On density of smooth functions in Sobolev-Orlicz spaces with variable exponent.

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Setting of the problem

- $\Omega \subset \mathbb{R}^n$ — bounded Lipschitz domain,
- $\rho \in L^1(\Omega)$ — weight,
- $p(\cdot) : \Omega \to \mathbb{R}_+$ — measurable exponent,

$$1 < \alpha < p(x) < \beta < \infty,$$

- $L^{p(\cdot)}(\Omega,\ \rho\,\mathrm{d}x)$ — variable exponent Lebesgue space with the Luxemburg norm

$$\|f\|_{p(\cdot),\,\rho\,\mathrm{d}x} = \inf\left\{\lambda > 0\ :\ \int |\lambda^{-1} f|^{p(x)} \rho\,\mathrm{d}x \leq 1\right\}.$$

  This space is a reflexive separable Banach space, its dual is $L^{p'(\cdot)}(\Omega,\ \rho\,\mathrm{d}x)$, $p'(x) = \frac{p(x)}{p(x)-1}$.

- W — Sobolev-Orlicz space

$$W = \left\{ u \in W_0^{1,1}(\Omega) \, : \, \int_\Omega |\nabla u|^{p(x)} \rho \, \mathrm{d}x < \infty \right\},$$
$$\|u\|_W := \|\nabla u\|_{p(\cdot), \rho \, \mathrm{d}x}.$$

From now on we additionally assume that

$$\rho^{-1/p} \in L^{p'}(\Omega; \ \mathrm{d}x), \quad \text{where} \quad p'(x) = \frac{p(x)}{p(x) - 1}.$$

Then, the space W is complete due to the generalized Hölder inequality:

$$\int_\Omega |\nabla u| \, \mathrm{d}x \leq 2 \|\nabla u\|_{L^p(\Omega, \rho \, \mathrm{d}x)} \|\rho^{-1/p}\|_{L^{p'}(\Omega, \ \mathrm{d}x)}.$$

- H — the closure of $C_0^\infty(\Omega)$ in W.

The Key Problem

whether smooth functions are dense in the Sobolev space, or

$$H = W \ ?$$

Why important - Lavrentiev's phenomenon.

Minimize the integral functional

$$J[u] = \int |\nabla u|^p \rho \, \mathrm{d}x, \quad u = \varphi \quad \text{on} \quad \partial\Omega.$$

If $H \neq W$, we can take inf over H or over W. This can give two different values:

$$\inf_{u \in W} J[u] < \inf_{u \in H} J[u].$$

Which one we get? Or an intermediate value? (if codimension of H in W is greater than 1)

$p = const$, the classics.

- $\rho \equiv 1$ — N. Meyers and J. Serrin, 1964, $H = W$. Density of $C^\infty(\Omega)$ in $W^{k,p}(\Omega)$. No smoothness of $\partial\Omega$ is required.
- $\rho \equiv 1$ — Under mild additional assumptions on the structure of $\partial\Omega$ we also have density of $C^\infty(\bar{\Omega})$ in the classical Sobolev spaces $W^{k,p}(\Omega)$.
- $\rho \in A_p$ (Muckenhoupt classes) — Meyers-Serrin result still holds, $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega,\ \rho\,\mathrm{d}x)$. The proof repeats the proof of Meyers and Serrin and is based on the uniform boundedness of the classical smoothing operators

$$T_\varepsilon f(x) = f * \varphi_\varepsilon(x) = \int f(y)\rho_\varepsilon(x-y)\,\mathrm{d}y$$

  in $L^p(\Omega,\ \rho\,\mathrm{d}x)$.

Natural limit of the classical method.

By classical results from the theory of the Muckenhoupt spaces, the uniform boundedness of T_ε in $L^p(\Omega,\ \rho\,\mathrm{d}x)$ is equivalent to $\rho \in A_p$.

Thus, using the classical averaging we cannot go beyond the Muckenhoupt classes A_p. The need for a new technique arises.

Lipschitz truncations.

Step 1. For $u \in W_0^{1,1}(\Omega)$ the following two estimates are valid:

$$|u(x) - u(y)| \leq C\{M(\nabla u)(x) + M(\nabla u)(y)\}\,|x-y|, \quad \text{a.e.} \quad x, y \in \Omega,$$
$$|u(x)| \leq Cd(x)M(\nabla u)(x), \quad u \in W_0^{1,1}(\Omega), \quad \text{a.e.} \quad x \in \Omega.$$

Here $d(x) = dist(x, \partial\Omega)$ and Mf is the Hardy-Littlewood maximal function:

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f| \,\mathrm{d}y, \quad \text{for} \quad f \in L^1_{loc}(\mathbb{R}^n),$$

the supremum is over all open balls which contain x.
Step 2. Combining the above estimates,

$$|u(x) - u(y)| \leq C\lambda|x-y| \quad \text{for} \quad x, y \in \{M(\nabla u) \leq \lambda\} \cup (\mathbb{R}^n \setminus \Omega).$$

Step 3. McShane extension theorem: we extend the restriction of u from the set $\{M(\nabla u) \leq \lambda\} \cup (\mathbb{R}^n \setminus \Omega)$ to the whole space $\mathbb{R}^n$.

Result. We have obtained u_λ — a Lipschitz function which coincides with u on the set $\{M(\nabla u) \leq \lambda\}$ and vanishes outside Ω. It is called the *Lipschitz trunction* of the original function. As λ increases, u_λ gets closer and closer to u as the set $\{M(\nabla u) = \infty\}$ has Lebesgue measure zero.

Zhikov's theorem

Recently, V.V. Zhikov proved the following interesting theorem:

Theorem

Let $p = 2$, $\rho = \omega\omega_0$, *where* $\omega_0 \in A_2$ *and*

$$\liminf_{t\to\infty} \frac{\left(\int_\Omega \omega^t \omega_0 \,\mathrm{d}x\right)^{1/t} \cdot \left(\int_\Omega \omega^{-t} \omega_0 \,\mathrm{d}x\right)^{1/t}}{t^2} < \infty.$$

Then $H = W$*.*

This theorem has a nice corollary: if

$$\exists t_0 : \quad \exp(t_0\omega),\ \exp(t_0\omega^{-1}) \in L^1(\Omega,\ \omega_0 \,\mathrm{d}x)$$

then $H = W$.

Example 1. Integrability to any power of ρ and ρ^{-1} is not enough. In $\Omega = \{|x| < 1/2\}$ take

$$\rho_\alpha(x) = \begin{cases} \left(\ln \frac{1}{|x|}\right)^{\alpha}, & x_1 x_2 > 0, \\ \left(\ln \frac{1}{|x|}\right)^{-\alpha}, & x_1 x_2 < 0. \end{cases}$$

This weight is regular ($H = W$) for $\alpha \le 1$ and irregular for $\alpha > 1$. It is not hard to see that for $\alpha > 1$ there holds

$$\lim_{t\to\infty} \frac{\left(\int_\Omega \rho_\alpha^t \, \mathrm{d}x\right)^{1/t} \cdot \left(\int_\Omega \rho_\alpha^{-t} \, \mathrm{d}x\right)^{1/t}}{t^{2\alpha}} < \infty,$$

$$\lim_{t\to\infty} \frac{\left(\int_\Omega \rho_\alpha^t \, \mathrm{d}x\right)^{1/t} \cdot \left(\int_\Omega \rho_\alpha^{-t} \, \mathrm{d}x\right)^{1/t}}{t^{2}} = \infty.$$

Here it is useful to keep in mind that

$$\lim_{n\to\infty} \frac{1}{n} \left[\int_0^{1/2} r \left(\ln \frac{1}{r} \right)^n \mathrm{d}r \right]^{\frac{1}{n}} = \frac{1}{2e}.$$

More general examples of this type are built as follows:

$$\rho(x) = \begin{cases} a(|x|), & x_1 x_2 > 0, \\ (a(|x|))^{-1}, & x_1 x_2 < 0, \end{cases} \qquad \int_0^1 \frac{a(r)\,\mathrm{d}r}{r} < \infty,$$
$$a(r), a^{-1}(r) \geq c(\varepsilon) > 0, \quad r > \varepsilon.$$

Then it is possible to show that the function

$$u(x) = \begin{cases} 1, & x_1 > 0,\ x_2 > 0, \\ \sin\theta, & x_1 < 0,\ x_2 > 0, \\ 0, & x_1 < 0,\ x_2 < 0, \\ \cos\theta, & x_1 > 0,\ x_2 < 0 \end{cases}$$

belongs to W but not to H.

Indeed, take

$$\tilde{u} = u(-x_2, x_1), \quad g = \left\{ -\frac{\partial \tilde{u}}{\partial x_2}, \ \frac{\partial \tilde{u}}{\partial x_1} \right\}.$$

If $u_\varepsilon \in C^\infty(\overline{\Omega})$ approximates u in the norm of W, then

$$\int_\Omega \nabla u_\varepsilon \cdot g \,\mathrm{d}x = \int_{\partial\Omega} u_\varepsilon g \cdot n d\sigma = -\int_{\partial\Omega} u_\varepsilon \frac{\partial \tilde{u}}{\partial \theta} d\sigma$$
$$\to \int_{\partial\Omega} u \frac{\partial \tilde{u}}{\partial \theta} d\sigma = -\int_0^{\pi/2} \sin\theta \, d\theta = 1.$$

On the other hand,

$$\int_\Omega \nabla u_\varepsilon \cdot g \,\mathrm{d}x \to \int_\Omega \nabla u \cdot g \,\mathrm{d}x = 0.$$

Example 2. If the weight ρ is degenerate only on a set closed F of zero measure and zero capacity

$$\mathrm{cap}(F,\rho) = \inf \int_\Omega |\nabla u|^2 \rho \, \mathrm{d}x, \quad u \in C_0^\infty(\Omega),$$
$$u = 1 \quad \text{in the neighbourhood of} \quad F,$$

then ρ is regular. (Cut-off functions...)
In the ball $\{|x| < 1/2\}$ take ρ satisfying

$$1 \le \rho \le C \left(\ln \frac{1}{|x|} \right)^\alpha, \quad 0 < \alpha \le 2.$$

Then Zhikov's theorem gives $H = W$, i.e. $C_0^\infty(\Omega)$ is dense in W. On the other hand, one can check that $cap(\{0\}, \rho) > 0$, i.e. $C_0^\infty(\Omega \setminus \{0\})$ is not dense in W.

A related result says that ρ is regular if

$$\mathrm{cap} F = 0, \quad \rho(x) \leq \frac{\mathrm{const}}{\mathrm{cap} F_\varepsilon},$$
$$F_\varepsilon = \{x \in \Omega \,:\, \mathrm{dist}(x, F) \leq \varepsilon\}.$$

In particular, if $\mathrm{cap} F = |F| = 0$, then boundedness of ρ implies regularity.
For the one-point set $F = \{0\}$ this turns into

$$\sup_{|x| \geq \varepsilon} \rho(x) \leq \begin{cases} \ln \frac{1}{\varepsilon}, & n = 2, \\ \frac{1}{\varepsilon^{n-2}}, & n > 2. \end{cases}$$

Using the same technique, Zhikov's theorem can be extended to the case of any constant $p > 1$:

Theorem

Let $p = const > 1$, $\rho = \omega\omega_0$, where $\omega_0 \in A_p$ and

$$\liminf_{t\to\infty} \frac{\left(\int_\Omega \omega^t \omega_0 \, \mathrm{d}x\right)^{1/t} \cdot \left(\int_\Omega \omega^{-t} \omega_0 \, \mathrm{d}x\right)^{1/t}}{t^p} < \infty.$$

Then $H = W$.

Our goal is to obtain a similar result for the case of the variable exponent. To this end, we have to understand first the limitations on the exponent $p(x)$ and second what an analogue of the classical Muckenhoupt class for the variable exponent can be.

Proof for $p = const$.

The proof is by contradiction. If $H \neq W$ there exists a nontrivial $f \in W^*$ such that $< f, \varphi >= 0$ for any $\varphi \in H$.
Step 1. Solve the problem

$$\mathrm{div}\left(\rho|\nabla u|^{p-2}\nabla u\right) = f, \quad u \in W,$$

which means that

$$\int |\nabla u|^{p-2}\nabla u \nabla\varphi\, \rho\, \mathrm{d}x =< f, \varphi > \quad \forall \varphi \in W.$$

By the choice of f, we have

$$\int |\nabla u|^{p-2}\nabla u \nabla\varphi\, \rho\, \mathrm{d}x = 0 \quad \forall \varphi \in H.$$

Step 2. Denote

$$\rho\,\mathrm{d}x = \omega\,\mathrm{d}\mu, \quad \mathrm{d}\mu = \omega_0\,\mathrm{d}x,$$

$$g(x) = \max\left\{M(\nabla u)(x), \frac{|u(x)|}{d(x)}\right\}, \quad A = \left(\int |\nabla u|^p \rho\,\mathrm{d}x\right)^{1/p}.$$

The notation $\|\cdot\|_q$ stands for the norm in $L^q(\Omega, \mathrm{d}\mu)$.
By the maximal function estimate, Hardy's inequality and the Hölder for any small positive ε' we obtain

$$\|g\|_{p-\varepsilon'} \le C\|\nabla u\|_{p-\varepsilon'}$$
$$\le \|\omega^{1/p}\nabla u\|_p \cdot \|\omega^{-1/p}\|_{p(p-\varepsilon')/\varepsilon'}$$
$$= CA\|\omega^{-1}\|^{1/p}_{\frac{p-\varepsilon'}{\varepsilon'}}.$$

Step 3. Use u_λ — the Lipschitz truncation of u as a test function in the integral identity defining a solution:

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla u_\lambda \, \rho \, \mathrm{d}x = 0.$$

Introducing $F_\lambda = \Omega \cap \{g \leq \lambda\}$ and breaking the integral into two parts, we have

$$\begin{aligned} \int_{F_\lambda} |\nabla u|^p \, \rho \, \mathrm{d}x &= - \int_{\Omega \setminus F_\lambda} |\nabla u|^{p-2} \nabla u \nabla u_\lambda \, \rho \, \mathrm{d}x \\ &\leq C\lambda \int_{\Omega \setminus F_\lambda} |\nabla u|^{p-1} \, \rho \, \mathrm{d}x. \end{aligned}$$

Step 4. By Fubini's theorem,

$$\int f(g)|\nabla u|^p \rho \,\mathrm{d}x = -\int_0^\infty f'(\lambda) \int_{F_\lambda} |\nabla u|^p \rho \,\mathrm{d}x \,\mathrm{d}\lambda,$$
$$\int \Phi(g)|\nabla u|^{p-1} \rho \,\mathrm{d}x = \int_0^\infty \Phi'(\lambda) \int_{\Omega \setminus F_\lambda} |\nabla u|^{p-1} \rho \,\mathrm{d}x \,\mathrm{d}\lambda.$$

So, multiplying the formula we obtained by $f'(\lambda)$ and integrating over $(0, \infty)$ we arrive at

$$\int f(g)|\nabla u|^p \rho \,\mathrm{d}x \le C \int \Phi(g)|\nabla u|^{p-1} \rho \,\mathrm{d}x,$$

where $\Phi'(\lambda) = -\lambda f'(\lambda)$. By Hölder's inequality

$$\int f(g)|\nabla u|^p \rho \,\mathrm{d}x \le C A^{p-1} \left(\int \Phi^p(g) \rho \,\mathrm{d}x \right)^{1/p}.$$

Choosing $f(\lambda) = \lambda^{-\varepsilon}$ for $\varepsilon' \in (0, p\varepsilon)$

$$\int g^{-\varepsilon} |\nabla u|^p \rho \, \mathrm{d}x \le CA^{p-1} \frac{\varepsilon}{1-\varepsilon} \left(\int g^{p(1-\varepsilon)} \rho \, \mathrm{d}x \right)^{1/p}$$

$$\le CA^{p-1} \varepsilon \left(\int g^{p-\varepsilon'} \, \mathrm{d}\mu \right)^{\frac{1-\varepsilon}{p-\varepsilon'}} \left(\int \omega^{\frac{p-\varepsilon'}{p\varepsilon - \varepsilon'}} \, \mathrm{d}\mu \right)^{\frac{p\varepsilon - \varepsilon'}{p(p-\varepsilon')}}$$

$$\le CA^{p-\varepsilon} \left(\varepsilon^p \|\omega^{-1}\|^{1-\varepsilon}_{\frac{p-\varepsilon'}{\varepsilon'}} \|\omega\|^{1/p}_{\frac{p-\varepsilon'}{p\varepsilon-\varepsilon'}} \right)^{1/p} .$$

Step 5. Let $\varepsilon = \tau\varepsilon'$, where $\tau > 1/p$, and $t = \frac{p-\varepsilon'}{p\varepsilon-\varepsilon'}$. Passing to the limit,

$$A^p \leq CA^p \left(\liminf_{t\to\infty} \frac{\|\omega^{-1}\|_{(p\tau-1)t}^{1-\varepsilon(t)} \|\omega\|_t}{t^p} \right)^{1/p},$$

$$\text{where} \quad \varepsilon(t) = \frac{p\tau}{(p\tau-1)t+1}.$$

If the limit here is small enough, $A = 0$ which is what we want. Taking $\tau = 2/p$, we obtain the symmetric form of the condition

$$\liminf_{t\to\infty} \frac{\|\omega^{-1}\|_t^{1-\varepsilon(t)} \|\omega\|_t}{t^p} < C_0, \quad \varepsilon(t) = \frac{2}{t+1}.$$

Step 6. The smallness condition on the limit can be easily removed: indeed, take

$$\rho_\delta = \omega_0 \omega_\delta, \quad \omega_\delta = \begin{cases} \delta\omega, & \omega > 1/\delta, \\ \omega, & \delta \le \omega \le 1/\delta, \\ \omega/\delta, & \omega < \delta. \end{cases}$$

It is easy that

$$\|\omega_\delta\|_t \le \mu(\Omega)^{1/t} + \delta\|\omega\|_t, \quad \|\omega_\delta^{-1}\|_t \le \mu(\Omega)^{1/t} + \delta\|\omega^{-1}\|_t,$$

hence for ω_δ the limit above can be made as small as we wish by the choice of δ.

Basics on Sobolev spaces with variable exponent.

- For the variable exponent the classical example by V.V. Zhikov
$$p(x) = \begin{cases} \alpha > 2, & x_1 x_2 > 0, \\ \beta < 2, & x_1 x_2 < 0 \end{cases}$$
shows that in the absence of continuity of $p(x)$ some strange phenomena may arise. In this example smooth functions are not dense in the Sobolev space.
- Around middle of 1990s V.V. Zhikov and X.L. Fan introduced the famous log-condition
$$|p(x) - p(y)| \leq \frac{k_0}{\ln \frac{1}{|x-y|}}, \quad |x - y| < 1,$$
which guarantees the uniform boundedness of the family of smoothing operators T_ε and thus density of smooth functions in Sobolev spaces.
- Log-condition is also sufficient and *in some sense almost necessary* for other important properties, like boundedness of the maximal function on $L^{p(\cdot)}(\mathbb{R}^n)$, etc.

Advanced results on density

- Thus, if one wants to prove density of smooth functions by mollifications, the Log-condition for $p(\cdot)$ is the natural boundary.
- However, if one is interested in density of smooth functions the Log-condition can be relaxed (V.V. Zhikov): let $r(t)$ be the modulus of continuity of $p(\cdot)$. Then $C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ provided that

$$\int_0^1 t^{-1+\frac{r(t)n}{\alpha}}\,\mathrm{d}t = \infty.$$

- This condition is closed to optimal: for $r(t) = \frac{k \ln \ln \frac{1}{t}}{\ln \frac{1}{t}}$ with $0 < k \leq \alpha/n$ the integral diverges and thus $C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, and if k is large enough one can show that $H \neq W$.

Zhikov's $H \neq W$ example.

Let $A_{(1)} - A_{(4)}$ be four nonintersecting open sections of the unit disk taken in counterclockwise dicrection. Assume that simultaneously

$$\int_{A_{(1)} \cup A_{(3)}} |x|^{-p'(x)} \,\mathrm{d}x < \infty, \quad \int_{A_{(2)} \cup A_{(4)}} |x|^{-p(x)} \,\mathrm{d}x < \infty.$$

Then $H \neq W$.

The idea of this example is based on the following simple Hardy-like estimate:

$$\left| \frac{1}{|\Gamma_1|} \int_{\Gamma_1} (u(R,\varphi) - u(0))\, \mathrm{d}\varphi \right| \leq \frac{1}{|\Gamma_1|} \int_{A_{(1)} \cap B_R} r^{-1} |\nabla u|\, \mathrm{d}x$$

$$\leq \frac{1}{|\Gamma_1|} \left(\int_{A_{(1)} \cap B_R} r^{-p'(x)}\, \mathrm{d}x + \int_{B_R} |\nabla u|^{p(x)}\, \mathrm{d}x \right),$$

valid for $u \in Lip_0(B_1)$. The same also holds for the sector $A_{(3)}$. Hence,

$$\left| \frac{1}{|\Gamma_1|} \int_{\Gamma_1} u(R,\varphi)\, \mathrm{d}\varphi - \frac{1}{|\Gamma_3|} \int_{\Gamma_3} u(R,\varphi)\, \mathrm{d}\varphi \right|$$

$$\leq C \left(\int_{A^R_{(1)} \cup A^R_{(3)}} r^{-p'(x)}\, \mathrm{d}x + \int_{B_R} |\nabla u|^{p(x)}\, \mathrm{d}x \right).$$

By closure, the same is also valid for $u \in H$.

Therefore, for any $u \in H$ the difference

$$\frac{1}{|\Gamma_1|} \int_{\Gamma_1} u(R, \varphi)\,\mathrm{d}\varphi - \frac{1}{|\Gamma_3|} \int_{\Gamma_3} u(R, \varphi)\,\mathrm{d}\varphi$$

tends to zero as R goes to zero.
On the other hand, take $u_{ex} = (1 - r^2) f(\varphi)$, where f has values between 0 and 1, $f = 0$ on Γ_1, $f = 1$ on Γ_3 and the support of f' is contained in $\Gamma_2 \cup \Gamma_3$. It is easy that

$$\int |\nabla u_{ex}|^{p(x)}\,\mathrm{d}x < \infty,$$

hence $u_{ex} \in W$. At the same time,

$$\frac{1}{|\Gamma_1|} \int_{\Gamma_1} u(R, \varphi)\,\mathrm{d}\varphi \equiv 0, \quad \lim_{R \to 0} \frac{1}{|\Gamma_3|} \int_{\Gamma_3} u(R, \varphi)\,\mathrm{d}\varphi = 1,$$

which contradicts what was proved for functions from H.

Standard Muckenhoupt classes.

We remind that a weight w belongs to the Muckenhoupt class A_p, $p > 1$, if

$$\sup \frac{1}{|Q|} \int_Q w \, \mathrm{d}x \left(\frac{1}{|Q|} \int_Q w^{\frac{1}{1-p}} \, \mathrm{d}x \right)^{p-1} < \infty$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with faces parallel to the coordinate hyperplanes.

What to do if p is variable?

Muckenhopt classes with variable exponent.

Definition. We say that a nonnegative L^1_{loc} function (*weight*) $\omega \in A_{p(\cdot)}(\Omega)$, if

$$\sup_{x\in Q\subset\Omega} \left(\int_Q \omega \,\mathrm{d}y\right)^{1/p(x)} \frac{\|\omega^{-1/p}\|_{L^{p'}(Q)}}{|Q|} < \infty,$$

where the supremum is taken over all cubes $Q \subset \Omega$ with faces parallel to the coordinate hyperplanes.

Properties of $A_{p(\cdot)}(\Omega)$.

- $A_{p(\cdot)}(\Omega) \subset A_\infty(\Omega)$, i.e.

$$\gamma \left(\frac{|E|}{|Q|}\right)^{\beta} < \frac{\omega(E)}{\omega(Q)} < \gamma_1 \left(\frac{|E|}{|Q|}\right)^{\beta_1}.$$

  for any cube $Q \subset \Omega$ and measurable $E \subset Q$.
- If $q(\cdot)$ is Log-continuous and $\omega \in A_{p(\cdot)}(\Omega)$ then $\exists C > 0$ s.t. $\omega(Q)^{q(x)-q(y)} \leq C$ for all cubes $Q \subset \Omega$ and $x, y \in Q$.
- If $\omega \in A_{p(\cdot)}(\Omega)$ and $p(\cdot)$ satisfies the Log-condition then $\omega^{-p'/p} \in A_{p'(\cdot)}(\Omega)$.
- If $p(\cdot)$, $q(\cdot)$ satisfy the Log-condition and $p \leq q$, then $A_{p(\cdot)}(\Omega) \subset A_{q(\cdot)}(\Omega)$.

Open-endedness

Theorem

Let $\omega \in A_{p(\cdot)}(\Omega)$ and $p(\cdot)$ satisfy the Log-condition. Then $\exists \varepsilon > 0$ s.t. $\omega \in A_{p(\cdot)-\varepsilon}(\Omega)$. As a consequence, for any cube or ball $Q \subset \Omega$ there holds

$$\frac{1}{|Q|}\int_Q |f|\,\mathrm{d}x \geq 1 \Rightarrow \frac{1}{\omega(Q)}\int_Q |f|^{p(x)-\varepsilon}\omega\,\mathrm{d}x \geq \gamma$$

with some $\gamma, \varepsilon > 0$.

Main Result.

Theorem

Let $p(\cdot)$ satisfy the Log-condition and $\rho = \omega\omega_0$, where $\omega_0 \in A_{p(\cdot)}(\Omega)$ and

$$\liminf_{t\to\infty} \left(\int_\Omega \omega^{-t}\omega_0 \,\mathrm{d}x\right)^{1/t} \left(\int_\Omega \left(t^{-p(x)}\omega\right)^t \omega_0 \,\mathrm{d}x\right)^{1/t} < \infty.$$

Then $H = W$.

Proof for the variable exponent case

The starting point is the same as before:

$$\int_{F_\lambda} |\nabla u|^p \rho \,\mathrm{d}x = -\int_{\Omega\setminus F_\lambda} |\nabla u|^{p-2} \nabla u \nabla u_\lambda \,\rho \,\mathrm{d}x$$
$$\leq C\lambda \int_{\Omega\setminus F_\lambda} |\nabla u|^{p-1} \rho \,\mathrm{d}x.$$

Multiplying this by $\varepsilon\lambda^{-1-\varepsilon}$, integrating from K to ∞, using Fubini's theorem and the Young inequality we obtain

$$\int_\Omega \max(g,K)^{-\varepsilon} |\nabla u|^p \rho \,\mathrm{d}x \leq C\varepsilon \int_\Omega \left(g^{1-\varepsilon} - K^{1-\varepsilon}\right)_+ |\nabla u|^{p-1} \rho \,\mathrm{d}x$$
$$\leq \frac{1}{2} \int_\Omega |\nabla u|^p \rho \,\mathrm{d}x + C \int_\Omega \varepsilon^p \left(g^{1-\varepsilon} - K^{1-\varepsilon}\right)_+^p \rho \,\mathrm{d}x,$$

Denoting the last integral by I_ε, we estimate it as follows:

$$I_\varepsilon := \int_\Omega \varepsilon^p \left(g^{1-\varepsilon} - K^{1-\varepsilon}\right)_+^p \rho \,\mathrm{d}x$$
$$= \int_K^\infty \int_{\Omega\cap\{g>\lambda\}} p(1-\varepsilon)\left(\lambda^{1-\varepsilon} - K^{1-\varepsilon}\right)_+^{p-1} \lambda^{-\varepsilon}\varepsilon^p \rho \,\mathrm{d}x \,\mathrm{d}\lambda$$
$$\leq \beta \int_K^\infty \int_{\Omega\cap\{g>\lambda\}} \lambda^{p-1-p\varepsilon}\varepsilon^p \rho \,\mathrm{d}x \,\mathrm{d}\lambda.$$

Now, cover the set $E_\lambda = \Omega \cap \{g > \lambda\}$ by a family of balls B_z, $z \in E_\lambda$ such that

$$\frac{1}{|B_z|}\int_{B_z} |\nabla u| \,\mathrm{d}x > \lambda.$$

Extract from this family the Besikovitch covering $B_{k,j}(\lambda)$, $k = 1, \ldots, N$.

Denoting $B'_{k,j}(\lambda) = \Omega \cap B_{k,j}(\lambda)$, obtain

$$I_{2\varepsilon} \leq C \int_K^\infty \sum_{k=1}^N \sum_{j=1}^\infty \int_{B'_{k,j}(\lambda)} \lambda^{p-1-2p\varepsilon} \varepsilon^p \rho \, \mathrm{d}x \, \mathrm{d}\lambda.$$

It is also easy that

$$|B_{k,j}(\lambda)| \leq \frac{2}{\lambda} \int_{B_{k,j}(\lambda) \cap \{f > \lambda/2\}} f \, \mathrm{d}x$$

and

$$|B_{k,j}(\lambda)| \leq \frac{2}{\lambda} \left(\int_\Omega \frac{f^{p(x)}}{p(x)} \rho \, \mathrm{d}x + \int_\Omega \frac{\rho^{\frac{1}{1-p}}}{p'(x)} \, \mathrm{d}x \right) \leq \frac{C}{\lambda}.$$

By the Log-condition,

$$\lambda^{p(x)-p(y)} \leq \exp\left(\ln\lambda \frac{C}{\ln\lambda^{1/n}}\right) \leq C, \quad \forall x,y \in B_{k,j}(\lambda).$$

For $\mathrm{d}\mu = \omega_0\,\mathrm{d}x$ we have

$$\int_{B_{k,j}(\lambda)\cap\{f>\lambda/2\}} \left(\frac{f}{\lambda}\right)^{p(x)-\delta} \mathrm{d}\mu \geq C\mu(B_{k,j}(\lambda)).$$

Using the above facts and the Hölder inequality we obtain

$$\int_{B'_{k,j}(\lambda)} \lambda^{p-1-2p\varepsilon} \varepsilon^p \rho \,\mathrm{d}x$$

$$\leq C \left(\int_{B'_{k,j}(\lambda)\cap\{f>\lambda/2\}} \lambda^{p-1} (f\lambda^{-1})^{(p-\delta)(1+\varepsilon/2)} \rho \,\mathrm{d}x \right)^{\frac{2-\varepsilon}{2+\varepsilon}} \times$$

$$\times \left(\int_{B'_{k,j}(\lambda)\cap\{f>\lambda/2\}} \lambda^{-\alpha} \omega^{-2/\varepsilon} \,\mathrm{d}\mu \right)^{\frac{\varepsilon(1-\varepsilon/2)}{2+\varepsilon}} \left(\int_{B'_{k,j}(\lambda)} \lambda^{-\alpha} \left(\varepsilon^p \omega\right)^{2/\varepsilon} \,\mathrm{d}\mu \right)^{\varepsilon/2}$$

Now, we sum over all balls of the covering and use Hölders inequality again:

$$\sum_{k=1}^{N}\sum_{j=1}^{\infty}\int_{B'_{k,j}(\lambda)}\lambda^{p-1-2p\varepsilon}\varepsilon^{p}\rho\,\mathrm{d}x$$

$$\le C\left(\int_{\{f>\lambda/2\}}\lambda^{p-1}(f\lambda^{-1})^{(p-\delta)(1+\varepsilon/2)}\rho\,\mathrm{d}x\right)^{\frac{2-\varepsilon}{2+\varepsilon}}\times$$

$$\times\left(\int_{\{f>\lambda/2\}}\lambda^{-\alpha}\omega^{-2/\varepsilon}\,\mathrm{d}\mu\right)^{\frac{\varepsilon(1-\varepsilon/2)}{2+\varepsilon}}\left(\int_{\Omega\cap\{g>\lambda\}}\lambda^{-\alpha}\left(\varepsilon^{p}\omega\right)^{2/\varepsilon}\,\mathrm{d}\mu\right)^{\varepsilon/2}.$$

Integrating in λ from K to ∞ we obtain the estimate

$$I_{2\varepsilon} \leq C \left(\int_\Omega |\nabla u|^p \rho \, \mathrm{d}x \right)^{\frac{2-\varepsilon}{2+\varepsilon}} \left(\int_\Omega \omega^{-2/\varepsilon} \, \mathrm{d}\mu \right)^{\frac{\varepsilon(1-\varepsilon/2)}{2+\varepsilon}} \left(\int_\Omega (\varepsilon^p \omega)^{2/\varepsilon} \, \mathrm{d}\mu \right)^{\varepsilon/2}$$

From this estimate it follows that

$$\begin{aligned} \int_\Omega |\nabla u|^p \rho \, \mathrm{d}x &= \lim_{\varepsilon \to 0+} \int_\Omega \max(g, K)^{-2\varepsilon} |\nabla u|^p \rho \, \mathrm{d}x \\ &\leq \frac{1}{2} \int_\Omega |\nabla u|^p \rho \, \mathrm{d}x + C \lim_{\varepsilon \to 0+} I_{2\varepsilon} \\ &\leq \int_\Omega |\nabla u|^p \rho \, \mathrm{d}x \cdot \end{aligned}$$
$$\cdot \left[\frac{1}{2} + C \liminf_{\varepsilon \to 0+} \left(\int_\Omega \omega^{-2/\varepsilon} \, \mathrm{d}\mu \right)^{\frac{\varepsilon(1-\varepsilon/2)}{2+\varepsilon}} \left(\int_\Omega (\varepsilon^p \omega)^{2/\varepsilon} \, \mathrm{d}\mu \right)^{\varepsilon/2} \right].$$

To obtain the result of the theorem it remains to replace $2/\varepsilon$ by $t \to \infty$.

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