Higher integrability of generalized Stokes system under perfect slip boundary conditions.

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Contents

Incompressible generalized Stokes equations with perfect slip boundary conditions

We investigate properties of the weak solution $u : \Omega \to \mathbb{R}^n$ and $\pi : \Omega \to \mathbb{R}$ solving:

$Stokes_{\mathrm{p.slip}}$

$$-\operatorname{div} \mathcal{S}(Du) + \nabla\pi = \operatorname{div} F \; in \; \Omega, \tag{1}$$

$$\operatorname{div} u = 0 \; in \; \Omega, \tag{2}$$

$$u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau = 0 \; on \; \partial\Omega, \tag{3}$$

where u is velocity, Du symmetric part of velocity gradient, $\operatorname{div} F$ density of volume forces, π kinematic pressure and Cauchy stress tensor $\mathcal{T}$ has the form $\mathcal{T} = -\pi\mathcal{I} + \mathcal{S}$.

N-functions

Definition

A real function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called N-function if there exists the derivative Φ' which is right continuous for $s \geq 0$, positive for $s > 0$, non-decreasing and satisfies $\Phi'(0) = 0$ and $lim_{s\to\infty}\Phi'(s) = \infty$.

By $(\Phi')^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ we denote the function

$$(\Phi')^{-1}(s) := \sup\{t \in \mathbb{R}^+ : \Phi'(t) \leq s\}.$$

The complementary function of Φ (which is again an N-function) is defined as

$$\Phi^*(s) := \int_0^s (\Phi')^{-1}(t)\,\mathrm{d}t.$$

Definition

N-function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if there exists a positive constant C, such that $\Phi(2s) \leq C\Phi(s)$ for $s > 0$. By $\Delta_2(\Phi)$ we denote the smallest such constant C.

Assumptions

We can construct $\Phi : [0, \infty) \mapsto [0, \infty)$ to $\mathcal{S}$, i.e.

$$\mathcal{S}_{ij}(A) = \partial_{ij}\Phi(|A|) = \Phi'(|A|)\frac{A_{ij}}{|A|} \quad \forall A \in \mathbb{R}^{n\times n}_{sym}.$$

Assumption 1

We suppose that $\Phi \in \mathcal{C}^{1,1}(0,\infty) \cap \mathcal{C}^1[0,\infty)$ is N-function, $\Phi \in \Delta_2$, $\Phi^* \in \Delta_2$ and $\Phi'(s) \sim s\Phi''(s)$ holds for all $s > 0$, i.e. there exist constants $C, c > 0$ such that, for $s > 0$

$$C\Phi'(s) \leq s\Phi''(s) \leq c\Phi'(s).$$

Some results will be valid for almost monotone $\Phi''(s)$, i.e. either
$\Phi''(s) \leq C\Phi''(t)\ \forall 0 < s \leq t$ (almost increasing) or
$\Phi''(s) \geq C\Phi''(t)\ \forall 0 < s \leq t$ (almost decreasing).
Define $V(A) = \sqrt{\Phi'(|A|)|A|}\frac{A}{|A|}$. It holds $|V(A)|^2 \sim \Phi(|A|)$.

Examples

We can consider models with a great deal of disparity, for example power-law models

$$\mathcal{S}(Du) = \mu_0 (1 + |Du|^2)^{\frac{p-2}{2}} Du, \quad \Phi(|Du|) = \mu_0 \int_0^{|Du|} (1+s^2)^{\frac{p-2}{2}} s \, \mathrm{d}s,$$

$$\mathcal{S}(Du) = \mu_0 (1 + |Du|)^{p-2} Du, \quad \Phi(|Du|) = \mu_0 \int_0^{|Du|} (1+s)^{p-2} s \, \mathrm{d}s,$$

$\mu_0 \in \mathbb{R}^+$, $p \in (1, \infty)$. Also the singular case

$$\mu(Du) = \mu_0 |Du|^{p-2} Du, \quad \Phi(|Du|) = \mu_0 \int_0^{|Du|} s^{p-1} \, \mathrm{d}s$$

is included.

Main result

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^{2,1}$ domain, $\Omega_Q = \Omega \cap Q$. Let Assumption 1 be fulfilled and u be a weak solution to (1)–(3). Then the following implication holds

$$\Phi^*(|F|) \in L^q(\Omega_{8Q}) \Rightarrow \Phi(|Du|) \in L^q(\Omega_{\frac{1}{2}Q}),$$

provided $q \in (1, \infty)$ for $n = 2$ and $q \in \left(1, \frac{n}{n-2}\right)$, resp. $q \in \left(1, \frac{n}{n-2} + \delta\right)$ for $n > 2$ and some $\delta > 0$ in case Φ'' is almost monotone.
Moreover, it holds

$$\fint_{\Omega_{\frac{1}{2}Q}} \Phi(|Du|)^q \,\mathrm{d}x \leq c \left(\fint_{\Omega_{8Q}} \Phi^*(|F|)^q \,\mathrm{d}x + \fint_{\Omega_{8Q}} \Phi\left(|u|\right)^q \,\mathrm{d}x \right) + c \left(\fint_{\Omega_{8Q}} \Phi(|Du|) \,\mathrm{d}x \right)^q . \quad (4)$$

Remarks

- Our goal is the regularity up to the boundary for perfect slip boundary conditions, since the interior regularity was proven in

L. Diening, P. Kaplický: 2013

L^q theory for a generalized Stokes system, Manuscripta Mathematica

- **Key parts of the proof:** comparison with the homogeneous system, flattening boundary, extension solution beyond flat boundary.
- **Structure of the proof:** similar as in the paper by L. Diening and P. Kaplický. It is based on the approach published in

L. A. Caffarelli, I. Peral: 1998

On $W^{1,p}$ estimates of elliptic equation in divergence form, Comm. Pure and Appl. Math.

Lemma

Let $1 \le p < q < s < \infty$, $f \in L^{q/p}((4Q)^+)$, $g \in L^{q/p}((4Q)^+)$ and $w \in L^p((4Q)^+)^n$, where $(4Q)^+ = 4Q \cap \mathbb{R}^n_+$. Let Q_k be dyadic cubes obtained from Q with predecessor $\tilde{Q}_k$. If $\exists \varepsilon \in (0, \varepsilon_0)$ $\forall Q_k \subset Q$ $\exists w_a \in L^p((4\tilde{Q}_k)^+)^n$ with following properties:

$$\begin{aligned}
\left(\fint_{(2\tilde{Q}_k)^+} |w_a|^s \,\mathrm{d}x \right)^{\frac{1}{s}} &\le \frac{C}{2} \left(\fint_{(4\tilde{Q}_k)^+} |w_a|^p \,\mathrm{d}x \right)^{\frac{1}{p}}, \\
\fint_{(4\tilde{Q}_k)^+} |w_a|^p \,\mathrm{d}x &\le C \fint_{(4\tilde{Q}_k)^+} |w|^p \,\mathrm{d}x + C \fint_{(4\tilde{Q}_k)^+} |g| \,\mathrm{d}x, \\
\fint_{(4\tilde{Q}_k)^+} |w - w_a|^p \,\mathrm{d}x &\le \varepsilon \fint_{(4\tilde{Q}_k)^+} |w|^p \,\mathrm{d}x + C \fint_{(4\tilde{Q}_k)^+} |f| \,\mathrm{d}x,
\end{aligned}$$

then $w \in L^q(Q)^n$. Furthermore,

$$\fint_Q |w|^q \,\mathrm{d}x \le c \left(\fint_{(4Q)^+} |f|^{\frac{q}{p}} \,\mathrm{d}x + \fint_{(4Q)^+} |g|^{\frac{q}{p}} \,\mathrm{d}x + \left(\fint_{(4Q)^+} |w|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right).$$

Lemma

Let $1 \le p < q < s < \infty$, $f \in L^{q/p}((4Q)^+)$, $g \in L^{q/p}((4Q)^+)$ *and* $w \in L^p((4Q)^+)^n$, *where* $(4Q)^+ = 4Q \cap \mathbb{R}^n_+$. *Let* Q_k *be dyadic cubes obtained from* Q *with predecessor* $\tilde{Q}_k$. *If* $\exists \varepsilon \in (0, \varepsilon_0)$ $\forall Q_k \subset Q$ $\exists w_a \in L^p((4\tilde{Q}_k)^+)^n$ *with following properties:*

$$\begin{aligned}
\left(\fint_{(2\tilde{Q}_k)^+} |w_a|^s \,\mathrm{d}x \right)^{\frac{1}{s}} &\le \frac{C}{2} \left(\fint_{(4\tilde{Q}_k)^+} |w_a|^p \,\mathrm{d}x \right)^{\frac{1}{p}}, \\
\fint_{(4\tilde{Q}_k)^+} |w_a|^p \,\mathrm{d}x &\le C \fint_{(4\tilde{Q}_k)^+} |w|^p \,\mathrm{d}x + C \fint_{(4\tilde{Q}_k)^+} |g| \,\mathrm{d}x, \\
\fint_{(4\tilde{Q}_k)^+} |w - w_a|^p \,\mathrm{d}x &\le \varepsilon \fint_{(4\tilde{Q}_k)^+} |w|^p \,\mathrm{d}x + C \fint_{(4\tilde{Q}_k)^+} |f| \,\mathrm{d}x,
\end{aligned}$$

then $w \in L^q(Q)^n$. *Furthermore,*

$$\fint_Q |w|^q \,\mathrm{d}x \le c \left(\fint_{(4Q)^+} |f|^{\frac{q}{p}} \,\mathrm{d}x + \fint_{(4Q)^+} |g|^{\frac{q}{p}} \,\mathrm{d}x + \left(\fint_{(4Q)^+} |w|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right).$$

Consider the homogeneous system

$$\begin{aligned} -\operatorname{div}\mathcal{S}(Dv)+\nabla p &= 0 \qquad \text{in } (2Q)^+,\\ \operatorname{div} v &= 0 \qquad \text{in } (2Q)^+,\\ v\cdot\nu = 0, \quad [\mathcal{S}(Dv)\nu]\cdot\tau &= 0 \qquad \text{on } \Gamma_{(2Q)^+}. \end{aligned} \tag{5}$$

Theorem

Let $v \in W^{1,\Phi}((2Q)^+)^n$ be a local weak solution to (5). Then there exists a constant C independent of v and R such that

$$\left(\fint_{Q^+}|V(Dv)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{(2Q)^+}|V(Dv)|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

for $q \in \left[2, \frac{2n}{n-2}\right]$ provided $n>2$ and $q\in[2,\infty)$ for $n=2$. In case Φ'' is almost monotone, $n>2$, we can even allow $q=\frac{rn}{n-r}$ for some $r>2$.

Proof of the theorem 1/2

At first we extend the solution from $(2Q)^+$ to $2Q$. For $\alpha = 1, \dots, n-1$ define $\tilde{v}$ as follows

$$\tilde{v}_\alpha(x', x_n) = \begin{cases} v_\alpha(x', x_n) & \text{for } x_n > 0, \\ v_\alpha(x', -x_n) & \text{for } x_n < 0, \end{cases}$$

$$\tilde{v}_n(x', x_n) = \begin{cases} v_n(x', x_n) & \text{for } x_n > 0, \\ -v_n(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Lemma

$\tilde{v} \in W^{1,\Phi}(2Q)^n$ is a local weak solution to (5) extended to $2Q$.

$$⨍_Q |\nabla V(D\tilde{v})|^2 \, \mathrm{d}x \le \frac{C}{R^2} \Big(⨍_{2Q} |V(D\tilde{v})|^2 \, \mathrm{d}x \Big). \tag{6}$$

For almost monotone Φ'' the estimate (6) can be improved to

$$⨍_Q |\nabla V(D\tilde{v})|^2 \, \mathrm{d}x \le \frac{C}{R^2} \Big(⨍_{2Q} |V(D\tilde{v}) - \langle V(D\tilde{v}) \rangle_{2Q}|^2 \, \mathrm{d}x \Big). \tag{7}$$

Proof of the lemma

- By reflection we have the interior problem, which is proven in [DK].
- We focus only to generalization from $n = 3$ to arbitrary $n \in \mathbb{N} \setminus \{1\}$.
- We can't test the weak formulation (5) by $\varphi = \mathrm{curl}[\xi^2(\mathrm{curl}(\tilde{v} - q))]$, since $curl : (v_1, v_2, v_3) \to (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)$.
- We test by $\varphi = \left(\star d[\xi^2 \star d(\tilde{v} - q)^\flat]\right)^\sharp$.
  ($\flat$ converts the vector field $(\tilde{v} - q)$ into a 1-form $(\tilde{v} - q)^\flat$. The exterior derivative d computes something like a curl but expressed as a 2-form $d(\tilde{v} - q)^\flat$. The Hodge map $\star$ turns this 2-form into a $(n-2)$-form. After multiplication by ξ^2 and application of the derivative d we obtain $(n-1)-$form and Hodge star $\star$ create $1-$form, which is by $\sharp$ converted to the vector.
- In components:
  $\varphi = \sum_{i,j=1}^n \Big(-\xi^2 \partial_i^2 (\tilde{v})_j + 2\xi\partial_i\xi[-\partial_i(\tilde{v} - q)_j + \partial_j(\tilde{v} - q)_i] \Big) e_j,$

Proof of the theorem 2/2

- Follows from the lemma by application of Sobolev-Poincaré inequality (resp. Sobolev-Poincaré inequality and reverse Hölder inequality).
- $\tilde{v} \to v$ and $Q \to Q^+$, resp. $2Q \to (2Q)^+$.

Lemma

Let $1 \le p < q < s < \infty$, $f \in L^{q/p}((4Q)^+)$, $g \in L^{q/p}((4Q)^+)$ *and* $w \in L^p((4Q)^+)^n$, *where* $(4Q)^+ = 4Q \cap \mathbb{R}^n_+$. *Let* Q_k *be dyadic cubes obtained from* Q *with predecessor* $\tilde{Q}_k$. *If* $\exists \varepsilon \in (0, \varepsilon_0)$ $\forall Q_k \subset Q$ $\exists w_a \in L^p((4\tilde{Q}_k)^+)^n$ *with following properties:*

$$\left(\fint_{(2\tilde{Q}_k)^+} |w_a|^s \, \mathrm{d}x \right)^{\frac{1}{s}} \le \frac{C}{2} \left(\fint_{(4\tilde{Q}_k)^+} |w_a|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

$$\fint_{(4\tilde{Q}_k)^+} |w_a|^p \, \mathrm{d}x \le \fint_{(4\tilde{Q}_k)^+} |w|^p \, \mathrm{d}x + C \fint_{(4\tilde{Q}_k)^+} |g| \, \mathrm{d}x,$$

$$\fint_{(4\tilde{Q}_k)^+} |w - w_a|^p \, \mathrm{d}x \le \varepsilon \fint_{(4\tilde{Q}_k)^+} |w|^p \, \mathrm{d}x + C \fint_{(4\tilde{Q}_k)^+} |f| \, \mathrm{d}x,$$

then $w \in L^q(Q)^n$. *Furthermore,*

$$\fint_Q |w|^q \, \mathrm{d}x \le c \left(\fint_{(4Q)^+} |f|^{\frac{q}{p}} \, \mathrm{d}x + \fint_{(4Q)^+} |g|^{\frac{q}{p}} \, \mathrm{d}x + \left(\fint_{(4Q)^+} |w|^p \, \mathrm{d}x \right)^{\frac{q}{p}} \right).$$

Flattening the boundary

- $H_R : Q^+ \mapsto \Omega_R := H_R(Q^+) \subset \Omega$.
- It holds that $H_R(0) = x_0$ and $\nabla H_R(0) = I$.
- $\nabla H_R(x) - \nabla H_R(0) = R\omega$, $\nabla H_R^{-1}(x) - \nabla H_R^{-1}(0) = R\omega$.
- For $f : \Omega_R \mapsto \mathbb{R}$ we state $\overline{f} : Q^+ \mapsto \mathbb{R}$ defined as $\overline{f}(x) = f(H_R(x)) = f(y)$.

$$
\begin{aligned}
2D_y f &= \left(\nabla_x \overline{f} \nabla_x H_R^{-1}\right) + \left(\nabla_x \overline{f} \nabla_x H_R^{-1}\right)^T = 2(D_x \overline{f} + Z_{\overline{f}}), \\
Z_{\overline{f}} &= \frac{1}{2}\left(\nabla_x \overline{f}(\nabla_x H_R^{-1} - I) + (\nabla_x H_R^{-1} - I)^T (\nabla_x \overline{f})^T\right) \\
&= \frac{R}{2}\left(\nabla_x \overline{f}\omega + (\nabla_x \overline{f}\omega)^T\right), \\
\operatorname{div}_y f &= \operatorname{Tr}\left(\nabla_x \overline{f} \nabla_x H_R^{-1}\right) = \operatorname{div}_x \overline{f} + \operatorname{Tr}(\nabla_x \overline{f}(\nabla_x H_R^{-1} - I)) \\
&= \operatorname{div}_x \overline{f} + R\operatorname{Tr}(\nabla_x \overline{f}\omega).
\end{aligned}
$$

Transformation of the weak formulation

We can transform the weak formulation

$$\int_{\Omega_R} \mathcal{S}(Du) : D\varphi \,\mathrm{d}y - \int_{\Omega_R} \pi \operatorname{div} \varphi \,\mathrm{d}y = \int_{\Omega_R} F : D\varphi \,\mathrm{d}y, \tag{8}$$

which holds for all $\varphi \in W^{1,\Phi}(\Omega)^n$, $\varphi \cdot \nu = 0$ on $\partial\Omega$ and $\varphi = 0$ on $\partial\Omega_R \setminus \partial\Omega$ into

$$\begin{aligned} \int_{Q^+} &\mathcal{S}(D\overline{u} + Z_{\overline{u}}) : (D\psi + Z_\psi + \omega'\psi)(1 + R\omega'') \,\mathrm{d}x \\ &- \int_{Q^+} \overline{\pi}\big(\operatorname{div} \psi + R \operatorname{Tr}(\omega\nabla\psi) + \operatorname{Tr}(\omega'\psi)\big)(1 + R\omega'') \,\mathrm{d}x \\ &= \int_{Q^+} \overline{F} : (\nabla\psi + R\omega\nabla\psi + \omega'\psi)(1 + R\omega'') \,\mathrm{d}x, \end{aligned} \tag{9}$$

which holds for all $\psi \in W^{1,\Phi}(Q^+)$, $\psi \cdot e_n = 0$ on Γ_{Q^+} and $\psi = 0$ on $\partial Q^+ \setminus \Gamma_{Q^+}$.

Decomposition of $\overline{u}$

Since $\operatorname{div}\overline{u} \neq 0$ in Q^+ and $\overline{u}\cdot e_n \neq 0$ on Γ_{Q^+}, we define a function $\overline{u}_2$ as a solution to

$$\begin{aligned} \operatorname{div}\overline{u}_2 &= -R\,\mathrm{Tr}(\nabla\overline{u}\omega) \text{ in } Q^+, \\ \overline{u}_2\cdot e_n &= R(\omega\overline{u})\cdot e_n \text{ on } \Gamma_{Q^+}. \end{aligned}$$

The boundary condition (10) comes from the fact that we want

$$\overline{u}_2\cdot e_n = \overline{u}\cdot e_n = (I-(\nabla H_R)^T)\overline{u}\cdot e_n + (\nabla H_R)^T\overline{u}\cdot e_n = R(\omega\overline{u})\cdot e_n \text{ on } \Gamma_{Q^+},$$

To obtain estimates of $\overline{u}_2$ in terms of $\overline{u}$ we use

Lemma (Bogovskiĭ)

$\Delta_2(\{\Phi^*,\Phi\}) < \infty$, $g \in L^\Phi(Q^+)$, $h \in W^{1,\Phi}(Q^+)$

$$\operatorname{div} z = g \quad in\ Q^+, \tag{10}$$

$$z\cdot\nu = h\cdot\nu \quad on\ \Gamma_{Q^+}, \tag{11}$$

$$\int_{Q^+}\Phi(|\nabla z|)\,\mathrm{d}x \le C\left(\int_{Q^+}\Phi(|g|)\,\mathrm{d}x + \int_{Q^+}\Phi(|\nabla h|)\,\mathrm{d}x\right), \tag{12}$$

- Define $\overline{u}_1 = \overline{u} - \overline{u}_2$.
- $\overline{u}_1 \cdot e_n = 0$ on Γ_{Q^+} and $\operatorname{div} u_1 = 0$ in Q^+.
- Construct v in Q^+ such that

$$v = \overline{u}_1 \text{ on } \partial Q^+ \setminus \Gamma_{Q^+},$$
$$v \cdot e_n = 0,\ [\mathcal{S}(Dv)e_n] \cdot e_\alpha = 0 \text{ on } \Gamma_{Q^+},$$

- Test weak formulation of (5) and (9) by $\varphi = \overline{u}_1 - v$ and obtain:

Lemma

$$\int_{Q^+} |V(Dv)|^2 \,\mathrm{d}x \le C \int_{Q^+} |V(D\overline{u})|^2 \,\mathrm{d}x + CR^{\alpha} \int_{Q^+} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x,$$

for some $\alpha > 1$ *and* $\theta \in (0,1)$. *Furthermore, for all* δ *there exists a positive constant* C_δ *independent of* v, $\overline{u}$ *and* Q^+ *such that*

$$\int_{Q^+} |V(D\overline{u}) - V(Dv)|^2 \,\mathrm{d}x \le C_\delta \int_{Q^+} \Phi^*(|F|) \,\mathrm{d}x + (\delta + CR^{\alpha}) \int_{Q^+} |V(D\overline{u})|^2 \,\mathrm{d}x + C \int_{Q^+} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x.$$

The End

Thank you for your attention.