A Remark on Transport Equation with $\boldsymbol{b} \in BV$ and $\text{div}_x \boldsymbol{b} \in BMO$.

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INNOVATIVE ECONOMY
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EUROPEAN REGIONAL DEVELOPMENT FUND

Problem

Investigate:

$$\begin{cases} \partial_t u(t,x) + \boldsymbol{b}(t,x) \cdot \nabla u(t,x) = 0, \\ u(0,x) = \bar{u}(x), \end{cases}$$

where $\boldsymbol{b}(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\bar{u}(x) : \mathbb{R}^d \mapsto \mathbb{R}^d$ are given and $u(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ is unknown.

Problem: What regularity of $\boldsymbol{b}$ guarantees uniqueness of the solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$?

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Method of characteristics

Assume: $\boldsymbol{b}$ - smooth, $\bar{u}$ - smooth.

Associated flow: $x_0 \mapsto X(t; x_0)$ - diffeomorphism,

$$\begin{cases} \dot{X}(t; x_0) = \boldsymbol{b}(t, X(t; x_0)) \\ X(0; x_0) = x_0 \quad \forall_{x_0 \in \mathbb{R}} \forall_{t \in \mathbb{R}_+} \end{cases}$$

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Irregular transport

Motivation: Shock waves.

Weak formulation: Let $\bar{u} \in L^\infty((0,T)\times\mathbb{R}^d)$, $\boldsymbol{b}, \mathrm{div}\boldsymbol{b} \in L^1_{loc}(0,T;L^1_{loc}(\mathbb{R}^d))$. We say that $u \in L^\infty((0,T)\times\mathbb{R}^d)$ is a weak solution to transport equation iff the following integral identity holds

$$\int_0^T \int_{\mathbb{R}^d} u\{\partial_t\varphi + \boldsymbol{b}\cdot D_x\varphi + \varphi \mathrm{div}_x \boldsymbol{b}\} dtdx = -\int_{\mathbb{R}^d} \bar{u}(x)\varphi(0,x)dx$$

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DiPerna & Lions	Ambrosio	Mucha
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Renormalization generalization $W^{1,1} \Rightarrow BV$

Theorem (Ambrosio): Let $\boldsymbol{b} \in L^1_{loc}(0,T; BV_{loc}(\mathbb{R}^d))$, $\mathrm{div}\boldsymbol{b} \ll \mathcal{L}^d$. Then $\boldsymbol{b}$ has the renormalization property.

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Difference quotients decomposition

For $B \in BV(\mathbb{R}^d)$ we have $DB = D^a B + D^s B$ and $D^a B = \nabla B \mathcal{L}^d$.

Decomposition Lemma Let $B \in BV_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ and $z \in \mathbb{R}^d$. Then there exists a decomposition

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Additionally we have

$$\limsup_{\epsilon \to 0} \int_K |B_z^{2,\epsilon}(x)| dx \leq |D^s B \cdot z|(K)$$

for each compact $K \subset \mathbb{R}^d$.

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$$\limsup_{\epsilon\to 0} \int_I \int_K |R_{\epsilon}^2(x)| dxdt$$
$$\leq ||u||_{L^\infty} \int_{\mathbb{R}^d} |\nabla\eta(z) \cdot M(t,x) \cdot z| dz \; |D^s\boldsymbol{b}|(I \times K).$$

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Using the identity $D^s\boldsymbol{b}(t,\cdot) = M(t,x)|D^s\boldsymbol{b}(t,\cdot)|$ (where $M(t,x) : I \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is the Radon-Nikodym derivative of $D^s\boldsymbol{b}(t,\cdot)$ with respect to $|D^s\boldsymbol{b}(t,\cdot)|$). Estimate

$$\limsup_{\epsilon\to 0} \int_I \int_K |R_\epsilon^2(x)| dxdt$$
$$\le ||u||_{L^\infty} \int_{\mathbb{R}^d} |\nabla\eta(z) \cdot M(t,x) \cdot z| dz \; |D^s\boldsymbol{b}|(I \times K).$$

Defect measure

$$\partial_t \beta(u_\epsilon) + \boldsymbol{b} \cdot \nabla \beta(u_\epsilon) = \beta'(u_\epsilon) R_\epsilon,$$

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$$\mathcal{K} := \left\{\eta \in C_c^\infty(B_1(0)) : \eta \geq 0 \ \text{ even and } \int_{B_1(0)} \eta = 1\right\},$$

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We had:

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Steps of the proof of the logarithmic theorem: **dual space** to $\mathcal{H}^1(\mathbb{T}^d)$ is $BMO(\mathbb{T}^d)$, hence

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Main Theorem: Let
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Thank you!