Compactness of higher-order Sobolev embeddings

Lenka Slavíková

May 1, 2014

Introduction

Sobolev embeddings and isoperimetric inequalities

Sobolev embeddings and isoperimetric inequalities

Maz'ya (1960):
* isoperimetric inequalities imply first-order Sobolev embeddings

Sobolev embeddings and isoperimetric inequalities

Maz'ya (1960):
* isoperimetric inequalities imply first-order Sobolev embeddings

Cianchi, Pick, S. (2013, submitted):
* isoperimetric inequalities imply also (optimal) higher-order Sobolev embeddings

Sobolev embeddings and isoperimetric inequalities

Maz'ya (1960):
* isoperimetric inequalities imply first-order Sobolev embeddings

Cianchi, Pick, S. (2013, submitted):
* isoperimetric inequalities imply also (optimal) higher-order Sobolev embeddings

Our goal: to prove a result in the spirit of the previous one, concerning compact Sobolev embeddings

Perimeter

Let $\Omega \subseteq \mathbb{R}^n$ be a domain endowed with a probability measure ν.

Perimeter

Let $\Omega \subseteq \mathbb{R}^n$ be a domain endowed with a probability measure ν. We shall assume that

$$d\nu(x) = \omega(x)\,dx,$$

where ω is a Borel function fulfilling that for a.e. $x \in \Omega$ there is a ball $B_x \subseteq \Omega$ centered in x for which

$$\operatorname{ess\,inf}_{B_x} \omega > 0.$$

Perimeter

Let $\Omega \subseteq \mathbb{R}^n$ be a domain endowed with a probability measure ν. We shall assume that

$$d\nu(x) = \omega(x)\, dx,$$

where ω is a Borel function fulfilling that for a.e. $x \in \Omega$ there is a ball $B_x \subseteq \Omega$ centered in x for which

$$\operatorname{ess\,inf}_{B_x} \omega > 0.$$

The *perimeter* of a measurable set $E \subseteq \Omega$ is

$$P_\nu(E, \Omega) = \int_{\Omega \cap \partial^M E} \omega(x)\, d\mathcal{H}^{n-1}(x),$$

where $\partial^M E$ denotes the essential boundary of E.

Isoperimetric function

The *isoperimetric function* of (Ω, ν) is defined by

$$I_{\Omega,\nu}(s) = \inf \left\{ P_\nu(E, \Omega) : E \subseteq \Omega,\ s \leq \nu(E) \leq \frac{1}{2} \right\}$$

if $s \in [0, 1/2]$ and by $I_{(\Omega,\nu)}(s) = I_{(\Omega,\nu)}(1-s)$ if $s \in (1/2, 1]$.

Isoperimetric function

The *isoperimetric function* of (Ω, ν) is defined by

$$I_{\Omega,\nu}(s) = \inf \left\{ P_\nu(E, \Omega) : E \subseteq \Omega,\ s \leq \nu(E) \leq \frac{1}{2} \right\}$$

if $s \in [0, 1/2]$ and by $I_{(\Omega,\nu)}(s) = I_{(\Omega,\nu)}(1 - s)$ if $s \in (1/2, 1]$.

We have the *isoperimetric inequality*

$$P_\nu(E, \Omega) \geq I_{\Omega,\nu}(\nu(E))$$

for every measurable subset $E \subseteq \Omega$.

Isoperimetric function

Isoperimetric function

* Usually hard to count

Isoperimetric function

* Usually hard to count

* We are interested only in its asymptotic behaviour at 0

Isoperimetric function

* Usually hard to count

* We are interested only in its asymptotic behaviour at 0
… this is known for quite a wide class of domains

* Usually hard to count

* We are interested only in its asymptotic behaviour at 0
... this is known for quite a wide class of domains

* We always have $I_{\Omega,\nu}(s) \leq Cs^{1-\frac{1}{n}}$, $s \in [0, \frac{1}{2}]$

Isoperimetric function

* Usually hard to count

* We are interested only in its asymptotic behaviour at 0
... this is known for quite a wide class of domains

* We always have $I_{\Omega,\nu}(s) \leq Cs^{1-\frac{1}{n}}$, $s \in [0, \frac{1}{2}]$

* We will assume that $I_{\Omega,\nu}(s) \geq Cs$, $s \in [0, \frac{1}{2}]$

Isoperimetric function

* Usually hard to count

* We are interested only in its asymptotic behaviour at 0
... this is known for quite a wide class of domains

* We always have $I_{\Omega,\nu}(s) \leq Cs^{1-\frac{1}{n}}$, $s \in [0, \frac{1}{2}]$

* We will assume that $I_{\Omega,\nu}(s) \geq Cs$, $s \in [0, \frac{1}{2}]$
... this excludes "too bad domains"

Examples

Examples

* Ω = a domain having a Lipschitz boundary, ν = the Lebesgue measure

$$I_{\Omega,\nu}(s) \approx s^{1-\frac{1}{n}}, \quad s \in [0, \frac{1}{2}]$$

Examples

* Ω = a domain having a Lipschitz boundary, ν = the Lebesgue measure

$$I_{\Omega,\nu}(s) \approx s^{1-\frac{1}{n}}, \quad s \in [0, \frac{1}{2}]$$

... this is fulfilled also for more general domains (so called *John domains*)

Examples

* Ω = a domain having a Lipschitz boundary, ν = the Lebesgue measure

$$I_{\Omega,\nu}(s) \approx s^{1-\frac{1}{n}}, \quad s \in [0, \frac{1}{2}]$$

... this is fulfilled also for more general domains (so called *John domains*)

* $\Omega = \mathbb{R}^n$, ν = the Gauss measure

$$I_{\Omega,\nu}(s) \approx s\sqrt{\log \frac{2}{s}}, \quad s \in [0, \frac{1}{2}]$$

Rearrangement-invariant spaces

$X(\Omega, \nu)$ is a *rearrangement-invariant space* if

Rearrangement-invariant spaces

$X(\Omega, \nu)$ is a *rearrangement-invariant space* if
* it is a Banach space consisting of ν-measurable functions on Ω

$X(\Omega, \nu)$ is a *rearrangement-invariant space* if
* it is a Banach space consisting of ν-measurable functions on Ω
* whenever $f \in X(\Omega, \nu)$ and g is a ν-measurable function satisfying

$$\nu(\{x \in \Omega : |f(x)| > \lambda\}) = \nu(\{x \in \Omega : |g(x)| > \lambda\}), \quad \lambda \geq 0,$$

then $g \in X(\Omega, \nu)$ and $\|f\|_{X(\Omega,\nu)} = \|g\|_{X(\Omega,\nu)}$

$X(\Omega,\nu)$ is a *rearrangement-invariant space* if
* it is a Banach space consisting of ν-measurable functions on Ω
* whenever $f \in X(\Omega,\nu)$ and g is a ν-measurable function satisfying

$$\nu(\{x \in \Omega : |f(x)| > \lambda\}) = \nu(\{x \in \Omega : |g(x)| > \lambda\}), \quad \lambda \geq 0,$$

then $g \in X(\Omega,\nu)$ and $\|f\|_{X(\Omega,\nu)} = \|g\|_{X(\Omega,\nu)}$

Examples:

$X(\Omega, \nu)$ is a *rearrangement-invariant space* if

* it is a Banach space consisting of ν-measurable functions on Ω
* whenever $f \in X(\Omega, \nu)$ and g is a ν-measurable function satisfying

$$\nu(\{x \in \Omega : |f(x)| > \lambda\}) = \nu(\{x \in \Omega : |g(x)| > \lambda\}), \quad \lambda \geq 0,$$

then $g \in X(\Omega, \nu)$ and $\|f\|_{X(\Omega,\nu)} = \|g\|_{X(\Omega,\nu)}$

Examples:

* Lebesgue spaces $L^p(\Omega, \nu)$

$X(\Omega,\nu)$ is a *rearrangement-invariant space* if
* it is a Banach space consisting of ν-measurable functions on Ω
* whenever $f \in X(\Omega,\nu)$ and g is a ν-measurable function satisfying

$$\nu(\{x \in \Omega : |f(x)| > \lambda\}) = \nu(\{x \in \Omega : |g(x)| > \lambda\}), \quad \lambda \geq 0,$$

then $g \in X(\Omega,\nu)$ and $\|f\|_{X(\Omega,\nu)} = \|g\|_{X(\Omega,\nu)}$

Examples:
* Lebesgue spaces $L^p(\Omega,\nu)$
* Lorentz spaces $L^{p,q}(\Omega,\nu)$

$X(\Omega, \nu)$ is a *rearrangement-invariant space* if

* it is a Banach space consisting of ν-measurable functions on Ω
* whenever $f \in X(\Omega, \nu)$ and g is a ν-measurable function satisfying

$$\nu(\{x \in \Omega : |f(x)| > \lambda\}) = \nu(\{x \in \Omega : |g(x)| > \lambda\}), \quad \lambda \geq 0,$$

then $g \in X(\Omega, \nu)$ and $\|f\|_{X(\Omega,\nu)} = \|g\|_{X(\Omega,\nu)}$

Examples:

* Lebesgue spaces $L^p(\Omega, \nu)$
* Lorentz spaces $L^{p,q}(\Omega, \nu)$
* Orlicz spaces
* ...

Representation spaces

$X(0,1)$ is the *representation space* of the space $X(\Omega, \nu)$ if

Representation spaces

$X(0,1)$ is the *representation space* of the space $X(\Omega,\nu)$ if
* it is a rearrangement-invariant space over $(0,1)$ endowed with the Lebesgue measure $|\cdot|$

Representation spaces

$X(0,1)$ is the *representation space* of the space $X(\Omega,\nu)$ if
* it is a rearrangement-invariant space over $(0,1)$ endowed with the Lebesgue measure $|\cdot|$
* whenever $f \in X(\Omega,\nu)$ and g is a measurable function on $(0,1)$ satisfying

$$\nu(\{x \in \Omega : |f(x)| > \lambda\}) = |\{t \in (0,1) : |g(t)| > \lambda\}|, \quad \lambda \geq 0,$$

then $g \in X(0,1)$ and $\|f\|_{X(\Omega,\nu)} = \|g\|_{X(0,1)}$

Example

Example

Ω = domain in $\mathbb{R}^n$ fulfilling $|\Omega| = 1$
ν = the Lebesgue measure

Example

Ω = domain in $\mathbb{R}^n$ fulfilling $|\Omega| = 1$
ν = the Lebesgue measure

If $1 \leq p < \infty$ and

$$\|f\|_{X(\Omega,\nu)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

Example

Ω = domain in $\mathbb{R}^n$ fulfilling $|\Omega| = 1$
ν = the Lebesgue measure

If $1 \leq p < \infty$ and

$$\|f\|_{X(\Omega,\nu)} = \left(\int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

then

$$\|g\|_{X(0,1)} = \left(\int_0^1 |g(t)|^p \, dt \right)^{\frac{1}{p}}.$$

Sobolev spaces

Sobolev spaces

Let $X(\Omega, \nu)$ be a rearrangement-invariant space.

Sobolev spaces

Let $X(\Omega, \nu)$ be a rearrangement-invariant space. For $m \in \mathbb{N}$ consider the *Sobolev space*

$$V^m X(\Omega,\nu) = \{u : \ u \text{ is } m\text{-times weakly differentiable in } \Omega \text{ and its weak derivatives of order } m \text{ belong to } X(\Omega,\nu)\}$$

Let $X(\Omega,\nu)$ be a rearrangement-invariant space. For $m \in \mathbb{N}$ consider the *Sobolev space*

$$V^m X(\Omega,\nu) = \{u : \ u \text{ is } m\text{-times weakly differentiable in } \Omega \text{ and its weak derivatives of order } m \text{ belong to } X(\Omega,\nu)\}$$

equipped with the norm

$$\|u\|_{V^m X(\Omega,\nu)} = \sum_{k=0}^{m-1} \| |\nabla^k u| \|_{L^1(\Omega,\nu)} + \| |\nabla^m u| \|_{X(\Omega,\nu)},$$

where $\nabla^k u$ denotes the vector of all k-th order weak derivatives of u (if $k = 1, 2, \dots, m$) and $\nabla^0 u = u$.

Sobolev spaces

Let $X(\Omega,\nu)$ be a rearrangement-invariant space. For $m \in \mathbb{N}$ consider the *Sobolev space*

$$V^m X(\Omega,\nu) = \{u : \ u \text{ is } m\text{-times weakly differentiable in } \Omega \text{ and its weak derivatives of order } m \text{ belong to } X(\Omega,\nu)\}$$

equipped with the norm

$$\|u\|_{V^m X(\Omega,\nu)} = \sum_{k=0}^{m-1} \||\nabla^k u|\|_{L^1(\Omega,\nu)} + \||\nabla^m u|\|_{X(\Omega,\nu)},$$

where $\nabla^k u$ denotes the vector of all k-th order weak derivatives of u (if $k = 1, 2, \dots, m$) and $\nabla^0 u = u$.

Then $V^m X(\Omega,\nu)$ is a normed linear space.

The result

The result

Let $X(\Omega, \nu)$ and $Y(\Omega, \nu)$ be rearrangement-invariant spaces.

The result

Let $X(\Omega, \nu)$ and $Y(\Omega, \nu)$ be rearrangement-invariant spaces. We are going to find a sufficient condition for

$$V^m X(\Omega, \nu) \hookrightarrow\hookrightarrow Y(\Omega, \nu).$$

The result

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. We are going to find a sufficient condition for

$$V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu).$$

* We first describe an asymptotic behaviour of $I_{\Omega,\nu}$ in terms of its lower bound:

The result

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. We are going to find a sufficient condition for

$$V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu).$$

* We first describe an asymptotic behaviour of $I_{\Omega,\nu}$ in terms of its lower bound:

$$I_{\Omega,\nu}(s) \geq cI(cs), \quad s \in [0, \frac{1}{2}],$$

for some nondecreasing function I with $I(s) \geq c's$.

The result

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. We are going to find a sufficient condition for

$$V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu).$$

* We first describe an asymptotic behaviour of $I_{\Omega,\nu}$ in terms of its lower bound:

$$I_{\Omega,\nu}(s) \geq cI(cs), \quad s \in [0, \frac{1}{2}],$$

for some nondecreasing function I with $I(s) \geq c's$.
* We then consider certain one-dimensional operator H_I^m depending on I and m.

The result

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. We are going to find a sufficient condition for

$$V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu).$$

* We first describe an asymptotic behaviour of $I_{\Omega,\nu}$ in terms of its lower bound:

$$I_{\Omega,\nu}(s) \geq cI(cs), \quad s \in [0, \frac{1}{2}],$$

for some nondecreasing function I with $I(s) \geq c's$.
* We then consider certain one-dimensional operator H_I^m depending on I and m.
* The sufficient condition will be

$$H_I^m : X(0,1) \longrightarrow\longrightarrow Y(0,1).$$

The result

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. We are going to find a sufficient condition for

$$V^m X(\Omega,\nu) \hookrightarrow\hookrightarrow Y(\Omega,\nu).$$

* We first describe an asymptotic behaviour of $I_{\Omega,\nu}$ in terms of its lower bound:

$$I_{\Omega,\nu}(s) \geq cI(cs), \quad s \in [0, \frac{1}{2}],$$

for some nondecreasing function I with $I(s) \geq c's$.
* We then consider certain one-dimensional operator H_I^m depending on I and m.
* The sufficient condition will be

$$H_I^m : X(0,1) \rightarrow\rightarrow Y(0,1).$$

... reduction to a one-dimensional problem

The operator H_I^m

Let I be as above.

Let I be as above. Consider the operator

$$H_I f(t) = \int_t^1 \frac{|f(s)|}{I(s)}\,ds, \quad f \text{ measurable on } (0,1), \quad t \in (0,1).$$

The operator H_I^m

Let I be as above. Consider the operator

$$H_I f(t) = \int_t^1 \frac{|f(s)|}{I(s)}\, ds, \quad f \text{ measurable on } (0,1), \quad t \in (0,1).$$

For $m \in \mathbb{N}$ we define

$$H_I^m = \underbrace{H_I \circ \cdots \circ H_I}_{m-times}.$$

The operator H_I^m

Let I be as above. Consider the operator

$$H_I f(t) = \int_t^1 \frac{|f(s)|}{I(s)}\, ds, \quad f \text{ measurable on } (0,1), \quad t \in (0,1).$$

For $m \in \mathbb{N}$ we define

$$H_I^m = \underbrace{H_I \circ \cdots \circ H_I}_{m-times}.$$

Then

$$H_I^m f(t) = \frac{1}{(m-1)!} \int_t^1 \frac{|f(s)|}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds.$$

Two tools

Two tools

* Cianchi, Pick, S. (2013, submitted):

Two tools

* Cianchi, Pick, S. (2013, submitted):

If (Ω, ν) and I are as above, then

$$H_I^m : X(0,1) \to Y(0,1) \quad \Rightarrow \quad V^m X(\Omega,\nu) \hookrightarrow Y(\Omega,\nu)$$

Two tools

* Cianchi, Pick, S. (2013, submitted):

If (Ω, ν) and I are as above, then

$$H_I^m : X(0,1) \to Y(0,1) \quad \Rightarrow \quad V^m X(\Omega, \nu) \hookrightarrow Y(\Omega, \nu)$$

* almost-compact embeddings

Almost-compact embeddings

(R, μ) ... a totally σ-finite measure space

(R, μ) ... a totally σ-finite measure space

Definition. A rearrangement-invariant space $X(R, \mu)$ is *almost-compactly embedded* into a rearrangement-invariant space $Y(R, \mu)$, denoted $X(R, \mu) \overset{*}{\hookrightarrow} Y(R, \mu)$,

Almost-compact embeddings

(R, μ) ... a totally σ-finite measure space

Definition. A rearrangement-invariant space $X(R, \mu)$ is *almost-compactly embedded* into a rearrangement-invariant space $Y(R, \mu)$, denoted $X(R, \mu) \overset{*}{\hookrightarrow} Y(R, \mu)$, if

$$\lim_{n \to \infty} \sup_{\|f\|_{X(R,\mu)} \leq 1} \|f \chi_{E_n}\|_{Y(R,\mu)} = 0$$

for every sequence $(E_n)_{n=1}^{\infty}$ of subsets of R such that $\chi_{E_n} \to 0$ μ-a.e.

(R, μ) ... a totally σ-finite measure space

Definition. A rearrangement-invariant space $X(R, \mu)$ is *almost-compactly embedded* into a rearrangement-invariant space $Y(R, \mu)$, denoted $X(R, \mu) \overset{*}{\hookrightarrow} Y(R, \mu)$, if

$$\lim_{n \to \infty} \sup_{\|f\|_{X(R,\mu)} \leq 1} \|f \chi_{E_n}\|_{Y(R,\mu)} = 0$$

for every sequence $(E_n)_{n=1}^{\infty}$ of subsets of R such that $\chi_{E_n} \to 0$ μ-a.e.

for us: $(R, \mu) = (\Omega, \nu)$ or $(R, \mu) = ((0, 1), |\cdot|)$

(R, μ) ... a totally σ-finite measure space

Definition. A rearrangement-invariant space $X(R, \mu)$ is *almost-compactly embedded* into a rearrangement-invariant space $Y(R, \mu)$, denoted $X(R, \mu) \overset{*}{\hookrightarrow} Y(R, \mu)$, if

$$\lim_{n\to\infty} \sup_{\|f\|_{X(R,\mu)}\leq 1} \|f\chi_{E_n}\|_{Y(R,\mu)} = 0$$

for every sequence $(E_n)_{n=1}^{\infty}$ of subsets of R such that $\chi_{E_n} \to 0$ μ-a.e.

for us: $(R, \mu) = (\Omega, \nu)$ or $(R, \mu) = ((0, 1), |\cdot|)$

Observation. $X(\Omega, \nu) \overset{*}{\hookrightarrow} Y(\Omega, \nu) \quad \Leftrightarrow \quad X(0, 1) \overset{*}{\hookrightarrow} Y(0, 1)$

Key one-dimensional results

Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces.

Key one-dimensional results

Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. We can characterize

$$H_I^m : X(0,1) \rightarrow\rightarrow Y(0,1)$$

by an almost-compact embedding in two different ways:

Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. We can characterize

$$H_I^m : X(0,1) \to\to Y(0,1)$$

by an almost-compact embedding in two different ways:

The first way: $X_{m,I}^r(0,1) \overset{*}{\hookrightarrow} Y(0,1)$,
where $X_{m,I}^r(0,1)$ is the smallest rearrangement-invariant space for which

$$H_I^m : X(0,1) \to X_{m,I}^r(0,1).$$

Key one-dimensional results

Suppose that $X(0,1)$ and $Y(0,1)$ are rearrangement-invariant spaces. We can characterize

$$H_I^m : X(0,1) \to\to Y(0,1)$$

by an almost-compact embedding in two different ways:

The first way: $X^r_{m,I}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$,
where $X^r_{m,I}(0,1)$ is the smallest rearrangement-invariant space for which

$$H_I^m : X(0,1) \to X^r_{m,I}(0,1).$$

The second way: $X(0,1) \overset{*}{\hookrightarrow} Y^d_{m,I}(0,1)$,
where $Y^d_{m,I}(0,1)$ is the largest rearrangement-invariant space for which

$$H_I^m : Y^d_{m,I}(0,1) \to Y(0,1).$$

Thank you for your attention.