On local strong solutions of the non-homogeneous Navier-Stokes Equations

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Joint work with
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Workshop on
Regularity theory for elliptic and parabolic systems
and problems in continuum mechanics

Telč, 30. April - 3. May 2014

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Non-homogeneous Navier-Stokes Equations (NNSE)

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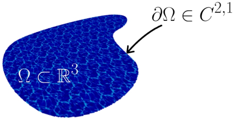
Here:

$0 < T \leq \infty$
$\Omega \subset \mathbb{R}^3$ bounded
$\partial\Omega \in C^{2,1}$



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E (with pressure h) solves the Linear Stokes Equations

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v (with pressure $\tilde{p}$) solves the Perturbed Navier-Stokes Equations

$$\begin{array}{rcll} v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla\tilde{p} & = & f & \text{in} \quad (0,T)\times\Omega \\ \nabla\cdot v & = & 0 & \text{in} \quad (0,T)\times\Omega \\ v & = & 0 & \text{on} \quad (0,T)\times\partial\Omega \\ v(0) & = & v_0 = u_0 & \text{in} \quad \Omega \end{array}$$

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(PNSE)

Then $u = v + E$ (with pressure $p = \tilde{p} + h$) solves (NNSE)

Function Spaces

Function Spaces

- Test spaces:
  - $C_0^\infty(\Omega)$
  - $C_{0,\sigma}^\infty(\Omega) := \{\Phi \in C_0^\infty(\Omega) \,|\, \nabla \cdot \Phi = 0\}$
  - $C_{0,\sigma}^\infty\left(\overline{\Omega}\right) := \left\{\Phi \in C_{0,\sigma}^\infty\left(\mathbb{R}^3\right) \,|\, \Phi = 0 \text{ on } \partial\Omega\right\}$

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- Lebesgue spaces:
  - $L^q(\Omega)$ with norm $\|\cdot\|_q\,, \quad 1 \leq q \leq \infty$
  - $L_\sigma^q(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}, \quad 1 \leq q < \infty$

Function Spaces

- Sobolev spaces:
  - $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{k;q}$, $\quad 1 \leq q < \infty, \quad k \in \mathbb{N}_0$
  - $W_0^{k,q}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k;q}}$
  - $W_{0,\sigma}^{k,q}(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{k;q}}$

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- Sobolev trace spaces:
  - $W^{\beta,q}(\partial\Omega)$ with norm $\|\cdot\|_{\beta;q}\,, \quad 1 < q < \infty, \quad 0 < \beta < 1$
  - $W^{-\beta,q'}(\partial\Omega) := \left(W^{\beta,q}(\partial\Omega)\right)', \quad \frac{1}{q} + \frac{1}{q'} = 1$ with norm
    - $\|\cdot\|_{-\beta;q'} := \sup\limits_{0 \neq \Phi \in W^{\beta,q}(\partial\Omega)} \frac{|\langle \cdot, \Phi \rangle_{\partial\Omega}|}{\|\Phi\|_{\beta;q}}$

Function Spaces

- Bochner spaces ($1 \leq q, s \leq \infty$):
  - $L^s(0, T; L^q(\Omega))$ with norm
    - $\|\cdot\|_{q,s;T} := \left(\int_0^T \|\cdot\|_q^s \, dt \right)^{1/s}, \quad 1 \leq s < \infty$
    - $\|\cdot\|_{q,\infty;T} := \operatorname*{ess\,sup}_{0 \leq t < T} \|\cdot(t)\|_q \quad , \quad s = \infty$

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  - $L^s\big(0, T; W^{\beta,q}(\partial\Omega)\big)$ with norm $\quad (0 < |\beta| < 1)$
    - $\|\cdot\|_{\beta,q,s;T} := \left(\int_0^T \|\cdot(t)\|_{\beta,q}^s \, dt \right)^{1/s}, \quad 1 \leq s < \infty$
    - $\|\cdot\|_{\beta,q,\infty;T} := \operatorname*{ess\,sup}_{0 \leq t < T} \|\cdot(t)\|_{\beta,q} \quad , \quad s = \infty$

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- $A_q = -P_q\Delta : \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$ Stokes operator ($A := A_q$)
  - $\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$
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- Semigroup $e^{-tA_q},\ t \geq 0$ generated by A_q in $L^q_\sigma(\Omega)$

Fractional Powers A_q^α

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Let $1 < q < \infty$, $-1 \le \alpha \le 1$:

- $A_q^\alpha : \mathcal{D}\big(A_q^\alpha\big) \to L^q_\sigma(\Omega)$ fractional power of A_q
  - $\mathcal{D}\big(A_q^\alpha\big) \subset L^q_\sigma(\Omega)$
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Moreover:

- $\big(A_q^\alpha\big)^{-1} = A_q^{-\alpha}$
- $\left(A_q\right)' = A_{q'} \quad$ with $\frac{1}{q} + \frac{1}{q'} = 1$

Very weak solutions for (LSE) and (NNSE)

- Amann (2002, 2003)
- Berselli – Galdi (2004)
- Galdi – Simader – Sohr (2005)
- Farwig – Galdi – Sohr (2005, 2006)
- Farwig – Kozono – Sohr (2007, 2011)
- Schumacher (2008)
- Kim (2009)
- Amrouche – Necasova – Raudin (2009)
- Farwig – Sohr – V. (2009, 2011, 2012, 2013)
- Amrouche – Rodriguez-Bellido (2010, 2010, 2010)
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Let $\quad k \in L^s(0, T; L^r(\Omega)), \quad g \in L^s\left(0, T; W^{-1/q, q}(\partial\Omega)\right)$
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- $-\langle E, \Phi_t \rangle_{\Omega, T} - \langle E, \Delta\Phi \rangle_{\Omega, T} + \langle g, (N \cdot \nabla)\Phi \rangle_{\partial\Omega, T} = 0$
  for all $\Phi \in C_0^\infty\left([0, T); C_{0,\sigma}^\infty(\overline{\Omega})\right)$

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$$\langle -\Delta E, \Phi \rangle_\Omega = \langle \nabla E, \nabla \Phi \rangle_\Omega = \langle E, -\Delta\Phi \rangle_\Omega + \langle E, (N \cdot \nabla)\Phi \rangle_{\partial\Omega}$$

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Proposition 1 (Farwig-Galdi-Sohr 2006)

Let $\quad 1 < s < \infty, \quad 1 < q < \infty, \quad {}^1\!/_q \leq {}^1\!/_r \leq {}^1\!/_q + {}^1\!/_3$

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- $(A^{-1}PE)_t \in L^s(0,T;L^q_\sigma(\Omega))$,
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- $A^{-1}PE \in C([0, T); L^q_\sigma(\Omega))$, $A^{-1}PE|_{t=0} = 0$,
- $\|(A^{-1}PE)_t\|_{q,s,T} + \|E\|_{q,s,T} \;\leq\; C\left(\|k\|_{r,s,T} + \|g\|_{-1/q,q,s,T}\right)$
  with $C = C(\Omega, q, r, s) > 0$.

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Definition 2

Let $\quad 2 < s < \infty, \quad 3 < q < \infty, \quad {}^2\!/_s + {}^3\!/_q = 1, \quad {}^1\!/_r = {}^1\!/_q + {}^1\!/_3$

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$$\text{(iv)} \quad \frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \, d\tau \leq \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F, \nabla v \rangle_\Omega \, d\tau + \int_0^t \langle (v+E)E, \nabla v \rangle_\Omega \, d\tau + \frac{1}{2}\int_0^t \langle k(v+2E), v \rangle_\Omega \, d\tau$$

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From (iii) : $\quad \cdots + \langle (v+E)(v+E), \nabla w \rangle_{\Omega,T} + \langle k(v+E), w \rangle_{\Omega,T} \cdots$

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$$(\text{iv}) \quad \frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \, d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F, \nabla v \rangle_\Omega \, d\tau + \int_0^t \langle (v+E)E, \nabla v \rangle_\Omega \, d\tau + \frac{1}{2}\int_0^t \langle k(v+2E), v \rangle_\Omega \, d\tau$$

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Strong solutions for (PNSE) and (NNSE)

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If, in addition,

$$v \in L^s(0, T; L^q(\Omega)), \quad \frac{2}{s} + \frac{3}{q} = 1,$$

then v is called a strong solution of (PNSE)
and $u = v + E_{k,g}$ a strong solution of (NNSE).

Main result: Existence of strong solutions

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$$b(T) := \|v_0\|_{B_T^{q,s}(\Omega)} + \|F\|_{\frac{q}{2},\frac{s}{2},T} + \|k\|_{q,s,T} + \|g\|_{-\frac{1}{q},q,s,T}$$

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$$b(T) \leq \varepsilon^*,$$

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Then:

$$v \in L^\infty_{\text{loc}}([0,T); W^{1,2}_0(\Omega)) \cap L^2_{\text{loc}}([0,T); W^{2,2}(\Omega)),$$
$$v_t \in L^2_{\text{loc}}([0,T); L^2_\sigma(\Omega)),$$
$$E \in L^s(0,T; W^{2,q}(\Omega)),\ E_t \in L^s(0,T; L^q(\Omega)),$$
$$u \in L^2_{\text{loc}}([0,T); W^{2,2}(\Omega)),\ u_t \in L^2_{\text{loc}}([0,T); L^2(\Omega)).$$

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Since $b(T) \to 0$ as $T \to 0$, each $[0, T^*)$ with $0 < T^* \leq T$ and $b(T^*) \leq \varepsilon^*$, defines an existence interval of uniquely determined strong solutions v of (PNSE) and $u = v + E$ of (NNSE), with T replaced by T^*.

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Remark 2

Let u be a strong solution as in Theorem 1.

Then for smooth data $f, k, g, v_0 \in C^\infty$ we obtain that v and $u = v + E_{k,g}$ satisfy $v, u \in C^\infty((0, T) \times \Omega)$.

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Remark 3

Let v be a strong solution of (PNSE) as in Definition 3.

Then we can replace the energy inequality (iv) by the corresponding energy equality as in the known case $k = 0, g = 0$.

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Let D_r be the data set from Assumption $(*)$ under the restriction

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Hence there is some $0 < T^* \leq T$ such that in $[0, T^*)$ the solution set VD_r of very weak solutions coincides with the solution set SD_r of strong solutions:

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This shows, at least for slightly restricted data, that the solutions in VD_r have the same regularity as the solutions in SD_r, and the notion "very weak" seems to be no longer justified!

Thank you very much

for your attention!