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Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

Telc, May 2014

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$L_{\text{THE SETUP}}$

 L_{NOTATION}

- $\Omega \subset \mathbb{R}^n$, (preferably $n=3, \Omega$ solid body)
- f density of the body forces
- p external loading
- t (time-like) loading parameter

 $x \mapsto x + u(x, t)$ displacement field

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 $-$ [The setup](#page-2-0)

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- f density of the body forces
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- t (time-like) loading parameter

 $x \mapsto x + u(x, t)$ displacement field

state of the deformed material: u, σ small deformations, balance of forces

 $-$ div $\sigma - f$

Boundary conditions

clamped part: external loading:

$$
u|_{\Gamma} = 0, \ \Gamma \subset \partial \Omega,
$$

$$
\sigma \cdot n = p \text{ on } \partial \Omega \setminus \Gamma
$$

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 $L_{\text{THE SETUP}}$

 L_{NOTATION}

Yield condition involves hardening variables ξ

 $F(\sigma, \xi) \leq 0$, F : convex function

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 $F(\sigma, \xi) \leq 0$, F : convex function

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von Mises: $\sigma_D = \sigma - \frac{\text{tr}\,\sigma}{n}$ $\frac{r\sigma}{n}$ \mathbb{I} (Deviatoric part)

$$
F(\sigma,\xi) = \begin{cases} |\sigma_D| - (\xi + \kappa) & \text{isotropic hardening} \\ |\sigma_D - \xi_D| - \kappa & \text{kinematic hardening} \end{cases}
$$

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Yield condition involves hardening variables ξ

 $F(\sigma, \xi) < 0$, F : convex function

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Relation between the stress-rate σ and the strain rate $\dot{\varepsilon}$: involves the compliance tensor \overline{A} (symmetric rank 4 tensor) L_{NOTATION}

Yield condition involves hardening variables ξ

 $F(\sigma, \xi) \leq 0$, F : convex function

von Mises: $\sigma_D = \sigma - \frac{\text{tr}\,\sigma}{n}$ $\frac{r\sigma}{n}$ \mathbb{I} (Deviatoric part)

$$
F(\sigma,\xi) = \begin{cases} |\sigma_D| - (\xi + \kappa) & \text{isotropic hardening} \\ |\sigma_D - \xi_D| - \kappa & \text{kinematic hardening} \end{cases}
$$

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Relation between the stress-rate σ and the strain rate ϵ : involves the compliance tensor \overline{A} (symmetric rank 4 tensor) flow rule for ξ : involves the hardening tensor $H \in \mathbb{R}^{m \times m}$ A, H positive definite

 \Box [Mathematical formulation as a variational inequality](#page-7-0)

 \Box ADMISSIBLE STRESSES AND HARDENING VARIABLES

 $\mathbb{K}(t)$: set of all pairs (τ, η) with

$$
\tau \in L^{2}(\Omega; \mathbb{R}^{n \times n}_{sym}), \ \eta \in L^{2}(\Omega, \mathbb{R}^{m})
$$
 (1)

 τ fulfills the balance of forces in the weak form:

$$
(\tau, \nabla \varphi)_{\Omega} = (f, \varphi)_{\Omega} + \int_{\partial \Omega} p \varphi \, d\sigma \text{ for all } \varphi \in H^1_{\Gamma}(\Omega). \qquad \text{(BF)}
$$

For isotropic hardening: $m = 1$,

$$
\eta \in L^2(\Omega; \mathbb{R}), \qquad |\tau_D| - \eta \le \kappa, \tag{YCI}
$$

for *kinematic* hardening: $m = n(n + 1)/2$

$$
\eta \in L^{2}(\Omega; \mathbb{R}^{n \times n}_{sym}), \qquad |\tau_D - \eta_D| \leq \kappa. \qquad \text{(YCK)}
$$

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 $\mathrel{\sqsubseteq}$ MATHEMATICAL FORMULATION AS A VARIATIONAL INEQUALITY

 $\mathrel{\sqsubseteq}$
ADMISSIBLE STRESSES AND HARDENING VARIABLES

Given:

$$
\begin{aligned} f, \dot{f}&\in L^\infty(0,\,T;L^\infty(\Omega)),\\ &\quad p,\dot{p}&\in L^\infty(0,\,T;L^\infty(\partial\Omega)),\qquad \ddot{p}&\in L\\ (\sigma_0,0)&\in \mathbb{K}(0) \end{aligned}
$$

$$
\begin{aligned}\n&\infty(\Omega)), &\qquad \ddot{f} \in L^1(0, T; L^2), \\
&\infty(\partial\Omega)), &\qquad \ddot{p} \in L^1(0, T; L^2(\partial\Omega)),\n\end{aligned}
$$

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 $\mathrel{\sqsubseteq}$ MATHEMATICAL FORMULATION AS A VARIATIONAL INEQUALITY

 $\mathrel{\sqsubseteq}$
ADMISSIBLE STRESSES AND HARDENING VARIABLES

Given:

$$
f, \dot{f} \in L^{\infty}(0, T; L^{\infty}(\Omega)), \qquad \ddot{f} \in L^{1}(0, T; L^{2}),
$$

\n
$$
\rho, \dot{\rho} \in L^{\infty}(0, T; L^{\infty}(\partial \Omega)), \qquad \ddot{\rho} \in L^{1}(0, T; L^{2}(\partial \Omega)),
$$

\n
$$
(\sigma_{0}, 0) \in \mathbb{K}(0)
$$

Find $\sigma\in L^\infty(L^2),\ \xi\in L^\infty(L^2)$ such that

$$
\dot{\sigma} \in L^2(L^2), \quad \dot{\xi} \in L^2(L^2)
$$

\n
$$
(\sigma(t), \xi(t)) \in \mathbb{K}(t), \ t \in [0, T]
$$

\n
$$
\sigma(0) = \sigma_0, \quad \xi(0) = 0
$$

\n
$$
(A\dot{\sigma}, \sigma - \tau) + (H\dot{\xi}, \xi - \eta) \le 0 \quad \text{a.e. in } [0, T]
$$

\nfor all $(\tau, \eta) \in \mathbb{K}(t)$.

[Mathematical formulation as a variational inequality](#page-10-0)

 $T_{\text{HE SARE LOAD CONDITION}}$

DEFINITION (SAFE LOAD CONDITION) Exist $\hat{\sigma} \in L^{\infty}(L^2)$, $\hat{\xi} \in L^{\infty}(L^2)$: $\dot{\hat{\sigma}} \in L^{\infty}(L^2),\; \ddot{\hat{\sigma}} \in L^1(L^2),\; \dot{\hat{\xi}} \in L^{\infty}(L^2)$ $(\hat{\sigma}(0), 0) \in \mathbb{K}(0), \ \hat{\xi}|_{t=0} = 0$ $(\hat{\sigma}(t, .), \hat{\xi}(t, .)) \in \mathbb{K}(t),$

and exists $\delta > 0$

$$
|\hat{\sigma}_D| - \xi \le \kappa - \delta \text{ or } |\hat{\sigma}_D - \hat{\xi}_D| \le \kappa - \delta, \text{ respectively.}
$$

4 0 > 4 4 + 4 3 + 4 3 + 5 + 9 4 0 +

[Mathematical formulation as a variational inequality](#page-11-0)

 L_{Eyrernor}

Johnson 78: Exists $u \in L^{\infty}(H^1_{\Gamma})$ with $u \in L^{\infty}(H^1_{\Gamma}),$ and a multiplier $\dot{\lambda}\in L^{\infty}(0,\,T;L^2(\Omega,\mathbb{R}))$ (Frehse & Loebach 08) s.t. for isotropic hardening:

$$
\frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}) = A\dot{\sigma} + \dot{\lambda}\sigma_D|\sigma_D|^{-1}
$$

$$
0 = H\dot{\xi} - \dot{\lambda}
$$

4 0 > 4 4 + 4 3 + 4 3 + 5 + 9 4 0 +

where $\lambda \geq 0$ a.e. and $\lambda(|\sigma_D| - \xi - \kappa) = 0$,

[Mathematical formulation as a variational inequality](#page-12-0)

 L_{Eyrernor}

Johnson 78: Exists $u \in L^{\infty}(H^1_{\Gamma})$ with $u \in L^{\infty}(H^1_{\Gamma}),$ and a multiplier $\dot{\lambda}\in L^{\infty}(0,\,T;L^2(\Omega,\mathbb{R}))$ (Frehse & Loebach 08) s.t. for kinematic hardening:

$$
\frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}) = A\dot{\sigma} + \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|
$$

$$
0 = H\dot{\xi} - \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|.
$$

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where $\dot{\lambda} \ge 0$ a.e. and $\dot{\lambda}(|\sigma_D - \xi_D| - \kappa) = 0$,

 $\mathrel{\sqsubseteq}$ MATHEMATICAL FORMULATION AS A VARIATIONAL INEQUALITY

 $L_{\text{EXISTENCE}}$

$$
\frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}) = A\dot{\sigma} + \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|
$$

$$
0 = H\dot{\xi} - \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|.
$$

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 $\mathrel{\sqsubseteq}$ MATHEMATICAL FORMULATION AS A VARIATIONAL INEQUALITY

 $L_{\text{EXISTENCE}}$

$$
\frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}) = A\dot{\sigma} + \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|
$$

$$
0 = H\dot{\xi} - \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|.
$$

 $strain=elastic strain + plastic strain$

$$
\dot{\varepsilon} = \dot{\lambda} \frac{\partial}{\partial \sigma} F(\sigma, \xi), \text{ where } \dot{\varepsilon}_{pl} = 0, \text{ if } F < 0,
$$
\n
$$
H\dot{\xi} = -\dot{\lambda} \frac{\partial}{\partial \xi} F(\sigma, \xi)
$$

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EREGULARITY RESULTS

[Known results: displacement, stresses and strains](#page-15-0)

Johnson 78 Seregin 94

Alber & Nessenenko '09 Knees 08 $\sigma,\xi\in L^\infty(H^{\frac{1}{2}-\delta}$ F rehse $\&$ Löbach '09

Löbach '09

 F rehse $&$ I öbach '11

 ${}^{\infty}$ (L^2) ${}^{\infty} (H^1_{loc})$ $\nabla(\varepsilon) \in L^{\infty}(\mathcal{C}_{\textit{lc}}^*)$ isotropic hard. ${}^{\infty}$ (H $^{\frac{1}{3}-\delta}$) kinematic hard) kinematic hard ${}^{\infty}$ (H $^{\frac{1}{2}-\delta}$) kinem. & isotr. hard ${}^{\infty}$ (H $^{\frac{1}{2}+\delta}$) kinem. & isot. hard $\dot{\xi}\in L^{\infty}(L^{2+2\delta}$) kinem. & isotr. hard

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 $\mathrel{{\sqsubseteq}_{\text{REGULARITY}}$ results

 $\mathrel{\rule{0pt}{.4ex}\rule{0pt}{1.5ex}}$ [Regularity for the velocities](#page-16-0)

$$
\Delta_t^s w(t,x) = w(t+h,x) - w(t,x),
$$

$$
\Delta_i^s w(t,x) = w(t,x + se_i) - w(t,x).
$$

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 $L_{\text{REGULARITY RESULTS}}$

L REGULARITY FOR THE VELOCITIES

$$
\Delta_t^s w(t,x) = w(t+h,x) - w(t,x),
$$

\n
$$
\Delta_i^s w(t,x) = w(t,x + s e_i) - w(t,x).
$$

THEOREM (Regularity in time, FREHSE $&$ Sp. 2012)

$$
h^{-2}\int\limits_{0}^{h}\int\limits_{0}^{T-h}\int\limits_{\Omega}\left[|\Delta_{t}^{s}\dot{\sigma}|^{2}+|\Delta_{t}^{s}\dot{\xi}|^{2}\right]\leq C
$$

uniformly for $0 < h < h_0$. \Rightarrow for kinematic hardening:

$$
h^{-2}\int\limits^{T-h}_{0}\int\limits^h_{0}\int\limits_{\Omega}|\Delta^s_t\nabla\dot{u}|^2\leq C
$$

 $\mathrel{\mathop{\rule{.15pt}{\mathop{\rule{.5pt}{0.5pt}}}}\mathrel{<}{\mathop{\rule{.5pt}{0.5pt}}}}$ REGULARITY RESULTS

 $\mathrel{\rule{0pt}{1.5ex}\rule{0pt}{1.5ex}}$ REGULARITY FOR THE VELOCITIES

$$
\Delta_t^s w(t,x) = w(t+h,x) - w(t,x),
$$

\n
$$
\Delta_i^s w(t,x) = w(t,x + s e_i) - w(t,x).
$$

\n
$$
h^{-2} \int_{0}^{h} \int_{\Omega} \int_{\Omega} \left[|\Delta_t^s \dot{\sigma}|^2 + |\Delta_t^s \dot{\xi}|^2 \right] \leq C
$$

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uniformly for $0 < h < h_0$.

 $\mathrel{\sqsubseteq}_{\mathrel{\mathit{REGULARITY}}}\mathrel{\mathit{RESULTS}}$

 $L_{\text{REGULARITY FOR THE VELOCITIES}}$

$$
\Delta_t^s w(t,x) = w(t+h,x) - w(t,x),
$$

\n
$$
\Delta_i^s w(t,x) = w(t,x + s e_i) - w(t,x).
$$

\n
$$
h^{-2} \int_{0}^{h} \int_{\Omega} \int_{\Omega} \left[|\Delta_t^s \dot{\sigma}|^2 + |\Delta_t^s \dot{\xi}|^2 \right] \leq C
$$

uniformly for $0 < h < h_0$.

Comment: Even prolongation in time: $\sigma: [-T, T] \to \mathbb{R}_{sym}^{n \times n}$ periodic,

$$
\sigma = \sum_{m=-\infty}^{\infty} c_m(x) \exp(\frac{im\pi}{2T}t) \implies
$$

$$
\sum_{m=-\infty}^{\infty} m^{1-\delta} \int_{\Omega} |c_m(x)|^2 dy \le C_{\delta} \quad \text{for all } \delta > 0.
$$

 $\mathrel{\sqsubseteq}_{\mathrel{\mathit{REGULARITY}}}\mathrel{\mathit{RESULTS}}$

L REGULARITY FOR THE VELOCITIES

THEOREM (Local regularity in space)

$$
\sup_{0\leq h\leq h_0} h^{-1}\int\limits^{T-h}_{0} \int\limits_{\Omega_0} |\Delta_i^h\dot{\sigma}|^2 + |\Delta_i^h\dot{\xi}|^2 \leq C, \ i=1,\ldots,n
$$

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for any domain Ω_0 such that $\overline{\Omega}_0 \subset \Omega$ and $h_0 \leq \text{dist}(\partial \Omega, \partial \Omega_0)$.

 $\mathrel{\mathop{\longleftarrow}}$ [Main ingredients of the proof](#page-21-0)

 $\mathrel{\sqsubseteq}$ [The penalty problem](#page-21-0)

Penalty potential

$$
\mathsf{G}_\mu(\sigma,\xi)=\frac{1}{2\mu}[\mathsf{F}(\sigma,\xi)]_+^2\qquad\Rightarrow\qquad
$$

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$$
\nabla G_{\mu} = \begin{cases} \frac{1}{\mu}[F]_{+} \begin{pmatrix} \sigma_{D}|\sigma_{D}|^{-1} \\ -1 \end{pmatrix} & \text{isotr. h.} \\ \frac{1}{\mu}[F]_{+} \frac{\sigma_{D} - \xi_{D}}{|\sigma_{D} - \xi_{D}|} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{kinem. h.} \end{cases}
$$

 $\mathrel{\mathop{\longleftarrow}}$ [Main ingredients of the proof](#page-22-0)

 $\mathrel{\sqsubseteq}$ [The penalty problem](#page-22-0)

Penalty potential

$$
G_{\mu}(\sigma,\xi)=\frac{1}{2\mu}[F(\sigma,\xi)]_{+}^{2}\qquad \Rightarrow
$$

$$
\nabla G_{\mu} = \begin{cases} \frac{1}{\mu} [F]_{+} \begin{pmatrix} \sigma_{D} |\sigma_{D}|^{-1} \\ -1 \end{pmatrix} & \text{isotr. h.} & |\partial_{\xi} G_{\mu}| = |\partial_{\sigma} G_{\mu}| \\ \frac{1}{\mu} [F]_{+} \frac{\sigma_{D} - \xi_{D}}{|\sigma_{D} - \xi_{D}|} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{kinem. h.} & \partial_{\xi} G_{\mu} = -\partial_{\sigma} G_{\mu} \end{cases}
$$

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[Some regularity results for plasticity problems.](#page-0-0) $\mathrel{\mathop{\longleftarrow}}$ [Main ingredients of the proof](#page-23-0)

 L [The penalty problem](#page-23-0)

Find
$$
\sigma_{\mu}, \xi_{\mu} \in L^{\infty}(L^2)
$$
 with $\sigma_{\mu}, \dot{\xi}_{\mu} \in L^{\infty}(L^2)$,

$$
(\sigma_{\mu}, \xi_{\mu})|_{t=0} = (\sigma_0, 0) \tag{IC}
$$

$$
(\sigma_{\mu}, \nabla \varphi)_{\Omega} = (f, \varphi)_{\Omega} + \int_{\partial \Omega} p\varphi \, d\sigma \text{ for all } \varphi \in H_{\Gamma}^{1}(\Omega). \qquad \text{(Bof)}
$$

$$
0 = (A\dot{\sigma}_{\mu} + \partial_{\sigma} G_{\mu}, \tau)_{\Omega} \qquad \qquad \text{(P1)}
$$
for all symmetric $\tau \in {\{\nabla \varphi : \varphi \in H_{\Gamma}^{1}\}}^{\perp}$

$$
0 = H\dot{\xi}_{\mu} + \partial_{\xi} G_{\mu} \tag{P2}
$$

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[Some regularity results for plasticity problems.](#page-0-0) [Main ingredients of the proof](#page-24-0)

L THE PENALTY PROBLEM

Find
$$
\sigma_{\mu}, \xi_{\mu} \in L^{\infty}(L^2)
$$
 with $\dot{\sigma}_{\mu}, \dot{\xi}_{\mu} \in L^{\infty}(L^2)$,

$$
(\sigma_{\mu}, \xi_{\mu})|_{t=0} = (\sigma_0, 0) \tag{IC}
$$

 ϵ

$$
(\sigma_{\mu}, \nabla \varphi)_{\Omega} = (f, \varphi)_{\Omega} + \int_{\partial \Omega} p\varphi \, d\sigma \text{ for all } \varphi \in H_{\Gamma}^{1}(\Omega). \qquad \text{(Bof)}
$$

$$
0 = (A\dot{\sigma}_{\mu} + \partial_{\sigma} G_{\mu}, \tau)_{\Omega} \qquad (P1)
$$

for all symmetric $\tau \in \{\nabla \varphi : \varphi \in \mathcal{H}_{\mathsf{\Gamma}}^1\}^\perp$

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$$
0 = H\dot{\xi}_{\mu} + \partial_{\xi} G_{\mu} \tag{P2}
$$

Well known: Problem has a unique solution, along with a sequence of handy priori estimates independent on μ .

 $\mathrel{\mathop{\longleftarrow}}$ [Main ingredients of the proof](#page-25-0)

 $\mathrel{\sqsubseteq}_{\mathrel{\mathsf{BASIC}}}$ ideas for the time regularity

For the time regularity

$$
0 = (A\dot{\sigma}_{\mu} + \partial_{\sigma} G_{\mu}, \tau)_{\Omega} \tag{P1}
$$

$$
0 = H\dot{\xi}_{\mu} + \partial_{\xi} G_{\mu} \tag{P2}
$$

\n- test (P1) with
$$
\int_{0}^{h} \Delta_t^s \dot{\sigma}_{\mu} ds
$$
 and (P2) with $\int_{0}^{h} \Delta_t^s \dot{\xi}_{\mu} ds$
\n- use the elementary relation
\n

$$
A\tau : \Delta_t^s \tau = -\frac{1}{2} A \Delta_t^s \tau : \Delta_t^s \tau + \frac{1}{2} \Delta_t^s (A\tau : \tau),
$$

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 $\mathrel{\mathop{\rule{0pt}{\text{}}}}$ –[Main ingredients of the proof](#page-26-0)

 $\mathrel{\sqsubseteq}_{\mathrel{\mathsf{BASIC}}}$ ideas for the time regularity

Arrive at

$$
\int_{t_1}^{t_2-h} \int_0^h (A\Delta_t^s \dot{\sigma}_\mu, \Delta_t^s \dot{\sigma}_\mu)_{\Omega} + (H\Delta_t^s \dot{\xi}_\mu, \Delta_t^s \dot{\xi}_\mu)_{\Omega}
$$
\n
$$
= \int_{t_1}^{t_2-h} \int_0^h \int_{\Omega} \Delta_t^s (A\dot{\sigma}_\mu : \dot{\sigma}_\mu) + \Delta_t^s (H\dot{\xi}_\mu : \dot{\xi}_\mu)
$$
\n
$$
+ \text{ term with } G_\mu - 2 \int_0^{t_2-h} \int_0^h (\nabla \dot{u}_\mu, \Delta_t^s \dot{\sigma}_\mu)_{\Omega}
$$

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 $-MAIN INGREDIENTS OF THE PROOF$

L BASIC IDEAS FOR THE TIME REGULARITY

In the limit $\mu \to 0$, $t_1 \to 0$ $t_2 \to T$:

$$
\blacktriangleright \{\dots\} \ge C \int_{0}^{h} \int_{0}^{T-h} \int_{\Omega} |\Delta_t^s \dot{\sigma}_{\mu}|^2 + |\Delta_t^s \dot{\xi}_{\mu}|^2
$$

$$
\blacktriangleright \{\ldots\} \leq C(\|\sigma_{\mu}\|_{L^{\infty}(L^2)} + \|\xi_{\mu}\|_{L^{\infty}(L^2)})h^2
$$

If lim sup $\{ \dots \} \leq 0$ (use the convexity of the the penalty potential and the following convergence result

$$
\int\limits_0^T\int\limits_\Omega G_\mu(\sigma_\mu,\xi_\mu)\to 0\ \ \text{as}\ \mu\to 0,
$$

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 $\blacktriangleright \, \{ \ldots \} \leq C h^2$ (use the safe load and bounds for $\| \nabla u \|_{L^\infty(L^2)})$

 $-MAIN INGREDIENTS OF THE PROOF$

[Local regularity in space](#page-28-0)

Local regularity in space

Test (P1), (P2) with $\zeta^2 (E_t^s E_i^h - I) \dot{\sigma}_{\mu} = \dot{\sigma}_{\mu} (t + s, x + he_i) - \dot{\sigma}_{\mu} (t, x), \ \zeta^2 \dots \dot{\zeta}_{\mu}$ ζ: Localization function In principle the arguments are similar, but in detail even more tricky as for the time direction.

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[Main ingredients of the proof](#page-29-0)

[Local regularity in space](#page-29-0)

Local regularity in space

Test (P1), (P2) with $\zeta^2 (E_t^s E_i^h - I) \dot{\sigma}_{\mu} = \dot{\sigma}_{\mu} (t + s, x + he_i) - \dot{\sigma}_{\mu} (t, x), \ \zeta^2 \dots \dot{\zeta}_{\mu}$ ζ: Localization function In principle the arguments are similar, but in detail even more tricky as for the time direction.

In case you wonder:

$$
\begin{aligned} |(E_t^s E_i^h - I)\dot{\sigma}|^2 &= |(E_t^s E_i^h - E_i^h + E_i^h - I)\dot{\sigma}|^2 \\ &\ge \frac{7}{8} |\Delta_i^h \dot{\sigma}|^2 - \frac{1}{8} |\Delta_t^s E_i^h \dot{\sigma}|^2 \end{aligned}
$$

i.e. for the space regularity one has to use the estimates in time also.

 $-MAIN INGREDIENTS OF THE PROOF$

LOCAL REGULARITY IN SPACE

Estimates up to the boundary:

- \triangleright W. I. o. g.: Boundary flat,
- \blacktriangleright tangential derivatives like in the interior case
- \blacktriangleright use integrated embedding theorems
- \blacktriangleright still work in progress!

Thank you!

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