Conformal composition operators and Brennan's conjecture

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In the present talk we consider a connection between problems of the classic complex analysis, like Brennan's Conjecture, and problems of the modern analysis, like Sobolev embedding theorems and solvability of elliptic equations.

This connection is given by means the theory of composition operators on Sobolev spaces.

Composition operators on Sobolev spaces have applications in the Sobolev type embedding theorems. The main idea of these applications can be demonstrated by the next diagram (V. Gol'dshtein and L. Gurov, 1994):

$$\begin{array}{ccc} W^1_p(\Omega') & \xrightarrow{\varphi^*} & W^1_q(\Omega) \\ \downarrow & & \downarrow \\ L_s(\Omega') & \xleftarrow{(\varphi^{-1})^*} & L_r(\Omega) \end{array}$$

In this diagram Ω is a regular domain supported by the Poincaré-Sobolev inequality (for, example, the unit ball) and Ω' is a non-regular domain.

This approach was suggested by V. Gol'dshtein and L. Gurov in 1994 for exact Sobolev embedding theorems in non-regular domains and was applied by V. Gol'dstein and A. Ukhlov (2009) to weighted embedding theorems. Note that such type embedding theorems lead to solvability of elliptic equations in Sobolev spaces.

The main difficulty in this scheme is to construct an appropriate homeomorphism (the change of variables) $\varphi : \Omega \to \Omega'$ which generate by the composition rule $\varphi^*(f) = f \circ \varphi$ a bounded operator

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$$

Recall the analytic description of homeomorphisms which generate bounded composition operators (A. Ukhlov, 1993).

Theorem. *A homeomorphism $\varphi:\Omega\to\Omega'$ between two domains $\Omega,\Omega'\subset\mathbb{R}^n$, $n\geq 2$, induces a bounded composition operator*

$$\varphi^*: L^1_p(\Omega')\to L^1_q(\Omega),\ \ 1\leq q<p<\infty,$$

if and only if $\varphi\in W^1_{1,loc}(\Omega)$, has finite distortion, and

$$K_{p,q}(f;\Omega)=\left(\int\limits_{\Omega}\left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}<\infty.$$

If $p=q$ then the sufficient and necessary analytic condition is:

$$K_p(f;\Omega)=\operatorname*{ess\,sup}_{x\in\Omega}\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}<\infty.$$

In the case $p=n$ we have the definition of mappings of bounded distortion (S. K. Vodop'yanov and V. Gol'dshtein, 1975) and we call mappings that generate bounded composition operators as mappings of bounded (p,q)-distortion.

The homeomorphisms that generate bounded composition operators of Sobolev spaces $L^1_1(\Omega')$ and $L^1_1(\Omega)$ were introduced by V. G. Maz'ya (1969) as a class of sub-areal mappings. This pioneering work established a connection between geometrical properties of homeomorphisms and corresponding Sobolev spaces.

The composition operators on Sobolev spaces were studied by
S. K. Vodop'yanov and V. Gol'dshtein, 1975, for $p = n$,
S. K. Vodop'yanov and V. Gol'dshtein, 1976, for $p > n$,
V. Gol'gshtein and A. S. Romanov, 1984, for $n - 1 < p < n$,
I. Markina, 1990, for $1 \leq p < n$.

The detailed study of composition operators on Sobolev spaces was carried by A. Ukhlov and S. K. Vodop'yanov in 2002.

In the recent decade composition operators on Sobolev spaces were studied by S. Hencl, P. Koskela and L. Kleprlik. For example, L. Kleprlik (2012) proved (by a different method) that homeomorphisms of bounded (p, q)-distortion generate a bounded composition operators on Sobolev spaces.

The seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, consists of locally summable, weakly differentiable functions $f : \Omega \to \mathbb{R}$ with the finite seminorm:

$$\|f \mid L^1_p(\Omega)\| = \||\nabla f| \mid L_p(\Omega)\|.$$

Note that every element $[f] \in L^1_p(\Omega) \subset L_{1,\mathrm{loc}}(\Omega)$ is an equivalence class of locally integrable functions that coincide a. e. in Ω. In the case $p > n$ the Lebesgue redefinition $\tilde{f}$ of a function $f \in [f]$:

$$\tilde{f}(x) = \lim_{r \to 0} \frac{1}{B(x,r)} \int\limits_{B(x,r)} f(y)\, dy, \;\; f \in [f],$$

is defined at every point $x \in \Omega$ and by the Sobolev embedding theorem is a continuous function.

Hence for all functions $f \in [f]$ we have a unique redefined continuous function $\tilde{f} \in L^1_p(\Omega)$ and we take this function as an element of Sobolev space $L^1_p(\Omega)$ for $p > n$.

Define the Sobolev space $W^1_p(\Omega)$, $1 \leq p < \infty$, as a normed space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \Bigg(\int\limits_{\Omega} |f(x)|^p \, dx \Bigg)^{1/p} + \Bigg(\int\limits_{\Omega} |\nabla f(x)|^p \, dx \Bigg)^{1/p}.$$

The Sobolev space $\overset{\circ}{W}{}^1_p(\Omega)$, $1 \leq p < \infty$, is defined as the closure of the space of smooth functions with compact supports $C^\infty_0(\Omega)$ in the norm of $W^1_p(\Omega)$.

In the present talk we consider the planar case $n = 2$. We have used the following notations: $z = x + iy$ is a complex number, $f(z) = f(x, y)$ is a real-valued function, $\nabla f(z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is the weak gradient of f, $d\mu = dxdy$ is the Lebesgue measure.
We study the weighted Poincaré-Sobolev inequalities

$$\left(\iint\limits_{\Omega} |f(z)|^r h(z)\, dxdy\right)^{\frac{1}{r}} \leq K \left(\iint\limits_{\Omega} |\nabla f(z)|^p dxdy\right)^{\frac{1}{p}} \qquad (1)$$

for functions f of the Sobolev space $\overset{\circ}{W}{}^1_p(\Omega)$ and special weights $h(z) := J(z, \varphi) = |\varphi'(z)|^2$ induced by conformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$.
The weight $h(z)$ can be defined also in the terms of the hyperbolic geometry of Ω and $\mathbb{D}$. Namely

$$h(z) = \lambda_{\Omega}^2(z) / \lambda_{\mathbb{D}}^2(\varphi(z))$$

when $\lambda_{\mathbb{D}}$ and λ_{Ω} are hyperbolic metrics in $\mathbb{D}$ and Ω.

On the first part of the talk we study the "conformal" case $p = 2$. In this case we have the following well known property of conformal mappings:

Lemma. *Let Ω and Ω' be two plane domains. Any conformal homeomorphism $w = \varphi(z) : \Omega \to \Omega'$ induces an isometry of spaces $L^1_2(\Omega')$ and $L^1_2(\Omega)$.*

Definition. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain with non-empty boundary and let φ be a conformal homeomorphism of Ω onto the unit disc $\mathbb{D}$. We call the smooth positive real-valued function $h(z) = J(z, \varphi) = |\varphi'(z)|^2$ the universal conformal weight in Ω (or simply the conformal weight).*

Note, that two different conformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$ and $\tilde{\varphi} : \Omega \to \mathbb{D}$ can be connected by a conformal automorphism $\eta : \mathbb{D} \to \mathbb{D}$ (i.e., $\varphi = \tilde{\varphi} \circ \eta$), the conformal weights induced by φ and $\tilde{\varphi}$ are equivalent. It means that

$$h(z) = J(z, \varphi) \lesssim J(z, \tilde{\varphi}) \lesssim J(z, \varphi) = h(z)$$

and therefore the weighted Lebesgue space $L_q(\Omega, h)$ does not depend on the choice of a conformal homeomorphism and depends only on the conformal structure (hyperbolic geometry) of Ω. This is a reason to call the weight $h(z)$ the (universal) conformal weight on Ω.

As a consequence of the Lemma we have the universal Sobolev-Poincaré inequality in simply commected plane domains:

Theorem. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain with non-empty boundary. Then the inequality*

$$\|f \mid L_r(\Omega, h)\| \leq K_r \|\nabla f \mid L_2(\Omega)\|$$

holds for every function $f \in C_0^\infty(\Omega)$ and for any $1 \leq r < \infty$. Here K_r is a constant depending only on r.

Remark. The constant K_r is equal to the exact constant for the corresponding Poincaré-Sobolev inequality in the unit disc $\mathbb{D} \subset \mathbb{C}$, i. e., for

$$\|f \mid L_r(\mathbb{D})\| \leq K_r \|\nabla f \mid L_2(\mathbb{D})\|, \ \ f \in C_0^\infty(\mathbb{D}).$$

This inequalities leads to

Theorem. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain with non-empty boundary. Then the embedding operator*

$$j_r : \overset{\circ}{W}{}^1_2(\Omega) \hookrightarrow L_r(\Omega, h)$$

is compact for any $1 \leq r < \infty$. Here h is the universal conformal weight.

Now we give some examples of conformal weights.

Example 1. Let $\Omega_{pl} = \mathbb{C} \setminus \overline{\mathbb{D}} = \{z \in \mathbb{C} : x^2 + y^2 > 1\}$ be the plane without the unit disc. The diffeomorphism

$$w = \varphi(z) = \frac{1}{z}, \;\; z = x + iy,$$

is conformal and maps Ω_{pl} onto the unit disc $\mathbb{D}$. The conformal weight is

$$h(z) = \frac{1}{|z^2|^2} = \frac{1}{(x^2 + y^2)^2}.$$

Example 2. Let $\Omega_h = \mathbb{C}_+ = \{z \in \mathbb{C} : y > 0\}$ be the upper half-plane. The diffeomorphism

$$w = \varphi(z) = \frac{z - i}{z + i}, \;\; z = x + iy,$$

is conformal and maps Ω_h onto the unit disc $\mathbb{D}$. Then the conformal weight is

$$h(z) = \frac{4}{|z + i|^4} = \frac{4}{(x^2 + (y + 1)^2)^2}.$$

Example 3. Let $\Omega_s = \{z \in \mathbb{C} : -\frac{\pi}{4} < Re\ z < \frac{\pi}{4}\}$ be a strip. The diffeomorphism

$$w = \varphi(z) = \frac{1}{i}\frac{e^{2iz}-1}{e^{2iz}+1} = \tan z, \quad z = x + iy,$$

is conformal and maps Ω onto the unit disc $\mathbb{D}$. Then the conformal weight is

$$h(z) = \frac{1}{|z^2+1|^2} = \frac{1}{(x^2+y^2)^2 + x^2 - y^2 + 1}.$$

Example 4. Let Ω_c be the interior of the cardioid: $r = \frac{1}{2}(1+\cos\theta)$. The diffeomorphism

$$w = \varphi(z) = \sqrt{z} - 1, \;\; z = x + iy,$$

is conformal and maps Ω_c onto the unit disc $\mathbb{D}$. Then the conformal weight

$$h(z) = \frac{1}{2\sqrt{|z|}} = \frac{1}{2\sqrt[4]{x^2+y^2}}.$$

As an application we consider the solvability of the classical Dirichlet problem for the degenerate Laplace operator on an arbitrary simply connected plane domain $\Omega \subset \mathbb{R}^2$ with non-empty boundary.

Define the Sobolev space $W_p^1(\Omega, h, 1)$, $1 \leq p < \infty$, as the normed space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W_p^1(\Omega)\| = \left(\iint\limits_{\Omega} |f|^p(z)\, h(z)\, dxdy \right)^{1/p} + \left(\iint\limits_{\Omega} |\nabla f(z)|^p\, dxdy \right)^{1/p}.$$

Here h is the universal conformal weight.

The Sobolev space $\overset{\circ}{W}{}^1_p(\Omega, h, 1)$, $1 \leq p < \infty$, is defined as the closure of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in the norm of $W^1_p(\Omega, h, 1)$.

We shall use a short notation for the inner products:

$$< u, v >:= \iint_\Omega u(z)v(z)\ dxdy \ \text{ in } \ L_2(\Omega)$$

and

$$< u, v >_h:= \iint_\Omega u(z)v(z)h(z)dxdy \ \text{ in } \ L_2(\Omega, h)$$

and also the notation $[u, v] :=< \nabla u, \nabla v >$.

The problem is as follows:

$$\Delta u = fh, \tag{2}$$
$$u|_{\partial\Omega} = 0. \tag{3}$$

in Ω.

The weak statement of this Dirichlet problem is as follows:

A function u solves the previous problem iff $u \in \overset{\circ}{W}{}^1_2(\Omega, h, 1)$ and

$$[u, v] =< \nabla u, \nabla v >= \iint_{\Omega} f(z)v(z)h(z)\ dxdy$$

for all $v \in \overset{\circ}{W}{}^1_2(\Omega, h, 1)$

Theorem. *Let Ω be a simply connected plane domain with non-empty boundary and let $1 < p < \infty$. If $f \in L_p(\Omega, h)$ then there exists the unique weak solution $u \in \overset{\circ}{W}{}^1_2(\Omega, h, 1)$ of the problem (2,3).*

For study of embeddings of Sobolev spaces $W^1_p(\Omega)$ into weighted Lebesgue spaces we use a connection between composition operators and Brennan's conjecture.

Brennan's conjecture (J. Brennan, 1978) is that for a conformal homeomprphism $\varphi : \Omega \to \mathbb{D}$

$$\int\limits_{\Omega} |\varphi'(z)|^s \, d\mu < +\infty, \quad \text{for all} \quad \frac{4}{3} < s < 4. \tag{4}$$

For $4/3 < s < 3$, it is a comparatively easy consequence of the Koebe distortion theorem. J. Brennan (1978) extended this range to $4/3 < s < 3 + \delta$, where $\delta > 0$, and conjectured it to hold for $4/3 < s < 4$. The example of $\Omega = \mathbb{C} \setminus (-\infty, -1/4]$ shows that this range of s cannot be extended. The upper bound of those s for which (4) is known to hold has been increased to $s \leq 3.399$ by Ch. Pommerenke, to $s \leq 3.421$ by D. Bertilsson, and then to $s \leq 3.752$ by Hedenmalm and Shimorin (2005).

We have the following connection between Brennan's Conjecture and composition operators (V. Gol'dshtein and A. Ukhlov, 2012):

Equivalence Theorem. *Brennan's Conjecture (4) holds for a number* $s \in (4/3; 4)$ *if and only if any conformal homeomorphism* $\varphi : \Omega \to \mathbb{D}$ *induces a bounded composition operator*

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_{q(p,s)}(\Omega)$$

for any $p \in (2; +\infty)$ *and* $q(p, s) = ps/(p + s - 2)$.

Remark. Brennan's Conjecture is proved for some special classes of domains: starlike domains, bounded domains which boundaries are locally graphs of continuous functions etc.

Using the Brennan's Conjecture and the theory of composition operators on Sobolev spaces we prove:

Theorem. *Suppose that $\Omega \subset \mathbb{C}$ is a simply connected domain with non empty boundary, Inverse Brennan's Conjecture holds and $h(z) = J(z, \varphi)$ is the conformal weight defined by a conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$. Then for every $p \in (4/3, 2)$ and every function $f \in C_0^\infty(\Omega)$, the inequality (1)*

$$\left(\iint\limits_{\Omega} |f(z)|^r h(z)\, dxdy\right)^{\frac{1}{r}} \leq K\left(\iint\limits_{\Omega} |\nabla f(z)|^p dxdy\right)^{\frac{1}{p}}$$

holds for any r such that $1 \leq r < p/(2-p)$.

For conformal homeomorphisms $\psi : \mathbb{D} \to \Omega$, Brennan's Conjecture can be reformulated as the Inverse Brennan's Conjecture

$$\int\limits_{\mathbb{D}} |\psi'(w)|^{\alpha} \, d\mu < +\infty, \quad \text{for all} \quad -2 < \alpha < 2/3$$

where $\alpha = 2 - s$.

The Inverse Brennan's Conjecture leads to the conjecture on the existence of bounded composition operators of $\overset{\circ}{W}{}^1_p(\Omega)$ to $\overset{\circ}{W}{}^1_q(\mathbb{D})$ for all $4/3 < p < 2$ and all $1 \leq q < 2p/(4-p)$. As a corollary, we obtain a conjecture about the existence of compact embeddings of $\overset{\circ}{W}{}^1_p(\Omega)$ into $L_r(\Omega, h)$ for all $1 \leq r < p/(2-p)$.

If $\alpha_0 > -2$ is the best known estimate in the Inverse Brennan's Conjecture, i. e. the Inverse Brennan's Conjecture holds for any $\alpha \in [\alpha_0, \frac{2}{3})$, then

$$1 \leq r \leq \frac{2p}{2-p} \cdot \frac{|\alpha_0|}{2+|\alpha_0|} < \frac{p}{2-p} \tag{5}$$

is the best estimate for these embeddings,
$p \in ((|\alpha_0| + 2)/(|\alpha_0| + 1), 2)$.

The "transfer" scheme leads us to:

Theorem. *Suppose that $\Omega \subset \mathbb{C}$ is a simply connected domain with non empty boundary, the Inverse Brennan's Conjecture holds for the interval $[\alpha_0, 2/3)$ where $\alpha_0 \in \big(-2, 0\big)$ and $h(z) = J(z,\varphi)$ is the conformal weight defined by a conformal mapping $\varphi : \Omega \to \mathbb{D}$. Then for any $p \in \big((|\alpha_0|+2)/(|\alpha_0|+1), 2\big)$ the embedding operator*

$$j_r : \overset{\circ}{W}{}^1_p(\Omega) \hookrightarrow L_r(\Omega, h)$$

is compact for every r such that

$$1 \leq r < \frac{2p\,|\alpha_0|}{2\,|\alpha_0| + 4 - p(2 + |\alpha_0|)} < \frac{p}{2-p}.$$

Remark. Of course, we can choose the best known estimate $\alpha_0 > -2$ in the Inverse Brennan's Conjecture.

Note, that the inequality (1) leads to the two-dimensional weighted Hardy type inequalities. For example, consider the domain

$$\Omega_{qr} = \{z = x + iy \in \mathbb{C} : x > 0, y > 0\}.$$

Then the mapping

$$w = \varphi(z) = \frac{z^2 - i}{z^2 + i}$$

is the conformal homeomorphism of Ω_{qr} onto the unit disc $\mathbb{D}$ with the Jacobian

$$J(z,\varphi) = \frac{16|z|^2}{|z^2+i|^4} = \frac{16(x^2+y^2)}{((x^2+y^2)^2+4xy+1)^4} \cong \frac{(x^2+y^2)}{((x^2+y^2)^2+1)^4}.$$

Hence, we have the two-dimensional weighted Hardy type inequality

$$\left[\int_0^\infty \int_0^\infty \frac{(x^2+y^2)|f(x,y)|^r\ dxdy}{((x^2+y^2)^2+1)^4}\right]^{\frac{1}{r}} \leq K\left[\int_0^\infty \int_0^\infty |\nabla f(x,y)|^p\ dxdy\right]^{\frac{1}{p}}$$

As an application, we show the solvability of the classical Dirichlet problem for the p-Laplace operator on an arbitrary simply connected plane domain $\Omega \subset \mathbb{C}$ with non-empty boundary.
The Dirichlet problem for the p-Laplace operator is

$$-\Delta_p(u) = h(x)|u|^{r-2}u \quad \text{in } \Omega, \tag{6}$$

$$u \in \overset{\circ}{W}{}^1_{r,p}(\Omega, h, 1), \tag{7}$$

for $p, r \in (1, +\infty)$.

We say that $u \in \overset{\circ}{W}{}^1_{r,p}(\Omega, h, 1)$ is a weak solution of the problem (6,7) if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_\Omega h(x)|u|^{q-2}uv \, dx, \quad \forall v \in \overset{\circ}{W}{}^1_{r,p}(\Omega, h, 1),$$

This problem was studied by V. Gol'dshtein, V. V. Motreanu, A. Ukhlov (2011) in a more general case of the weighted p-Laplace operator.

Denote by $\big(\overset{\circ}{W}{}^1_{r,p}(\Omega,h,1), \|\cdot|L^1_p(\Omega)\|\big)$ the Sobolev space $\overset{\circ}{W}{}^1_{r,p}(\Omega,h,1)$ endowed with the norm $\|\cdot|L^1_p(\Omega)\|$.
Theorem. Let $\Omega\subset\mathbb{C}$ be a simply connected domain such that $\mu(\mathbb{C}\setminus\Omega)>0$ and $\varphi:\Omega\to\mathbb{D}$ be a conformal homeomorphism. Suppose that the Inverse Brennan's Conjecture holds for the interval $(\alpha_0, 2/3)$ where $\alpha_0\in\big(-2,0\big)$ and $p\in\big((|\alpha_0|+2)/(|\alpha_0|+1),2\big)$. Then for every r, $r\neq p$, such that

$$1<r<\frac{2p\,|\alpha_0|}{2\,|\alpha_0|+4-p(2+|\alpha_0|)}<\frac{p}{2-p},$$

the problem (6,7) has at least two nontrivial weak solutions $u_1\geq 0,\ u_2\leq 0$.

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