Some regularity results for plasticity problems.

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Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

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$\Omega \subset \mathbb{R}^n$, (preferably $n = 3$, Ω solid body)
f density of the body forces
p external loading
t (time-like) loading parameter

$$x \mapsto x + u(x, t) \quad \text{displacement field}$$

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state of the deformed material: u, σ
small deformations, balance of forces

$$-\operatorname{div} \sigma = f$$

Boundary conditions

clamped part:	$u\|_\Gamma = 0, \ \Gamma \subset \partial\Omega,$
external loading:	$\sigma \cdot n = p \text{ on } \partial\Omega \setminus \Gamma$

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$$F(\sigma,\xi) = \begin{cases} |\sigma_D| - (\xi + \kappa) & \text{isotropic hardening} \\ |\sigma_D - \xi_D| - \kappa & \text{kinematic hardening} \end{cases}$$

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Relation between the stress-rate $\dot{\sigma}$ and the strain rate $\dot{\varepsilon}$:
involves the compliance tensor A (symmetric rank 4 tensor)
flow rule for ξ:
involves the hardening tensor $H \in \mathbb{R}^{m\times m}$
A, H positive definite

$\mathbb{K}(t)$: set of all pairs (τ, η) with

$$\tau \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{sym}}), \;\; \eta \in L^2(\Omega, \mathbb{R}^m) \tag{1}$$

τ fulfills the balance of forces in the weak form:

$$(\tau, \nabla\varphi)_\Omega = (f, \varphi)_\Omega + \int_{\partial\Omega} p\varphi \; do \;\; \text{for all } \varphi \in H^1_\Gamma(\Omega). \tag{BF}$$

For isotropic hardening: $m = 1$,

$$\eta \in L^2(\Omega; \mathbb{R}), \qquad |\tau_D| - \eta \le \kappa, \tag{YCI}$$

for *kinematic* hardening: $m = n(n+1)/2$

$$\eta \in L^2(\Omega; \mathbb{R}^{n\times n}_{sym}), \qquad |\tau_D - \eta_D| \le \kappa. \tag{YCK}$$

Given:

$$f, \dot{f} \in L^\infty(0,T;L^\infty(\Omega)), \qquad \ddot{f} \in L^1(0,T;L^2),$$

$$p, \dot{p} \in L^\infty(0,T;L^\infty(\partial\Omega)), \qquad \ddot{p} \in L^1(0,T;L^2(\partial\Omega)),$$

$$(\sigma_0, 0) \in \mathbb{K}(0)$$

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$$(\sigma_0, 0) \in \mathbb{K}(0)$$

Find $\sigma \in L^\infty(L^2)$, $\xi \in L^\infty(L^2)$ such that

$$\dot{\sigma} \in L^2(L^2), \quad \dot{\xi} \in L^2(L^2)$$
$$(\sigma(t), \xi(t)) \in \mathbb{K}(t), \ \ t \in [0,T]$$
$$\sigma(0) = \sigma_0, \quad \xi(0) = 0$$
$$(A\dot{\sigma}, \sigma - \tau) + (H\dot{\xi}, \xi - \eta) \leq 0 \quad \text{a.e. in } [0,T]$$
$$\text{for all } (\tau, \eta) \in \mathbb{K}(t).$$

Definition (Safe load condition)

Exist $\hat{\sigma} \in L^\infty(L^2)$, $\hat{\xi} \in L^\infty(L^2)$:

$$\dot{\hat{\sigma}} \in L^\infty(L^2),\ \ddot{\hat{\sigma}} \in L^1(L^2),\ \dot{\hat{\xi}} \in L^\infty(L^2)$$
$$(\hat{\sigma}(0), 0) \in \mathbb{K}(0),\ \hat{\xi}|_{t=0} = 0$$
$$(\hat{\sigma}(t,.), \hat{\xi}(t,.)) \in \mathbb{K}(t),$$

and exists $\delta > 0$:

$$|\hat{\sigma}_D| - \xi \leq \kappa - \delta \text{ or } |\hat{\sigma}_D - \hat{\xi}_D| \leq \kappa - \delta, \text{ respectively.}$$

Johnson 78: Exists $u \in L^\infty(H^1_\Gamma)$ with $\dot{u} \in L^\infty(H^1_\Gamma)$,

and a multiplier $\dot{\lambda} \in L^\infty(0, T; L^2(\Omega, \mathbb{R}))$ (Frehse & Loebach 08)

s.t. for isotropic hardening:

$$\begin{aligned} \frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}) &= A\dot{\sigma} + \dot{\lambda}\sigma_D|\sigma_D|^{-1} \\ 0 &= H\dot{\xi} - \dot{\lambda} \end{aligned}$$

where $\dot{\lambda} \geq 0$ a.e. and $\dot{\lambda}(|\sigma_D| - \xi - \kappa) = 0$, Johnson 78: Exists $u \in L^\infty(H^1_\Gamma)$ with $\dot{u} \in L^\infty(H^1_\Gamma)$,

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s.t. for kinematic hardening:

$$\begin{aligned} \frac{1}{2}(\nabla \dot{u} + \nabla \dot{u}) &= A\dot{\sigma} + \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D| \\ 0 &= H\dot{\xi} - \dot{\lambda}(\sigma_D - \xi_D)|\sigma_D - \xi_D|. \end{aligned}$$

where $\dot{\lambda} \geq 0$ a.e. and $\dot{\lambda}(|\sigma_D - \xi_D| - \kappa) = 0$,

$$
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strain=elastic strain + plastic strain

$$\dot{\varepsilon} = \dot{\lambda}\frac{\partial}{\partial\sigma}F(\sigma,\xi), \text{ where } \dot{\varepsilon}_{pl} = 0, \textit{if}\, F < 0,$$

$$H\dot{\xi} = -\dot{\lambda}\frac{\partial}{\partial\xi}F(\sigma,\xi)$$

Johnson 78	$\nabla \dot{u} \in L^\infty(L^2)$	
Seregin 94	$\sigma, \xi \in L^\infty(H^1_{loc})$	
	$\nabla(\varepsilon) \in L^\infty(C^*_{loc})$	isotropic hard.
Alber & Nessenenko '09	$\sigma, \xi \in L^\infty(H^{\frac{1}{3}-\delta})$	kinematic hard
Knees 08	$\sigma, \xi \in L^\infty(H^{\frac{1}{2}-\delta})$	kinematic hard
Frehse & Löbach '09	$\sigma, \xi \in L^\infty(H^{\frac{1}{2}-\delta})$	kinem. & isotr. hard
Löbach '09	$\sigma, \xi \in L^\infty(H^{\frac{1}{2}+\delta})$	kinem. & isot. hard
Frehse & Löbach '11	$\nabla \dot{u}, \dot{\sigma}, \dot{\xi} \in L^\infty(L^{2+2\delta})$	kinem. & isotr. hard

$$\Delta_t^s w(t,x) = w(t+h,x) - w(t,x),$$
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Theorem (Regularity in time, Frehse & Sp. 2012)

$$h^{-2}\int_0^h \int_0^{T-h} \int_\Omega \left[|\Delta_t^s \dot\sigma|^2 + |\Delta_t^s \dot\xi|^2\right] \le C$$

uniformly for $0 < h < h_0$.
$\Rightarrow$ *for kinematic hardening:*

$$h^{-2}\int_0^{T-h} \int_0^h \int_\Omega |\Delta_t^s \nabla \dot u|^2 \le C$$

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uniformly for $0 < h < h_0$.

Comment: Even prolongation in time: $\sigma : [-T, T] \to \mathbb{R}^{n\times n}_{sym}$ periodic,

$$\sigma = \sum_{m=-\infty}^{\infty} c_m(x) \exp(\frac{im\pi}{2T}t) \;\Rightarrow$$

$$\sum_{m=-\infty}^{\infty} m^{1-\delta} \int_\Omega |c_m(x)|^2 dy \leq C_\delta \quad \text{for all } \delta > 0.$$

Theorem (Local regularity in space)

$$\sup_{0\leq h\leq h_0} h^{-1} \int_0^{T-h} \int_{\Omega_0} |\Delta_i^h \dot{\sigma}|^2 + |\Delta_i^h \dot{\xi}|^2 \leq C, \ \ i = 1, \ldots, n$$

for any domain Ω_0 *such that* $\overline{\Omega}_0 \subset \Omega$ *and* $h_0 \leq dist(\partial\Omega, \partial\Omega_0)$.

Penalty potential

$$G_\mu(\sigma,\xi) = \frac{1}{2\mu}[F(\sigma,\xi)]_+^2 \qquad \Rightarrow$$

$$\nabla G_\mu = \begin{cases} \frac{1}{\mu}[F]_+ \begin{pmatrix} \sigma_D|\sigma_D|^{-1} \\ -1 \end{pmatrix} & \text{isotr. h.} \\ \frac{1}{\mu}[F]_+ \frac{\sigma_D-\xi_D}{|\sigma_D-\xi_D|} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{kinem. h.} \end{cases}$$

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Find $\sigma_\mu, \xi_\mu \in L^\infty(L^2)$ with $\dot\sigma_\mu, \dot\xi_\mu \in L^\infty(L^2)$,

$$(\sigma_\mu, \xi_\mu)|_{t=0} = (\sigma_0, 0) \tag{IC}$$

$$(\sigma_\mu, \nabla\varphi)_\Omega = (f, \varphi)_\Omega + \int_{\partial\Omega} p\varphi \, do \quad \text{for all } \varphi \in H^1_\Gamma(\Omega). \tag{Bof}$$

$$0 = (A\dot\sigma_\mu + \partial_\sigma G_\mu, \tau)_\Omega \quad \text{for all symmetric } \tau \in \{\nabla\varphi : \varphi \in H^1_\Gamma\}^\perp \tag{P1}$$

$$0 = H\dot\xi_\mu + \partial_\xi G_\mu \tag{P2}$$

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$$0 = H\dot\xi_\mu + \partial_\xi G_\mu \tag{P2}$$

Well known: Problem has a unique solution, along with a sequence of handy priori estimates independent on μ.

For the time regularity

$$0 = (A\dot{\sigma}_\mu + \partial_\sigma G_\mu, \tau)_\Omega \quad \text{(P1)}$$

$$0 = H\dot{\xi}_\mu + \partial_\xi G_\mu \quad \text{(P2)}$$

- test (P1) with $\int_0^h \Delta_t^s \dot{\sigma}_\mu ds$ and (P2) with $\int_0^h \Delta_t^s \dot{\xi}_\mu ds$
- use the elementary relation

$$A\tau : \Delta_t^s \tau = -\frac{1}{2} A \Delta_t^s \tau : \Delta_t^s \tau + \frac{1}{2} \Delta_t^s (A\tau : \tau),$$

Arrive at

$$\int_{t_1}^{t_2-h}\int_0^h (A\Delta_t^s\dot{\sigma}_\mu, \Delta_t^s\dot{\sigma}_\mu)_\Omega + (H\Delta_t^s\dot{\xi}_\mu, \Delta_t^s\dot{\xi}_\mu)_\Omega$$

$$= \int_{t_1}^{t_2-h}\int_0^h\int_\Omega \Delta_t^s(A\dot{\sigma}_\mu : \dot{\sigma}_\mu) + \Delta_t^s(H\dot{\xi}_\mu : \dot{\xi}_\mu)$$

$$+ \text{term with } G_\mu - 2\int_{t_1}^{t_2-h}\int_0^h (\nabla\dot{u}_\mu, \Delta_t^s\dot{\sigma}_\mu)_\Omega$$

In the limit $\mu \to 0$, $t_1 \to 0$ $t_2 \to T$:

- $\{\dots\} \geq C \int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s \dot{\sigma}_\mu|^2 + |\Delta_t^s \dot{\xi}_\mu|^2$
- $\{\dots\} \leq C(\|\dot{\sigma}_\mu\|_{L^\infty(L^2)} + \|\dot{\xi}_\mu\|_{L^\infty(L^2)})h^2$
- $\limsup \{\dots\} \leq 0$ (use the convexity of the the penalty potential and the following convergence result

$$\int_0^T \int_\Omega G_\mu(\sigma_\mu, \xi_\mu) \to 0 \quad \text{as } \mu \to 0,$$

- $\{\dots\} \leq Ch^2$ (use the safe load and bounds for $\|\nabla u\|_{L^\infty(L^2)}$)

Local regularity in space

Test (P1), (P2) with
$\zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu = \dot{\sigma}_\mu(t+s, x+he_i) - \dot{\sigma}_\mu(t,x)$, $\zeta^2 \ldots \dot{\xi}_\mu$
ζ: Localization function
In principle the arguments are similar, but in detail even more tricky as for the time direction.

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In case you wonder:

$$\begin{aligned} |(E_t^s E_i^h - I)\dot{\sigma}|^2 &= |(E_t^s E_i^h - E_i^h + E_i^h - I)\dot{\sigma}|^2 \\ &\geq \frac{7}{8}|\Delta_i^h \dot{\sigma}|^2 - \frac{1}{8}|\Delta_t^s E_i^h \dot{\sigma}|^2 \end{aligned}$$

i.e. for the space regularity one has to use the estimates in time also.

Estimates up to the boundary:

- W. l. o. g.: Boundary flat,
- tangential derivatives like in the interior case
- use integrated embedding theorems
- ... still work in progress!

Thank you!