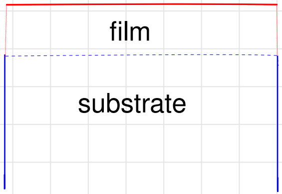
# Motion of elastic thin films by anisotropic surface diffusion with curvature regularization

(work in collaboration with I. Fonseca, G. Leoni and M. Morini)

Nicola Fusco

# A model for epitaxial growth: Static theory

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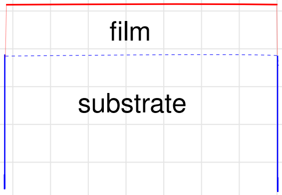


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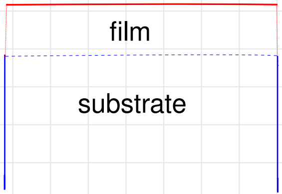


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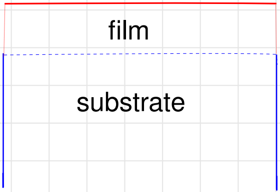


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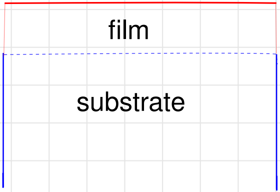


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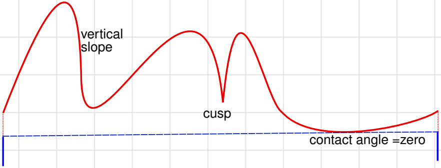

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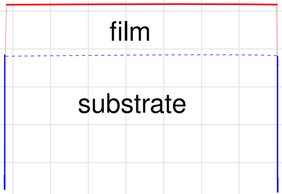


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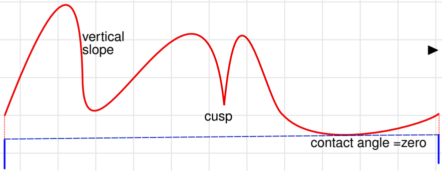


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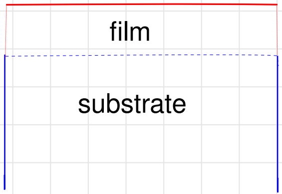


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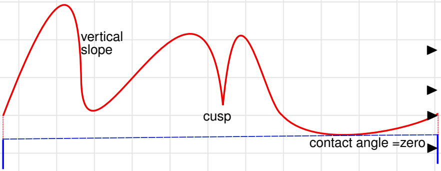


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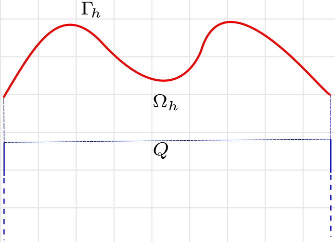
- B.Spencer, D.Meiron (Acta Metal. Mater., 1994)
- B.Spencer, J.Tersoff (Phy. Rev. Letter, 1997)
- further numerical results: Chiu, H. Gao, W. Nix

# Static theory: variational formulation




$h : \mathbb{R}^2 \to [0, \infty)$ $Q$-periodic, Lipschitz

$$\Omega_h = \{(x, y) \in Q \times \mathbb{R} \colon 0 < y < h(x)\}$$

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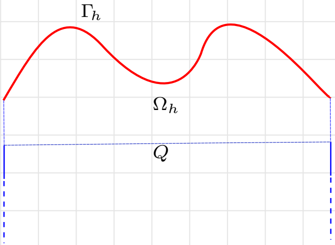


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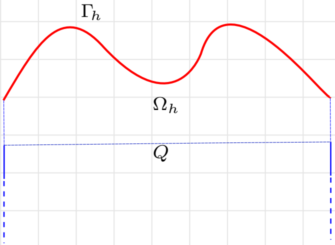


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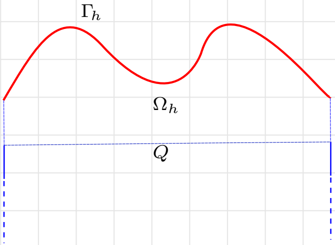


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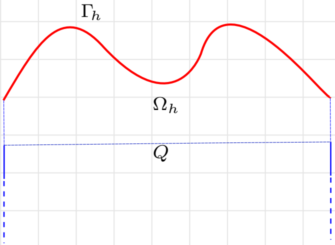


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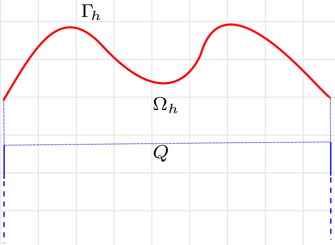


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- $e_0 > 0$ measure the mismatch between the two lattices

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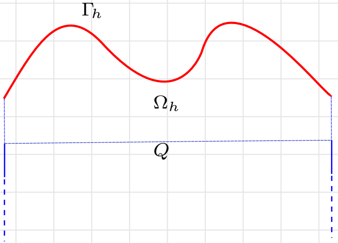

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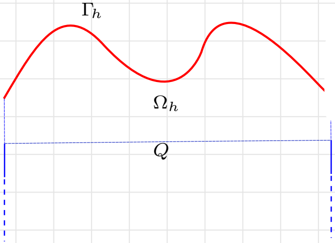


$$F(h,u) := \int_{\Omega_h} W(E(u))\,dxdy + \int_{\Gamma_h} \psi(\nu)d\sigma$$

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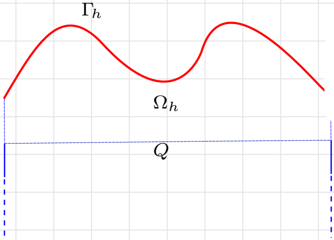


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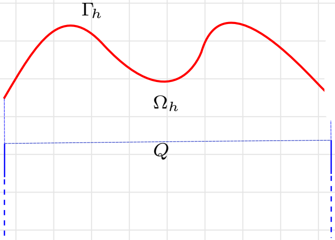


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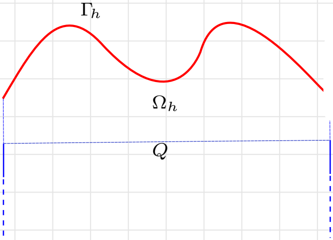


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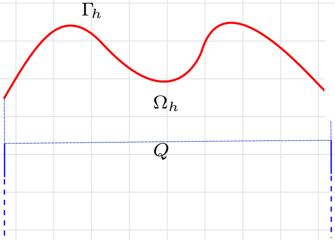


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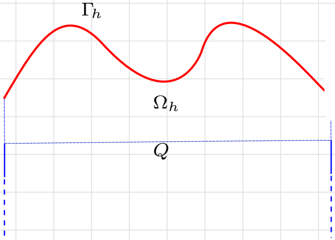


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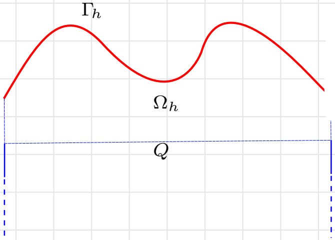


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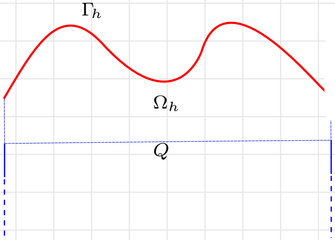


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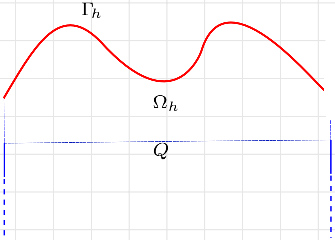


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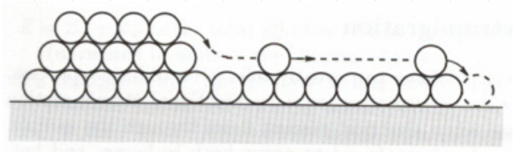

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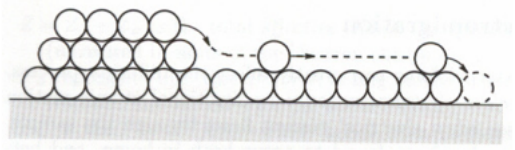

Einstein-Nernst law:

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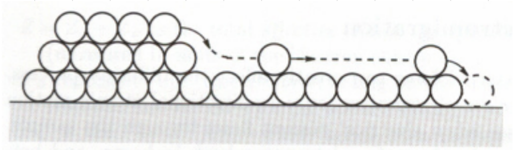

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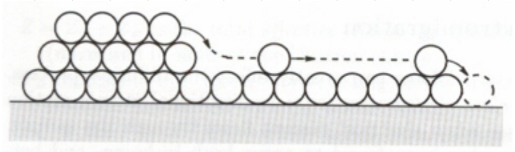

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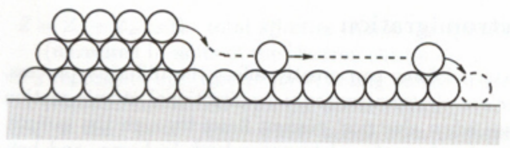

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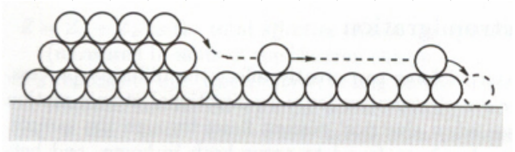

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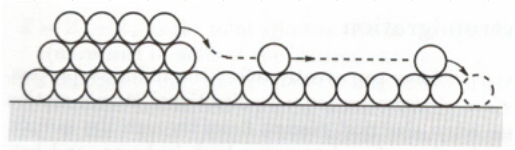

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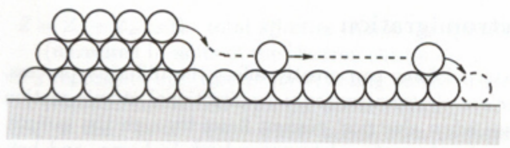

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# Highly anisotropic surface energies

For highly anisotropic $\psi$ it may happen

$$D^2\psi(\nu)[\tau,\tau] < 0 \quad \text{for some } \tau \perp \nu$$

$$\Downarrow$$

the evolution becomes backward parabolic

Remedy: add a curvature regularization

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$$V = \Delta_\Gamma \Big[\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) - \varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\Big(\kappa_1^2 + \kappa_2^2 - \frac{1}{p}H^2\Big)\Big)\Big]$$

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Regularized energy:

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Given $W$, we wish to find $h\colon \mathbb{R}^2 \times [0, T_0] \to (0, +\infty)$ s.t.

$$\begin{cases} \dfrac{1}{J}\dfrac{\partial h}{\partial t} = \Delta_\Gamma \Big[ \mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) \\ \qquad -\varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\Big(\kappa_1^2 + \kappa_2^2 - \dfrac{1}{p}H^2\Big)\Big)\Big], \quad \text{in } \mathbb{R}^2 \times (0, T_0), \\ \mathrm{div}\,\mathbb{C}E(u) = 0 \quad \text{in } \Omega_h, \\ \mathbb{C}E(u)[\nu] = 0 \quad \text{on } \Gamma_h, \qquad u(x,0,t) = e_0(x,0)\,, \\ h(\cdot,t) \text{ and } Du(\cdot,t) \quad \text{are } Q\text{-periodic}, \\ h(\cdot,0) = h_0\,, \end{cases}$$

Here $J := \sqrt{1 + |Dh|^2}$.

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- no analytical results for the sharp interface evolution with elasticity

# The gradient flow structure

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$$\dot{h} = -\nabla_{H^{-1}}\overline{F}(h)$$

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- First observed by Cahn &Taylor (1994) in the context of surface diffusion

# The minimizing movements scheme in our case

- Given $T > 0$, $N \in \mathbb{N}$, we set $\tau := \frac{T}{N}$. For $i = 1, \dots, N$ we define inductively $(h_i, u_i)$ as the solution of the incremental minimum problem

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# The discrete Euler-Lagrange equation

- The Euler-Lagrange equation of the incremental problem is

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\begin{aligned}
\frac{1}{\tau} w_{h_i} &= \operatorname{div}_{\Gamma_{h_i}}(D\psi(\nu)) + W(E(u_i)) \\
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which is a discrete version of the continuous evolution law.

# Estimates

- $h_N(\cdot, t) = h_{i-1} + \frac{t-(i-1)\tau}{\tau}(h_i - h_{i-1}) \quad$ if $(i-1)\tau \leq t \leq i\tau$

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# Higher regularity (difficult!)

Set $\tilde{H}_N(\cdot, t) = H_i$ , for $(i-1)\Delta T \leq t < i\Delta T$, the sum of the principal curvatures of $h_i(\cdot)$, then we have

$$(*) \qquad \int_o^{T_o} \int_Q |D^2(|\tilde{H}_N|^{p-2}\tilde{H}_N)|^2 \, dxdt \leq C$$

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&- \frac{\varepsilon}{p}\int_Q |H_i|^p\frac{DH_i\cdot D\varphi}{J_i} - \int_Q D\psi(-DH_i,1)\cdot(-D\varphi,0)\,dx \\
&- \int_Q W(E(u_i(x,H_i(x))))\varphi\,dx + \frac{1}{\tau}\int_Q v_{h_i}\varphi\,dx = 0\,,
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## Lemma

*Let* $p > 2$, $u \in L^{\frac{p}{p-1}}(Q)$ *such that*

$$\int_Q u\, A\, D^2\varphi\, dx + \int_Q b \cdot D\varphi + \int_Q c\varphi\, dx = 0 \quad \forall \varphi \in C^\infty(Q) \text{ with } \int_Q \varphi\, dx = 0,$$

*where* $A \in W^{1,p}(Q; \mathbb{M}^{2\times 2}_{sym})$, $b \in L^1(Q;\mathbb{R}^2)$, *and* $c \in L^1(Q)$. *Then* $u \in L^q(Q)$ *for all* $q \in (1,2)$. *Moreover, if* $b$, $u$ *div*$A \in L^r(Q;\mathbb{R}^2)$ *and* $c \in L^r(Q)$ *for some* $r > 1$, *then* $u \in W^{1,r}(Q)$.

# Local in time existence of weak solutions

$$\frac{1}{J_{i-1}}\frac{h_i - h_{i-1}}{\tau} = \Delta_{\Gamma_{h_{i-1}}}\Big[\mathrm{div}_{\Gamma_{h_i}}(D\psi(\nu)) + W(E(u_i))$$
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- Previous estimates+ compactness argument $\rightsquigarrow$ $h_N \to h$ up to subsequences
- $h$ is a weak solution in the following sense:

## Theorem (Local existence)

$h \in L^\infty(0,T_0;W^{2,p}_{\#}(Q)) \cap H^1(0,T_0;H^{-1}_{\#}(Q))$ *is a weak solution in* $[0,T_0]$ *in the following sense:*

(i) $\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) - \varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - \frac{1}{p}|H|^p H + |H|^{p-2}H\big(\kappa_1^2 + \kappa_2^2 - \frac{1}{p}H^2\big)\Big) \in L^2(0,T_0;H^1_{\#}(Q))$,

(ii) *for a.e.* $t \in (0,T_0)$

$$\frac{1}{J}\frac{\partial h}{\partial t} = \Delta_\Gamma\Big[\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u))$$
$$-\varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\big(\kappa_1^2 + \kappa_2^2 - \tfrac{1}{p}H^2\big)\Big)\Big] \quad \textit{in } H^{-1}_{\#}(Q).$$

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- The proof is based on the following estimate.

## Proposition

*Let $h \in H^5$, $h \geq c_0 > 0$, and let $u$ be corresponding elastic equilibrium. Then, there exists a constant $C$ depending only on $\|h\|_{H^2}$, $c_0$, and $\|E(u)\|_{L^\infty(\Omega_h)}$ s.t.*

$$\int_{\Gamma_h} |DE(u)|^2 \, d\mathcal{H}^1 \leq C \int_0^b \left(1 + |h^{(iv)}|^2\right) dx$$

# Uniqueness and regularity in 2D

## Theorem

*In two dimensions:*

(i) *The weak solution is unique.*

(ii) *If $h_0 \in H^3$, $\psi \in C^4$, then the solution is in* $H^1(0,T_0;L^2) \cap L^2(0,T_0;H^6)$.

- The proof is based on the following estimate.

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*and*

$$\int_{\Gamma_h} |D^2 E(u)|^2 \, d\mathcal{H}^1 + \int_{\Gamma_h} |D_\sigma(E(u))|^4 \, d\mathcal{H}^1 \leq C \int_0^b \left(1 + |h^{(v)}|^2\right) dx \, .$$

# Second variation approach

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## Theorem (F.-Morini, 2012)

*Let $(h,u)$ be a critical configuration, $h \in C^2$, $h > 0$ s.t.*

$$\partial^2 G(h,u)[\varphi] > 0 \qquad\qquad \forall \varphi \neq 0.$$

*Then, there exists $\delta > 0$ s.t.*

$$G(h,u) < G(g,v)$$

*for all admissible $(g,v)$, with $|\Omega_g| = |\Omega_h|$ and $0 < \|g-h\|_{L^\infty} < \delta$.*

# Local minimality of the 2D flat configuration

- the flat configuration in the $[0, b]$

$$\left(\frac{d}{b}, u_0\right) \qquad u_0(x,y) = e_0\left(x, -\frac{\lambda y}{2\mu + \lambda}\right)$$

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## Theorem (F., Morini, 2012)

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*- if* $b > \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$, *the flat configuration is an isolated local minimizer for* $0 < d < d_{loc}(b)$, *where* $d_{loc}(b)$ *is the unique solution to*

$$K\left(\frac{2\pi d_{loc}(b)}{b^2}\right) = \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}\frac{1}{b}, \qquad K \text{ explicit}$$

*while for* $d > d_{loc}(b)$ *the flat configuration is never an isolated local minimizer*

# Local minimality of the 3D flat configuration: anisotropic case

Let

$$G(h,u) = \int_{\Omega_h} W(E(u))\,dxdy + \int_{\Gamma_h} \psi(\nu)d\mathcal{H}^2$$

and as before

$$F(h,u) = G(h,u) + \frac{\varepsilon}{p}\int_{\Gamma_h} |H|^p\,d\mathcal{H}^2.$$

## Theorem (Bonacini, 2013)

*Assume that* $D^2\psi(e_3) > 0$ *on* $(e_3)^\perp$ *and* $\partial^2 G(d,u_0) > 0$*. Then there exists* $\varepsilon > 0$ *s.t.*

$$\int_Q h = d, \quad 0 < \|h-d\|_{C^{1,\alpha}} \leq \varepsilon \Longrightarrow G(d,u_d) < G(h,u_h).$$

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# Global in time existence and asymptotic stability

Consider the regularized surface diffusion equation

$$\frac{1}{J}\frac{\partial h}{\partial t} = \Delta_\Gamma \Big[ \operatorname{div}_\Gamma (D\psi(\nu)) + W(E(u)) \\ -\varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\Big(\kappa_1^2 + \kappa_2^2 - \frac{1}{p}H^2\Big)\Big)\Big]$$

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The main result is

## Theorem (Fonseca-F.-Leoni-Morini)

*Assume that* $D^2\psi(e_3) > 0$ *on* $e_3^\perp$ *and* $\partial^2 G(d, u_0) > 0$. *There exists* $\varepsilon > 0$ *s.t.*
*if* $\|h_0 - d\|_{W^{2,p}} \le \varepsilon$, *then:*

(i) *any variational solution* $h$ *exists for all times;*

(ii) $h(\cdot, t) \to d$ *in* $W^{2,p}$ *as* $t \to +\infty$.

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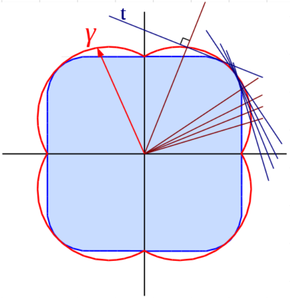
- By isolated minimality $h(t) \to d$ in $W^{2,p}$ as $t \to \infty$

# Liapunov stability in the highly non-convex case

Consider the Wulff shape

$$W_\psi := \{z \in \mathbb{R}^3 : \ z \cdot \nu < \psi(\nu) \text{ for all } \nu \in S^2\}$$



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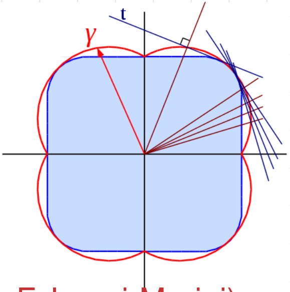


## Theorem (Fonseca-F.-Leoni-Morini)

*Assume that $W_\psi$ contains a horizontal facet. Then for every $d > 0$ the flat configuration $(d, u_d)$ is Liapunov stable, that is, for every $\sigma > 0$ there exists $\delta(\sigma) > 0$ s.t.*

$$\int_Q h_0 = d, \quad \|h_0 - d\|_{W^{2,p}} \leq \delta(\sigma) \quad \Longrightarrow \quad \|h(t) - d\|_{W^{2,p}} \leq \sigma \text{ for all } t > 0.$$

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THANK YOU FOR YOUR ATTENTION!