$l.o.t.?

$\frac{2n}{n+2} < p < \infty$

1. Result $X = C^{\beta}_{\mathrm{par}}$, β small, [Misawa '02].
2. Result $X = L^q$ for $p' \leq q < \infty$, [Acerbi, Mingione '07].

What happens in between?

Intrinsic Cylinders

$p \geq 2$
$Q_r^\lambda := (t, t - \lambda^{2-p} r^2) \times B_r(x)$ is called (sub)-intrinsic, if
$⨍_{Q_r^\lambda} |\nabla u|^p (\leq) = \lambda^p$.

Lemma

Let Q_R^λ be intrinsic.
For every $r \in (0, R]$. There exist $Q_r^{\lambda_r}$ sub-intrinsic, s. t.
$Q_{r_1}^{\lambda_{r_1}} \subset Q_{r_2}^{\lambda_{r_2}}$ for $r_1 \leq r_2$.
For every $r \in (0, R]$ there exist $\rho \in [r, R]$, s.t $Q_\rho^{\lambda_\rho}$ is intrinsic and
$\lambda_r \leq \left(\frac{r}{\rho}\right)^\beta \lambda_\rho$.

Closing the Gap

From now on $p \geq 2$. The transfered p-Laplace theory gives

Theorem (Schwarzacher '13)

Let $g \in L^\infty(2I, \mathrm{BMO}(2B))$*, then*

$$\sup_{Q_r^\lambda \subset I \times B,\ subintr.} \fint_{Q_r^\lambda} |\nabla u - \langle \nabla u \rangle_{Q_r^\lambda}|^p \leq c \|g\|^{p'}_{L^\infty(2I, \mathrm{BMO}(2B))} + l.o.t.$$

Not too nice. Use the weak time derivative to estimate

$$\fint_{\{t\} \times B_r} \left| \frac{u - \Pi^1_{B_r}(u)}{r} \right|^2 \leq c \|g\|^{\frac{2}{p-1}}_{L^\infty(2I, \mathrm{BMO}(2B))} + l.o.t.$$

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Main result

Analysis provides $\sup_{B_r \subset \Omega} \fint_{B_r} \left| \frac{f - \Pi^1_{B_r}(f)}{r} \right|^2 < \infty \iff f \in \mathcal{C}^1(\Omega)$ the Hölder-Zygmund space .

Theorem

Let $g \in L^\infty(2I, \mathrm{BMO}(2B))$*, then*

$$\|u\|_{L^\infty(I,\mathcal{C}^1(B))} \le c\|g\|^{\frac{1}{p-1}}_{L^\infty(2I,\mathrm{BMO}(2B))} + l.o.t.$$

Definition $f \in \mathcal{C}^1(\Omega)$ if $\sup_{[x,x+2h]\subset\Omega} \frac{|f(x+2h)-2f(x+h)+f(x)|}{|h|} < \infty$.

$$C^1(B) \subset W^{1,\mathrm{BMO}}(B) \subset \mathcal{C}^1(B) \subset \bigcap_{1\le q<\infty} W^{1,q}(B).$$

Theorem

$g \in L^\infty(I, C^{\frac{\beta}{p-1}}(B))$ *implies* $\nabla u \in C^\beta_{par}(I \times B)$.