Mathematical analysis of fluids in motion: From well-posedness to model reduction

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Abstract

This paper reviews some recent results on the Navier-Stokes-Fourier system governing the evolution of a general compressible, viscous, and heat conducting fluid. We discuss several concepts of weak solutions, in particular, using the implications of the Second law of thermodynamics. We introduce the concept of relative entropy and dissipative solution and show the principle of weak-strong uniqueness. The second part of the paper is devoted to problems of model reduction and the related singular limits. Several examples of singular limits are presented: The incompressible limit, the inviscid limit, the low Rossby number limit and their combinations.

Contents

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1 Introduction

Mathematical modeling of fluids in motion includes an enormous variety of problems ranging from the derivation of rigorous mathematical models from basic physical principles, mathematical analysis of complicated systems of partial differential equations, to the design of appropriate and efficient numerical schemes and their implementations, including scale analysis, and, last but not least, the methods and tools commonly known as *model reduction*. We review some recent results concerning the mathematical theory of complete fluid systems and the basic properties of solutions to the underlying equations, in particular, the problem of *well-posedness*, meaning existence, uniqueness, and continuous dependence (stability) of solutions to given system of equations with respect to the data.

We present a sufficiently robust existence theory in the framework of weak (distributional) solutions to the full Navier-Stokes-Fourier system of equations governing the motion of a general compressible, viscous, and heat conductive fluid. Then we illustrate the strength of the new theory by applications in the analysis of *singular limits* arising in the process of scale analysis, where some features of a given fluid flow are being accented or suppressed by aa appropriate choice of the characteristic numbers. Replacing the original (primitive) system by a reduced one obtained by means of the scale analysis is an example of the method of *model reduction* applied at the level of modeling. Another type of model reduction can be used in the process of mathematical analysis of a given problem, where only special types of distinguished solutions may be considered and/or the solution families restricted to various, typically finite-dimensional subsets of the corresponding phase space, like invariant manifolds or attractors. Last but not least, the model reduction is amply used in numerical analysis, where judiciously simplified schemes provide sufficiently accurate results minimizing the computational costs.

In all the afore-mentioned situations, the problem *stability* of a family of solutions with respect to data plays a crucial role. The paper is organized as follows. In Chapter 2, we introduce the basic equations arising in continuum fluid dynamics, starting with simple models of incompressible viscous and inviscid fluids. Then we derive a more complex model of a complete fluid - the Navier-Stokes-Fourier system - using the basic principles of classical thermodynamics, in particular the Second law. Chapter 3 is devoted to the mathematical problem of well-posedness, both in the classical and in the modern (weak) sense. In Chapter 4, we introduce the concept of relative entropy and apply it to showing the property of weak-strong uniqueness and conditional regularity for the Navier-Stokes-Fourier system. Chapter 5 introduces shortly the problem of model reduction for complex fluid systems based on scaling, while Chapter 6 contains several examples of singular limits.

2 Equations of continuum fluid mechanics

Probably the best known model in continuum fluid mechanics is the *Navier-Stokes system* of equations

$$\mathrm{div}_x \mathbf{u} = 0, \tag{2.1}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \nu \Delta \mathbf{u}, \tag{2.2}$$

where $\mathbf{u} = \mathbf{u}(t, x)$ is the velocity field and $\Pi = \Pi(t, x)$ is the pressure of an incompressible viscous fluid, where $\nu > 0$ denotes the viscosity coefficient. The inviscid counterpart of (2.1), (2.2) is the well-known *Euler system*

$$\mathrm{div}_x \mathbf{u} = 0, \tag{2.3}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = 0, \tag{2.4}$$

where the effect of viscosity on the fluid motion is neglected. Despite the apparent simplicity of the above systems, the basic questions of existence, uniqueness and stability of solutions remain largely open, representing one of the major challenges in the theory of partial differential equations, see the formulation of one of the *millennium prize problems* by Fefferman [18].

Yet the models represented by (2.1 - 2.4) are drastically simplified, the compressibility of the fluid is neglected as well as the thermal effects produced as an inevitable consequence of the internal viscous friction causing the irreversible transfer of the mechanical energy into heat enforced by the *Second law of thermodynamics.* A more accurate picture of reality is provided by the *barotropic* Navier-Stokes system, where the pressure p is expressed by means of a *state equation* $p = p(\varrho)$ as an explicit function of the (variable) fluid density $\varrho = \varrho(t,x)$:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.5}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \nabla_x(\eta \mathrm{div}_x \mathbf{u}), \tag{2.6}$$

where μ and ν are the shear and bulk viscosity coefficient, respectively.

Although apparently more complicated than their "incompressible" counterparts (2.1), (2.2), the equations (2.5), (2.6) look more "natural" from the point of view of the theory of evolutionary equations, as time changes of both unknowns ϱ, $\mathbf{u}$ are interrelated through a conventional nonlinear operator. Note that the unknown pressure Π in the "incompressible" equations (2.2) or (2.4) is determined *a posteriori*, playing the role of a "Lagrange multiplier" enforced by the incompressibility constraint (2.1). In particular, the pressure Π is a "non-local" quantity, the value of which at a given instant requires the knowledge of u in the entire physical space occupied by the fluid. Indeed, taking the divergence of (2.2), we get, formally,

$$\Delta \Pi = -\mathrm{div}_x \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}),$$

meaning the pressure is determined locally modulo a harmonic function. For further discussion on the interpretation of the pressure in the incompressible models see Bechtel, Rooney and Wang [3], Li, Li and Pego [47], [48], among others.

The model of a fluid represented by the barotropic system (2.5), (2.6) is still incomplete from the point of view of *thermodynamics*. Taking the scalar product of the momentum equation (2.6) with u, we deduce the corresponding *kinetic energy balance* in the form

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)+\mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right)+\mathrm{div}_x(p\mathbf{u})-\mathrm{div}_x\left(\mathbb{S}\mathbf{u}\right)=-\mathbb{S}:\nabla_x\mathbf{u}+p\mathrm{div}_x\mathbf{u}, \tag{2.7}$$

where we have introduced the viscous stress tensor

$$\mathbb{S}=\mu\left(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\right)+\eta\mathrm{div}_x\mathbf{u}\mathbb{I}.$$

Moreover, we can use (2.5) to deduce

$$p(\varrho)\mathrm{div}_x\mathbf{u}=-\partial_t\left(\varrho\mathcal{P}(\varrho)\right)+\mathrm{div}_x\left(\varrho\mathcal{P}(\varrho)\mathbf{u}\right),\ \mathcal{P}(\varrho)=\int_1^\varrho\frac{p(z)}{z^2}\ \mathrm{d}z, \tag{2.8}$$

and, going back (2.7) we recover a balance equation for the mechanical energy density

$$\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho\mathcal{P}(\varrho),$$

specifically,

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho\mathcal{P}(\varrho)\right)+\mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho\mathcal{P}(\varrho)\right)\mathbf{u}\right]+\mathrm{div}_x(p\mathbf{u})-\mathrm{div}_x\left(\mathbb{S}\mathbf{u}\right)=-\mathbb{S}:\nabla_x\mathbf{u}$$

containing a *sink* term proportional to $-\mathbb{S}:\nabla_x\mathbf{u}$. Thus the model (2.5), (2.6) features the dissipation of the mechanical ("elastic") energy.

In order to obtain a thermodynamically complete model, we have to introduce the (specific) *internal energy* e and admit that p, and possibly also μ, η and other quantities as the case may be, may depend on e. Under these circumstances, the *total energy density* reads

$$\mathcal{E}=\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e,$$

and e obeys the equation

$$\partial_t(\varrho e)+\mathrm{div}_x(\varrho e\mathbf{u})+\mathrm{div}_x\mathbf{q}=\mathbb{S}:\nabla_x\mathbf{u}-p\mathrm{div}_x\mathbf{u} \tag{2.9}$$

where the vector field $\mathbf{q}$ represents a diffusive flux of the internal energy. Note that the source term in (2.9) simply equals the sink in (2.7). The resulting *complete Navier-Stokes system* reads:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.10}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}, \tag{2.11}$$

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}. \tag{2.12}$$

Besides the three unknowns ϱ, $\mathbf{u}$, and e, the system (2.10 - 2.12) contains the pressure p, and the transport terms $\mathbb{S}$, $\mathbf{q}$ therefore certain *constitutive relations* or further *field equations* are needed to close the problem.

2.1 Second law of thermodynamics, entropy

There are several possibilities how to close the system (2.10 - 2.12). The theory of *extended thermodynamics* (see Müller and Ruggeri [56]) suggests to supplement the system by other field equations expressing the time evolution of the unknowns quantities like $\mathbb{S}$, $\mathbf{q}$ etc. Here, we pursue a different strategy closing the system at the level of three field equations (2.10 - 2.12) by a family of *constitutive relations* that characterize a specific material. To this end, we first evoke the Second law by introducing another thermodynamic function - the (specific) *entropy* s.

Following Callen [6] we report the following basic properties of s:

- The entropy s is an increasing function of the internal energy e,

  $$\frac{\partial s}{\partial e} = \frac{1}{\vartheta} > 0;$$

  the (positive) quantity ϑ is termed *absolute temparature*.

- The Third law of thermodynamics: The entropy approaches a constant value (set zero for convenience) $s \to 0$ if $\vartheta \to 0$.

- The entropy production rate is a non-negative quantity. The entropy is being produced in any physically admissible process.

Now assume that the internal energy depends only on the density ϱ and the entropy s. Accordingly, the equation (2.12) can be written in the form

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$
-\partial_t \varrho \frac{\varrho}{\vartheta}\frac{\partial e}{\partial \varrho} - \mathbf{u}\cdot\nabla_x \varrho \frac{\varrho}{\vartheta}\frac{\partial e}{\partial \varrho} - \frac{p}{\vartheta}\mathrm{div}_x \mathbf{u}.
$$

In accordance with the Second law, the last three terms on the right-hand side of the previous identity should mutually cancel which yields the relation

$$
\frac{\partial e(\varrho, s)}{\partial \varrho} = \frac{p}{\varrho^2} \tag{2.13}
$$

imposing certain restrictions on the choice of the thermodynamic functions p, e, and s. In physics, the processes under which the entropy remains unchanged are termed *reversible*. The message of (2.13) states that the changes of the internal energy under constant entropy are only due to the changes in the density and the pressure. It is customary to write (2.13) in a universal form as *Gibbs' relation*:

$$
\vartheta D s = D e + p D\left(\frac{1}{\varrho}\right). \tag{2.14}
$$

In what follows, it will be more convenient to express the thermodynamic functions e, p, and s in terms of the density ϱ and the absolute temperature ϑ playing hereafter the role of *state variables*. Accordingly, (2.13) reads

$$
\frac{\partial e(\varrho, \vartheta)}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta \frac{\partial p(\varrho,\vartheta)}{\partial \vartheta}\right), \tag{2.15}
$$

while the equation (2.12) may be replaced by the *entropy production equation*

$$
\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \tag{2.16}
$$

or, equivalently, the *thermal energy equation*

$$
\varrho c_v \Big(\partial_t \vartheta + \mathbf{u}\cdot\nabla_x\vartheta\Big) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \vartheta \frac{\partial p}{\partial \vartheta}\mathrm{div}_x \mathbf{u}, \tag{2.17}
$$

where the quantity

$$
c_v = c_v(\varrho, \vartheta) = \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}
$$

is the *specific heat at constant volume.*

Finally, revoking the fundamental statement of the *Second law*, we claim that the *entropy production rate*

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \tag{2.18}$$

must be non-negative. Similarly to (2.6), we restrict ourselves to the class of *Newtonian fluids*, where the viscous stress is given by *Newton's rheological law*:

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbb{I}, \tag{2.19}$$

where the viscosity coefficients μ and η are non-negative scalars that may depend on ϱ and ϑ.

Analogously, we assume that the heat flux $\mathbf{q}$ is linearly proportional to the temperature gradient, thus given by *Fourier's law*:

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{2.20}$$

where the *heat conductivity coefficient* κ is a non-negative quantity enjoying sharing similar structure with μ and λ. As a matter of fact, the non-negativity of the *transport coefficients* μ, η, and κ is enforced by the Second law.

We have obtained the *Navier-Stokes-Fourier system* governing the motion of a linearly viscous, compressible, and heat conducting fluid:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.21}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}, \tag{2.22}$$

$$\varrho c_v\Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta\Big) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \vartheta \frac{\partial p}{\partial \vartheta}\mathrm{div}_x \mathbf{u} + \mathcal{Q}, \tag{2.23}$$

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbb{I}, \tag{2.24}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{2.25}$$

In the above system, we have also introduced an *external driving force* $\mathbf{f}$ and a *heat source* $\mathcal{Q}$. More details and a detailed derivation of the model (2.21 - 2.25) may be found, for instance, in the monograph Gallavotti [27]. Systems of equations like (2.21 - 2.25) are called *complete* as they incorporate all the basic principles of thermodynamics.

3 Well posedness

Models in continuum fluid dynamics are (supposed to be) *deterministic*. The information about the *present* state of all phase variables determines completely the behavior of the system in the future. The initial time is usually taken $t = 0$ for definiteness. The concept of *well-posedness* in the sense of Hadamard entails the following properties of a system of (evolutionary) equations:

- The system in question admits, possibly *global-in-time*, solutions for any "admissible class" of data.
- The solutions are uniquely determined in terms of the data.
- The solutions depend continuously on the data and the time t.

To be honest, given the present state of knowledge, we are still very far from a complete solution of the problem of well posedness even for the "simply" looking models like (2.1), (2.2) or even (2.3), (2.4). In this section, we discuss briefly the recent state of the art of the well-posedness problem and suggest some directions to be pursued in future studies.

3.1 Data

We still have not specified what we mean by the *data* for the evolutionary systems like (2.21 - 2.25). Obviously, the *initial state* of all the relevant phase variables must be given, specifically we prescribe the *initial conditions*

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{3.1}$$

where this set is appropriately reduced for the simpler systems (2.5), (2.6) or (2.1), (2.2). Note that the driving force $\mathbf{f}$ as well as the heat source $\mathcal{Q}$, if any, may be considered as a kind of data as well.

3.2 The effect of the physical boundary

In the real world applications, the fluids are confined to a physical space, typically a (bounded) domain in the Euclidean space R^3. In many the cases, the boundary can be *free*, meaning determined by the motion itself and not known a priori. In more complicated but still physically very relevant cases, the boundaries may divide two or more qualitatively different fluids like gases and liquids. Of course, these are just a few examples of what may occur in the nature.

In this paper, we restrict ourselves to the simplest possible situation of a single fluid confined to a fixed domain $\Omega \subset R^N$, $N = 1, 2, 3$. Even in such a case, a proper choice of the boundary conditions is subject to discussion.

If, in addition, the boundary $\partial\Omega$ is *impermeable*, we easily deduce that

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.2}$$

where $\mathbf{n}$ denotes the (outer) normal vector to $\partial\Omega$. The impermeability condition (3.2) is sufficient for a complete description of the (ideal) inviscid fluids, the motion of which is governed by the Euler system (2.3), (2.4). Indeed using (3.2), together with (2.3), (2.4), we check easily that the total kinetic energy

$$\int_\Omega \frac{1}{2}|\mathbf{u}|^2(t, \cdot) \, \mathrm{d}x$$

is (formally) conserved in an ideal fluid, which therefore slips around the boundary without any resistance or interaction with the latter. We shall see below that certain *weak* solutions to the Euler system may actually *violate* the principle of energy conservation.

The situation becomes more delicate for *viscous fluids*, for which a condition relevant to the *complete slip* reads

$$[\mathbb{S} \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0, \tag{3.3}$$

meaning the tangential component of the normal (viscous) stress vanishes on $\partial\Omega$. The complete slip condition (3.3), however, contradicts strongly the observed boundary behavior of *viscous fluids* and leads to d'Alembert's paradox of the absence of the drag force on a body moving with a constant velocity with respect to an (incompressible) fluid.

Numerous practical experiments, however, seem to be at odds with (3.3) as *viscous* fluids are usually observed to adhere completely to the physical boundary - the behavior described by the *no-slip* boundary condition

$$[\mathbf{u}]_{\tan}|_{\partial\Omega} = 0 \tag{3.4}$$

provided the boundary is at rest.

Although frequently used in modeling of viscous fluids, the no-lip condition (3.3) gives rise to a number of paradoxes, among which the absence of collisions of rigid bodies moving in a viscous fluid, see Hesla [29], Hillairet [30] , Hillairet and Takahashi [31]. In the light of these arguments, a suitable compromise between the complete slip and its absence is offered by Navier's boundary condition

$$[\mathbb{S} \cdot \mathbf{n}]_{\tan} + \beta[\mathbf{u}]_{\tan} = 0, \tag{3.5}$$

where $\beta \geq 0$ plays a role of a friction coefficient, see Bulíček, Málek, and Rajagopal [4]. Note that the condition (3.5) reduces to the complete slip for $\beta \to 0$ and approaches the no-slip behavior for $\beta \to \infty$. The reader interested in the physical aspects of (3.5) may consult Priezjev and Troian [58], and Priezjev, Darhuber, and Troian [57], among others.

3.2.1 Far field behavior

In certain situations like modeling objects in *large* physical domains like gaseous stars in astrophysics, it is convenient to consider *unbounded domains*, in particular, the domains exterior to one or several compact sets. In such a case, the behavior of certain quantities for $|x| \to \infty$ must prescribed to avoid multiplicity of solutions for given data. Typically, we postulate the far field behavior of the state variables, specifically,

$$\varrho \to \varrho_\infty, \ \vartheta \to \vartheta_\infty, \ \mathbf{u} \to \mathbf{u}_\infty \text{ for } |x| \to \infty. \tag{3.6}$$

3.3 Strong (classical) vs. weak (distributional) solutions

Leray [41] was probably the first who applied the modern approach to the incompressible Navier-Stokes system (2.1), (2.2) and obtained what is now called *weak solution* of the problem in the natural $3D-$ setting. A Leray's type solution $\mathbf{u}$ is actually considerably "weaker" than what we may call a distributional solution since

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3)) \cap L^2(0,T; W^{1,2}(\Omega; R^3)), \ {\rm div}_x \mathbf{u} = 0,$$

satisfies the integral identity

$$\int_0^T \int_\Omega \left(\partial_t \mathbf{u} \cdot \partial_t \varphi + (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi \right) \ {\rm d}x \ {\rm d}t = \int_0^T \int_\Omega \nu \nabla_x \mathbf{u} : \nabla_x \varphi \ {\rm d}x \ {\rm d}t \tag{3.7}$$

for any test function

$$\varphi \in C^\infty_c((0,T) \times \Omega; R^3), \ {\rm div}_x \varphi = 0.$$

In particular, the pressure Π miraculously disappeared in (3.7) because of the specific class of *solenoidal* test functions. This elegant "trick" simplifies the problem considerably and makes the proof of existence quite easy, at least in the light of the modern tools (not known in Leray's time) based on Sobolev spaces and compactness arguments of Rellich-Kondrashev type. The pressure itself can be recovered *a posteriori* by means of a non-constructive representation theorem, where the specific form of Π depends also on the boundary conditions, see Sohr [66, Chapter 2, Lemma 2.2.1]. In general, very little is known about regularity of Π and it is exactly this aspect that makes the problem of *regularity* of Leray's solutions very difficult and in fact largely open till present times (cf. Fefferman [18]).

As a matter of fact, the solutions constructed in Leray's seminal work satisfy also the energy inequality (under suitable boundary and/or far field consitions) in the form

$$\frac{\rm d}{{\rm d}t} \int_\Omega \frac{1}{2} |\mathbf{u}|^2 \ {\rm d}x + \nu \int_\Omega |\nabla_x \mathbf{u}|^2 \ {\rm d}x \leq 0 \text{ in } \mathcal{D}'(0,T), \tag{3.8}$$

where the inequality in place of the expected equality sign is the price to pay for working in a large class of weak solutions, where the kinetic energy dissipation may not be (hypothetically) captured by the viscous term. Inequality (3.8) can be taken as an integral part of the definition of suitable or *admissible* weak solutions since, as uniqueness in this class is an open problem, there might be weak solutions that violate (3.8). Note that the existence of (unrealistic) solutions that "produce" energy was rigorously proved for the inviscid Euler system (2.3), (2.4), see Scheffer [61], Shnirelman [64], De Lellis and Székelyhidi [11], Wiedemann [70]. The reader may consult the papers of Cheskydov et al. [8], Duchon and Robert [14], Schvydkoy [65] for an interesting discussion concerning the possibility of the inertial energy dissipation in equations of fluid mechanics.

On the other hand, the existence of classical (smooth) and possibly global-in-time solutions is something we expect to be granted, at least for the equations and systems describing the motion of *viscous* fluids, where possible singularities should be outset by the dissipation. In the light of many nowadays standard results, see Prodi [59], Serrin [63], Caffarelli et al. [5], the set of hypothetical singularities in *incompressible* (viscous) fluid systems is expected to be in some sense small, and the singularities of concentration (blow up) type rather than discontinuities in the form of shock waves that would be stable and persisting in time. Note, however, that certain discontinuities may survive in the *compressible* Navier-Stokes system (2.5), (2.6) provided they were imposed through the choice of the initial data, see Hoff [32], Hoff and Santos [33].

3.3.1 Classical solutions for the complete fluid systems

Smooth, classical solutions for models of viscous and even inviscid fluids are known to exist, however, only *locally* in time, and (of course) for sufficiently smooth initial data. For future use, we quote the result of Valli [68] based on the energy method developed earlier in the seminal work of Matsumura and Nishida [54], [55] that applies to the complete Navier-Stokes-Fourier system (2.10), (2.11), and (2.17). The technique of energy estimates was put in an elegant unifying framework by Kato [35] in the context of the Euler flow. This approach is based on the scale of "energy" spaces of Sobolev type $W^{k,2}(\Omega)$ of functions having k distributional derivatives square integrable. For $k \geq 3$, these spaces form Banach algebras suitable for handling the nonlinear terms in the equations.

We consider regular initial data:

$$\varrho(0,\cdot) = \varrho_0,\ \inf_\Omega \varrho_0 > 0,\ \varrho_0 \in W^{3,2}(\Omega), \tag{3.9}$$

$$\vartheta(0,\cdot) = \vartheta_0,\ \inf_\Omega \vartheta_0 > 0,\ \vartheta_0 \in W^{3,2}(\Omega), \tag{3.10}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in W^{3,2}(\Omega; R^3). \tag{3.11}$$

We note that these regularity hypotheses are probably not optimal form the point of view of existence of local smooth solutions, however, solutions in the class are convenient for our future discussion, in particular in the part devoted to the problem of weak-strong uniqueness.

For definiteness, we focus on problems on bounded regular domains $\Omega \subset R^3$ and assume the no-slip boundary condition for the velocity

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{3.12}$$

together with the no-flux condition

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{3.13}$$

Moreover, since we are interested in classical (smooth) solutions, the compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0,\ \nabla_x \vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \nabla_x p(\varrho_0, \vartheta_0)|_{\partial\Omega} = \mathrm{div}_x \mathbb{S}(\varrho_0, \vartheta_0, \nabla_x \mathbf{u}_0) + \varrho_0 \mathbf{f}|_{\partial\Omega} \tag{3.14}$$

must be imposed.

For such a choice of initial data, the complete Navier-Stokes-Fourier system (2.21 - 2.25), supplemented with the boundary conditions (3.12), (3.13), is well-posed in the classical sense albeit only locally in time. More specifically, we report the following result of Valli [67], [68], and Valli, Zajaczkowski [69]:

Theorem 3.1 *Let $\Omega \subset R^3$ be a bounded domain of class C^4. Suppose that $p = p(\varrho, \vartheta)$, $\mu = \mu(\varrho, \vartheta)$, $\eta = \eta(\varrho, \vartheta)$, $c_v = c_v(\varrho, \vartheta)$, and $\kappa = \kappa(\vartheta)$ are C^3-functions of their arguments, satisfying*

$$0 < \underline{c}_v \le c_v(\varrho, \vartheta) \le \overline{c}_v,\ 0 < \underline{\mu} \le \mu(\varrho, \vartheta),\ \eta(\varrho, \vartheta) \ge 0,\ 0 < \underline{\kappa} \le \kappa(\vartheta)$$

for all $\varrho > 0$, $\vartheta > 0$. Let, moreover, $f \in C^3([0,T] \times \Omega)$.

Finally, let the initial data $\{\varrho_0, \vartheta_0, \mathbf{u}_0\}$ belong to the class $W^{3,2}$ and satisfy (3.9), (3.10), together with the compatibility conditions (3.14).

Then there exists a positive time T such that the problem (2.21 - 2.25), (3.1), (3.12), (3.13) admits a unique solution $\{\varrho, \vartheta, \mathbf{u}\}$ on the time interval $(0,T)$ in the class

$$\varrho, \vartheta \in C([0,T]; W^{3,2}(\Omega)),\ \mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)).$$

Moreover, the solution is classical in $(0,T) \times \Omega$, meaning all derivatives appearing in the system (2.21 - 2.23) are continuous in the open set $(0,T) \times \Omega$.

3.3.2 Weak solutions for the complete fluid system

We introduce the concept of *weak solution* used first in [15] and later developed in the monograph [21]. The main ingredients of this approach can be characterized as follows:

- The heat equation (2.23) is replaced by the entropy balance, where the entropy production rate satisfies the *inequality*:
$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right). \tag{3.15}$$

- The resulting system is supplemented with the *total energy balance*:
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right) \ \mathrm{d}x = \int_\Omega \Big(\varrho \mathbf{f} \cdot \mathbf{u} + \mathcal{Q}\Big) \ \mathrm{d}x. \tag{3.16}$$

The *inequality* sign in (3.15) anticipates the (hypothetical) presence of singularities in the weak solutions that would produce an extra piece of entropy not captured by the "classical" terms on the right-hand side. This loss of information is compensated by augmenting the resulting system by (3.16).

The weak formulation of the problem is standard. For the sake of simplicity, we take $\mathbf{f} = 0$, $\mathcal{Q} = 0$. We say that a trio $\{\varrho, \vartheta, \mathbf{u}\}$ is a *weak solution* to the Navier-Stokes-Fourier system (2.10), (2.11), (2.16), emanating from the initial data
$$\varrho(0,\cdot) = \varrho_0, \ \varrho\mathbf{u}(0,\cdot) = \varrho_0 \mathbf{u}_0, \ \varrho s(\varrho, \vartheta)(0,\cdot) = \varrho_0 s(\varrho_0, \vartheta_0), \quad \varrho_0 \geq 0, \ \vartheta_0 > 0, \tag{3.17}$$
and supplemented with the initial conditions (3.12), (3.13) if:

- the density and the absolute temperature satisfy $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ for a.a. $(t,x) \in (0,T) \times \Omega$, $\varrho \in C_{\rm weak}([0,T]; L^\gamma)$, $\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^\beta(\Omega; R^3))$ for certain γ, $\beta > 1$ specified below, $\vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,r}(\Omega))$, and $\mathbf{u} \in L^2(0,T; W^{1,2}_0(\Omega; R^3))$;

- the equation of continuity (2.10) is replaced by a family of integral identities
$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot) \ \mathrm{d}x - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x = \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \tag{3.18}$$
for any $\varphi \in C^1([0,T] \times \overline{\Omega})$, and any $\tau \in [0,T]$;

- the momentum equation (2.11) is satisfied in the sense of distributions, specifically,
$$\int_\Omega \varrho \mathbf{u}(\tau,\cdot) \cdot \varphi(\tau,\cdot) \ \mathrm{d}x - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0,\cdot) \ \mathrm{d}x \tag{3.19}$$
$$\int_0^\tau \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) {\rm div}_x \varphi - \mathbb{S} : \nabla_x \varphi\Big) \ \mathrm{d}x \ \mathrm{d}t$$
for any $\varphi \in C^1([0,T] \times \overline{\Omega}; R^3)$, $\varphi|_{\partial\Omega} = 0$, and any $\tau \in [0,T]$;

- the entropy balance (2.16) is replaced, in accordance with (3.15), by a family of integral inequalities

$$\int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\,{\rm d}x - \int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot)\,{\rm d}x + \int_0^\tau \int_\Omega \frac{\varphi}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\,{\rm d}x\,{\rm d}t \quad (3.20)$$

$$\leq - \int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}\cdot\nabla_x\varphi}{\vartheta}\right)\,{\rm d}x\,{\rm d}t$$

  for any $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \geq 0$, and a.a. $\tau \in [0,T]$;

- the total energy is conserved:

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)\,{\rm d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right)\,{\rm d}x \tag{3.21}$$

  for a.a. $\tau \in [0,T]$.

As shown in [21, Chapter 2] a weak solution is a strong one, in particular the entropy balance is satisfied with an *equality* sign, as soon as it is smooth.

3.3.3 Global existence for the complete Navier-Stokes-Fourier system

The main advantage of working in the framework of weak solutions is the fact that the complete fluid system admits global-in-time solutions for any physically relevant choice of data. We report here the existence result proved in detail in [21, Chapter 3]. To this end, we introduce several restrictions imposed on the constitutive equations interrelating the thermodynamic functions p, e, and s. The reader may consult the monograph Eliezer, Ghatak and Hora [16] as well as [21, Chapter 1] for the physical background. Note that all hypotheses listed below comply with the general physical principles discussed in the previous part of this paper, in particular with Gibbs' equation (2.14).

We suppose that the pressure p is given by the formula

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{3.22}$$

with $P \in C^1[0,\infty)\cap C^3(0,\infty)$ satisfying

$$P(0) = 0,\ P'(Z) > 0,\ 0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0, \tag{3.23}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} > 0. \tag{3.24}$$

Here, we assume that the pressure consists of the molecular component

$$p_M(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \text{ that obeys the mono-atomic gas state eqution,}$$

and of the radiation component

$$p_R(\vartheta) = \frac{a}{3}\vartheta^4.$$

The remaining, rather awkwardly looking assumptions, follow from the hypothesis of thermodynamics stability introduced and discussed in Section 4.1 below, see also [21, Chapter 1].

In agreement with Gibbs' equation (2.14), we take

$$e(\varrho, \vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4, \tag{3.25}$$

together with

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{3.26}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}, \ \lim_{Z\to\infty} S(Z) = 0. \tag{3.27}$$

Finally, we assume the the viscous stress $\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$ and the heat flux $\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$ are given through (2.19), (2.20), where the transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ are continuously differentiable functions of the temperature satisfying

$$\underline{\mu}(1 + \vartheta^\Lambda) \le \mu(\vartheta) \le \overline{\mu}(1 + \vartheta^\Lambda), \ |\mu'(\vartheta)| < c \text{ for all } \vartheta \in [0, \infty), \ \frac{2}{5} < \Lambda \le 1, \tag{3.28}$$

$$0 \le \eta(\vartheta) \le \overline{\eta}(1 + \vartheta^\Lambda) \text{ for all } \vartheta \in [0, \infty), \tag{3.29}$$

$$\underline{\kappa}(1 + \vartheta^3) \le \kappa(\vartheta) \le \overline{\kappa}(1 + \vartheta^3) \text{ for all } \vartheta \in [0, \infty). \tag{3.30}$$

Here, similarly to our choice of the thermodynamic functions p, e, and s, the growth restrictions imposed in (3.30) are motivated by the effect of thermal radiation.

We report the following result ([21, Chapter 3, Theorem 3.1]):

Theorem 3.2 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the thermodynamic functions p, e, s and the transport coefficients μ, η κ comply with the hypotheses (3.22 - 3.30). Let the initial data satisfy*

$$\varrho_0 \in L^{5/3}(\Omega),\ \ \vartheta_0 \in L^\infty(\Omega),\ \ \varrho_0, \vartheta_0 > 0, \mathbf{u}_0 \in W^{1,\infty}(\Omega; R^3). \tag{3.31}$$

Then the Navier-Stokes-Fourier system possesses a weak solution $\{\varrho, \vartheta, \mathbf{u}\}$ on an arbitrary time interval $(0, T)$ in the sense specified in Section 3.3.2. Specifically, the weak solution enjoys the following properties:

$$\varrho \geq 0 \ \textit{a.a. in}\ (0,T) \times \Omega,\ \ \varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^{5/3}(\Omega)) \cap L^\delta((0,T) \times \Omega) \tag{3.32}$$

for a certain $\delta > \frac{5}{3}$;

$$\vartheta > 0 \ \textit{a.a. in}\ (0,T) \times \Omega,\ \ \vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \tag{3.33}$$

$$\vartheta^3, \log(\vartheta) \in L^2(0,T; W^{1,2}(\Omega)); \tag{3.34}$$

$$\mathbf{u} \in L^2(0,T; W^\alpha_0(\Omega; R^3)),\ \ \alpha = \frac{8}{5 - \Lambda},\ \ \varrho\mathbf{u} \in C_{\rm weak}(0,T; L^{5/4}(\Omega; R^3)). \tag{3.35}$$

We remark that (3.32) is definitely not optimal, see [21, Chapter 3] for possible extensions. In particular, the initial density ϱ_0 may contain vacuum zones, meaning ϱ_0 may vanish on a non-empty proper subset of Ω.

4 Relative entropy and weak-strong uniqueness

In the previous part, we have discussed two basic concepts of solutions to problems in fluid dynamics. The strong solutions satisfying the underlying system of equations in the classical sense, and the weak solutions that comply with a family of integral identities corresponding to the original formulation of the problem in the form of conservation or balance laws. The strong solutions are uniquely determined by the data but exist, or at least are known to exist, for a possibly very short lap of time. The weak solutions, on the other hand, are not (known to) be uniquely determined by the data but exist globally in time, at least for a certain physically relevant choice of constitutive relations. We report that weak solutions are strong as soon as they are smooth enough.

In this part, we address a more delicate question, namely, the problem of *weak-strong* uniqueness: Do the weak and strong solutions corresponding to the same data coincide on their common existence time interval? Or, in other words, are the strong solutions unique in the class of weak solutions? To answer this question, we introduce an auxilliary functional termed *relative entropy* and discuss its basic properties.

4.1 Static solutions and the total dissipation balance

Static solutions are solutions of the Navier-Stokes-Fourier system minimizing the entropy production rate σ, cf. (3.15). Accordingly, we obtain

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} = 0, \text{ and } \mathbf{q}(\vartheta, \nabla_x \vartheta) : \nabla_x \vartheta = 0,$$

form which we immeadiately deduce that

$$\mathbf{u} \equiv 0, \text{ and } \vartheta = \tilde{\vartheta} > 0 \text{ - a positive constant.}$$

Note that we have used the fact that $\mathbf{u}$ satisfies the no-slip boundary condition (3.12), while $\nabla_x \vartheta$ complies with (3.13). In the absence of any external force action ($\mathbf{f} \equiv 0$), the static density distribution $\tilde{\varrho}$ therefore satisfies

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = 0 \text{ in } \Omega. \tag{4.1}$$

In general, equation (4.1) admits infinitely many (constant) solutions. To reduce the solution set, we suppose, in addition to Gibbs' equation (2.14), that the *thermodynamic stability hypothesis* holds:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{4.2}$$

In particular, we deduce from (4.1) that $\tilde{\varrho}$ must be constant that can be chosen in such a way that

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = \int_\Omega \varrho_0 \, \mathrm{d}x = M_0, \tag{4.3}$$

where M_0 is the total mass of the fluid.

Finally, the value of $\tilde{\vartheta}$ is uniquely determined by means of (4.2) and

$$\int_\Omega \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) \, \mathrm{d}x = E_0 = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right) \, \mathrm{d}x, \tag{4.4}$$

where E_0 is to total energy of the fluid. We remark that the relevant static solution for a specific motion is determined by the initial state of the fluid.

Summing up the previous discussion we can rewrite the total energy balance (3.21) together with the entropy production equation (3.20) in a concise form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\tilde{\vartheta}}(\varrho,\vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})}{\partial\varrho}(\varrho-\tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})\right)\ \mathrm{d}x = -\tilde{\vartheta}\int_\Omega \sigma\ \mathrm{d}x \leq 0, \tag{4.5}$$

where we have set

$$H_{\tilde{\vartheta}}(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \tilde{\vartheta}s(\varrho,\vartheta)\Big). \tag{4.6}$$

The quantity $H_{\tilde{\vartheta}}(\varrho,\vartheta)$ is called *ballistic free energy* (cf. Ericksen [17]), while the relation (4.5) is usually termed *total dissipation balance*. Note that

$$t \mapsto \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\tilde{\vartheta}}(\varrho,\vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})}{\partial\varrho}(\varrho-\tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})\right)(t,\cdot)\ \mathrm{d}x$$

is a *Lyapunov function* for the (homogeneous) Navier-Stokes-Fourier system and the total dissipation balance (4.5) may be viewed as a statement about *stability* of the static solution $\tilde{\varrho}$, $\tilde{\vartheta}$. Indeed, as a straightforward consequence of the thermodynamic stability hypothesis (4.2), we deduce that

- $\varrho \mapsto H_{\tilde{\vartheta}}(\varrho,\tilde{\vartheta})$ is a strictly convex function of ϱ,
- $\vartheta \mapsto H_{\tilde{\vartheta}}(\varrho,\vartheta)$ is decreasing for $\vartheta < \tilde{\vartheta}$ and increasing for $\vartheta > \tilde{\vartheta}$,

in particular

$$H_{\tilde{\vartheta}}(\varrho,\vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})}{\partial\varrho}(\varrho-\tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta}) \geq 0$$

with the equality sign only if $\varrho = \tilde{\varrho}$, $\vartheta = \tilde{\vartheta}$. A detailed discussion concerning the long-time behavior of the weak solutions as well as the asymptotic stability of the static states to the full Navier-Stokes-Fourier system may be found in the monograph [25].

4.2 Relative entropy

Motivated by the previous discussion, we introduce *relative entropy functional* in the form

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho-r) - H_\Theta(r,\Theta)\right)\ \mathrm{d}x. \tag{4.7}$$

As we have observed in the previous part, the relative entropy $\mathcal{E}(\varrho, \vartheta, \mathbf{u} | r, \Theta, \mathbf{U})$ represents a kind of distance between the triples $(\varrho, \vartheta, \mathbf{u})$ and $(r, \Theta, \mathbf{U})$. Note that the relative entropy is actually a relative "energy" and its definition is different from similar concepts of "genuine" relative entropies used in the theory of hyperbolic systems, see, for instance, Dafermos [10].

Our goal will be to derive an integral relation, similar to the total dissipation balance (4.5), where $\{\varrho, \vartheta, \mathbf{u}\}$ is a weak solution to the complete system and $\{r, \Theta, \mathbf{U}\}$ are arbitrary smooth functions satisfying

$$r > 0, \ \Theta > 0, \mathbf{U}|_{\partial\Omega} = 0. \tag{4.8}$$

To this end, we have to realize that

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right) = \sum_{j=1}^{5} I_j,$$

where

$$I_1 = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right) \ \mathrm{d}x,$$

$$I_2 = -\int_\Omega \varrho\mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x,$$

$$I_3 = \int_\Omega \varrho \left(\frac{1}{2}|\mathbf{U}|^2 - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}\right) \ \mathrm{d}x,$$

$$I_4 = -\int_\Omega \Theta\varrho s(\varrho, \vartheta) \ \mathrm{d}x,$$

and

$$I_5 = \int_\Omega \left(\frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} r + H_\Theta(r, \Theta)\right) \ \mathrm{d}x.$$

Since the functions r, Θ, and $\mathbf{U}$ are supposed to be smooth, all integrals $I_1 \dots I_5$ can be evaluated by means of the weak formulation (3.18 - 3.21). Consequently, after a bit tedious but still straightforward manipulation, we deduce the *relative entropy inequality* in the form

$$\begin{aligned} &\left[\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \\ &+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta}\right) \ \mathrm{d}x \ \mathrm{d}t \\ &\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t, \end{aligned} \tag{4.9}$$

with

$$\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \tag{4.10}$$

$$= \int_0^\tau \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\mathbf{U} - \mathbf{u} \Big) \cdot \nabla_x \Theta \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{U} \cdot \nabla_x \Theta + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \ \mathrm{d}x \ \mathrm{d}t,$$

cf. [23] for details.

4.2.1 Dissipative solutions

The relative entropy inequality reveals a number of interesting properties of the weak solutions to the Navier-Stokes-Fourier system. As a matter of fact, pursuing the strategy of DiPerna and Lions [45] in the context of the inviscid (Euler) system, we may define a new class of *dissipative solutions* to the Navier-Stokes-Fourier system by requiring only the relative entropy inequality (4.9) to be satified for any trio of smooth test functions $\{r, \Theta, \mathbf{U}\}$. As we have observed, any weak solutions is a dissipative solution, meaning satisfies (4.9), (4.10). Thus the dissipative solutions are solutions enjoying the same regularity (integrability) properties of the weak solutions and satisfying, in addition, the relative entropy inequality (4.9).

4.2.2 Weak-strong uniqueness

The first important consequence of the relative entropy inequality is the principle of *weak-strong uniqueness*. In this case, we take the test functions $\{r, \Theta, \mathbf{U}\}$ to be a (hypothetical) strong solution of the same problem emanating from the same initial data as the weak solution $\{\varrho, \vartheta, \mathbf{u}\}$. Now, it turns out that all integrals appearing in the remainder term $\mathcal{R}$ on the right-hand side of (4.9) can be "absorbed" by the left-hand side by means of a Gronwall type argument. Thus, after a bit tedious but straightforward manipulation carried over in [23], we deduce that the weak and strong solutions coincide as long as the latter exists. In other words, the strong solutions are *unique* in the class of weak solutions. More precisely, we have the following result, see [23, Theorem 2.1]:

Theorem 4.1 *Under the hypotheses of Theorem 3.2, suppose that* $\{\varrho, \vartheta, \mathbf{u}\}$ *is a dissipative (weak) solution to the Navier-Stokes-Fourier system emanating from the initial data* $\{\varrho_0, \vartheta_0, \mathbf{u}_0\}$ *that belong to the regularity class specified through (3.9 - 3.11), (3.14). Let*

$$\tilde{\varrho},\ \tilde{\vartheta} \in C([0,T] : W^{3,2}(\Omega)),\ \mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3))$$

be the strong solution of the same problem defined on an existence interval $[0, T_{\max})$, $0 < T < T_{\max}$.

Then

$$\varrho = \tilde{\varrho},\ \vartheta = \tilde{\vartheta},\ \mathbf{u} = \tilde{\mathbf{u}} \ \textit{in}\ [0,T] \times \Omega.$$

4.2.3 Conditional regularity

The weak-strong uniqueness principle may be used to derive a criterion of conditional regularity for the weak or even only dissipative solutions - [24, Theorem 2.1]:

Theorem 4.2 *Under the hypotheses of Theorem 4.1, assume that* $\{\varrho, \vartheta, \mathbf{u}\}$ *is a weak solution of the Navier-Stokes-Fourier system in* $(0,T) \times \Omega$ *such that*

$$\operatorname{ess}\sup_{(0,T)\times\Omega} \|\nabla_x \mathbf{u}\| < \infty.$$

Then $\{\varrho, \vartheta, \mathbf{u}\}$ *is a classical solution of the problem in* $(0,T) \times \Omega$.

5 Reducing complexity of the model

There are several possibilities how to approach the problem of *model reduction* in fluid mechanics. It can be viewed from the point of view of *mathematical modeling* as elaborating models and their simplified (reduced) versions that would give rise to a satisfactory mathematical theory and produce, at lower computational costs, the desired information on the fluid system in question. *Mathematical analysis* may see the model reduction processes as a purely theoretical task, where the formal passage from the primitive to target systems is rigorously justified by the tools of modern functional analysis. *Model reduction* at this level may also include the study of systems reduced to invariant manifolds or attractors as well as explicit solution formulas based on group symmetries and other physically relevant simplifications of a given problem. Probably the most specific use of the term *model reduction* occurs in numerical analysis and implementations of numerical schemes. Here *model reduction* or *model order reduction* is understood as an effective process of reducing the number of equations used for modeling a given system, without substantial changes in the accuracy of the expected output. Unlike researchers in the field of modeling and analysis, numerical analysts have usually very clear ideas concerning the specific methods and tools used in the model reduction process.

In the following text, we discuss several examples how the tools developed in the framework of the abstract theory of complete fluid systems may be used to perform effective model reduction. In particular, we rewrite the Navier-Stokes-Fourier system in the dimensionless form and perform several *singular limits* when certain characteristic parameters become small or very large.

5.1 Scaling and scale analysis

With an appropriate choice of the reference (characteristic) units, the parameters determining the behavior of a complete fluid system become explicit. Asymptotic analysis provides a useful tool in the situations when certain of these parameters called *characteristic numbers* vanish or become infinite. The *Navier-Stokes-Fourier system* in the standard form introduced in Section 3.3.2 does not reveal anything more than the balance laws of certain quantities characterizing the instantaneous state of a fluid. In order to obtain a deeper insight into the structure of possible solutions, it is necessary to identify the *characteristic values* of relevant physical quantities: the *reference time* $T_{\rm ref}$, the *reference length* $L_{\rm ref}$, the *reference density* $\varrho_{\rm ref}$, the *reference temperature* $\vartheta_{\rm ref}$, together with the *reference velocity* $U_{\rm ref}$, and the characteristic values of other composed quantities $p_{\rm ref}$, $e_{\rm ref}$, $\mu_{\rm ref}$, $\eta_{\rm ref}$, $\kappa_{\rm ref}$, and the source terms $\mathbf{f}_{\rm ref}$, $\mathcal{Q}_{\rm ref}$ as the case may be.

Accordingly, the resulting *scaled Navier-Stokes-Fourier system* reads as follows:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{5.1}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x F, \tag{5.2}$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}} \mathrm{div}_x\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \sigma, \tag{5.3}$$

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho F \Big) \mathrm{d}x = 0, \tag{5.4}$$

with the scaled entropy production rate

$$\sigma \geq \frac{1}{\vartheta} \Big(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \Big), \tag{5.5}$$

supplemented with a suitable set of boundary and initial conditions. Here, we have considered a potential driving force

$$\mathbf{f} = \nabla_x F(x)$$

and set $\mathcal{Q} = 0$, see Klein et al. [39].

The dimensionless *characteristic numbers* appearing in the preceding system (5.1 - 5.5) are defined as follows:

△ Symbol	△ Definition	△ Name
Sr	$L_{\rm ref}/(T_{\rm ref}U_{\rm ref})$	Strouhal number
Ma	$U_{\rm ref}/\sqrt{p_{\rm ref}/\varrho_{\rm ref}}$	Mach number
Re	$\varrho_{\rm ref}U_{\rm ref}L_{\rm ref}/\mu_{\rm ref}$	Reynolds number
Fr	$U_{\rm ref}/\sqrt{L_{\rm ref}f_{\rm ref}}$	Froude number
Pe	$p_{\rm ref}L_{\rm ref}U_{\rm ref}/(\vartheta_{\rm ref}\kappa_{\rm ref})$	Péclet number

It is easy to observe that *different choices* of characteristic physical quantities may give rise to the same sample of characteristic numbers. In the following part, we discuss several asymptotic limits when some of these numbers become infinitely small or large.

6 Examples of singular limits

We review some recent results concerning the asymptotic behavior of solutions to the scaled Navier-Stokes-Fourier system (5.1 - 5.5) for singular values of certain characteristic numbers.

6.1 Low Mach number limit: From compressible to incompressible fluid flows

In many real world applications, such as atmosphere-ocean flows, fluid flows in engineering devices, astrophysics, and many others, the velocities are small compared with the speed of sound proportional to $1/\sqrt{\rm Ma}$ in the *scaled Navier-Stokes-Fourier system*. Accordingly, we consider a scaled *Navier-Stokes-Fourier* system in the form:

$$\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0, \tag{6.1}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) + \frac{1}{\varepsilon}\varrho\nabla_x F, \tag{6.2}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_{\mathbf{x}}\vartheta)}{\vartheta}\right) = \sigma_\varepsilon, \tag{6.3}$$

supplemented with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\varepsilon}\left(\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varepsilon\varrho F\right)(t,\cdot)\ \mathrm{d}x = 0, \tag{6.4}$$

where the entropy production rate σ_ε satisfies

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\left(\varepsilon^2\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla_x\vartheta|^2\right) \geq 0. \tag{6.5}$$

The system is supplemented with conservative boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{6.6}$$

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{6.7}$$

The reader will have noticed that we consider the problem on a family of spatial domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ that depend on ε. As a matter of fact, our goal is to consider the behavior of solutions on "large domains" the radius of which approaches infinity for $\varepsilon\to 0$. The boundary conditions (6.6) are called the *complete slip* boundary conditions. They play a crucial role in the analysis as the lead to a very simple equations for the *acoustic waves* represented by the "compressible" component of the velocity assumed to disappear in the asymptotic limit $\varepsilon\to 0$.

The initial state of the fluid system is determined by the following conditions:

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon\varrho^1_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^1_{0,\varepsilon}, \tag{6.8}$$

where

$$\overline{\varrho},\ \overline{\vartheta} > 0,\ \int_{\Omega_\varepsilon}\varrho^1_{0,\varepsilon}\ \mathrm{d}x = \int_{\Omega_\varepsilon}\vartheta^1_{0,\varepsilon}\ \mathrm{d}x = 0 \text{ for all } \varepsilon > 0, \tag{6.9}$$

and

$$\{\varrho^1_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^1_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2\cap L^\infty(\Omega_\varepsilon). \tag{6.10}$$

It is essential that the aplitude of the perturbations of the data is of the same order as the Mach number. Such data are usually called *ill-prepared* and we will come to this issue later in this section.

In addition, we suppose

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{6.11}$$

where

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2 \cap L^\infty(\Omega_\varepsilon; R^3). \tag{6.12}$$

As already pointed out, the fluid flow is restricted to a family of *bounded* domains Ω_ε chosen to "mimick" the behavior of the fluid in a fictitious large (unbounded) domain Ω. Pursuing the philosophy that any real physical space is always bounded but possibly "large" with respect to the speed of sound in the medium, we consider a family of *bounded* domains $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset R^3$ such that $\Omega_\varepsilon \approx \Omega$ in a certain sense as $\varepsilon \to 0$. More specifically, we suppose that

$$\Omega \subset R^3 \text{ is an unbounded domain with a compact smooth boundary } \partial\Omega, \tag{6.13}$$

and set

$$\Omega_\varepsilon = B_{r(\varepsilon)} \cap \Omega, \tag{6.14}$$

where $B_{r(\varepsilon)}$ is a ball centered at zero with a radius $r(\varepsilon)$, with $\varepsilon r(\varepsilon) \to \infty$.

Our next goal will be to discuss the following topics:

- uniform bounds on the family of solution $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (6.1 - 6.7), independent of the parameter $\varepsilon \to 0$;
- strong (pointwise a.a.) convergence

$$\left\{ \begin{array}{c} \varrho_\varepsilon \to \overline{\varrho} \\ \\ \vartheta_\varepsilon \to \overline{\vartheta} \end{array} \right\} \text{a.a. in } (0,T) \times \Omega, \tag{6.15}$$

  and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ a.a. in } (0,T) \times \Omega \tag{6.16}$$

  at least for suitable subsequences.

As soon as the afore-mentioned issues are clarified, it is a routine matter to perform the limit for $\varepsilon \to \infty$ in the *weak formulation* of the Navier-Stokes-Fourier system. Under the specific scaling in (6.1 - 6.3) the limit problem is identified as the *Oberbeck-Boussinesq approximation*:

$$\mathrm{div}_x \mathbf{U} = 0, \tag{6.17}$$

$$\overline{\varrho}\Big[\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \times \mathbf{U})\Big] + \nabla_x \Pi = \mu(\vartheta)\Delta\mathbf{U} + r\nabla_x F \tag{6.18}$$

$$\overline{\varrho} c_p\Big[\partial\Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big] - \kappa(\overline{\vartheta})\Delta\Theta - \overline{\varrho}\ \overline{\vartheta}\alpha\mathrm{div}_x(\mathbf{U}F) = 0 \tag{6.19}$$

$$r + \overline{\varrho}\alpha\Theta = 0, \tag{6.20}$$

see [26] for the complete proof and more information on the limit system. In (6.17 - 6.20), the symbol Π denotes the pressure determined *a posteriori* by the motion, and Θ is the weak limit of the temperature deviations

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta.$$

Note there are two positive parameters $\alpha = \alpha(\overline{\varrho}, \overline{\vartheta})$ and $c_p = c_p(\overline{\varrho}, \overline{\vartheta})$ resulting from the process of scaling.

In the remaining part of this section, we focus on the two basic issues mentioned above, namely, finding uniform bounds on the solutions of the scaled system independent of the scaling parameter $\varepsilon \to 0$, and compactness of the family of the velocity fields $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ intimately related to the propagation of acoustic waves in the low Mach number regime.

6.1.1 Uniform bounds

The key ingredient in the proof of uniform bounds is the total dissipation balance (4.5) introduced in Section 4.1. Its scaled version reads:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \Big[H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} (\varrho_\varepsilon - \tilde{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) \right) \Big] \ \mathrm{d}x = -\frac{\overline{\vartheta}}{\varepsilon^2} \int_\Omega \sigma_\varepsilon \ \mathrm{d}x \leq 0, \tag{6.21}$$

where $(\tilde{\varrho}_\varepsilon, \overline{\vartheta})$ is the static solution with the mass and energy,

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\varrho}_\varepsilon \nabla_x F \text{ in } \Omega_\varepsilon. \tag{6.22}$$

Due to our specific choice of scaling, we may check that (6.22) implies

$$\tilde{\varrho}_\varepsilon \to \overline{\varrho} \text{ a.a. in } \Omega.$$

Our hypotheses imposed on the initial data (6.8 - 6.12) imply the the quantity

$$\int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \Big[H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} (\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) \right) \Big] \ \mathrm{d}x$$

evaluated in terms of the initial data remains bounded as $\varepsilon \to 0$. Consequently, we immediately deduce that

$$\operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \Big[H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} (\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) \Big] \right) \ \mathrm{d}x \leq c \tag{6.23}$$

and

$$\int_\Omega \sigma_\varepsilon \ \mathrm{d}x \leq \varepsilon^2 c \tag{6.24}$$

for $\varepsilon \to 0$.

The bounds (6.23), (6.24), together with the structural properties (3.22 - 3.27) of the functions p, e, and s, and (3.28 - 3.30) imposed of the transport coefficients μ, η, and κ, where we take, for the sake of simplicity,

$$\Lambda = 1$$

in (3.28), (3.29), give rise to a family of uniform estimates listed below (see [26] for details):

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2+L^{5/3}(\Omega_\varepsilon)} \leq c, \tag{6.25}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2+L^{4}(\Omega_\varepsilon)} \leq c, \tag{6.26}$$

$$\int_0^T \int_\Omega \left| \nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c, \ \int_0^T \int_\Omega \left| \nabla_x \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c, \tag{6.27}$$

and

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^2+L^{5/4}(\Omega;R^3)} \leq c, \ \int_0^T \int_\Omega \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c. \tag{6.28}$$

A crucial role in deriving the above estimates is played by the coercivity properties of the function $H_{\overline{\vartheta}}$ established in Section 4.1 that follow directly from the *thermodynamic stability hypothesis* (4.2). Another important factor is the effective presence of a dissipativ mechanism expressed by means of non-zero viscosity and heat conductivity. Last but not least, the validity of (6.25 - 6.28) is strictly conditioned by our choice of the initial data, specifically (6.8). Since (6.8), togethere with (6.10 - 6.12) seems to be the weakest assumption that guarantees the uniform bounds established above, this type of data is usually termed *ill-prepared*, in contrast with the *well-prepared* data for which the families $\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}$, $\{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}$ as well as the gradient components of the velocity fields tend to zero a.a. in Ω.

Note that (6.25), (6.26) imply the pointwise convergence claimed in (6.15). As for the family of velocity fields $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, one can deduce from (6.28) and a variant of Korn's inequality that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{6.29}$$

The pointwise (a.a.) convergence claimed in (6.16) represents a more delicate question because of possible *oscillatory* behavior of the velocities with respect to the *time* variable. This is the main issue to be discussed in the next section.

6.1.2 Acoustic waves

The velocity field $\mathbf{u}_\varepsilon$, or rather the momentum $\varrho_\varepsilon \mathbf{u}_\varepsilon$, can be written in the form

$$\varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] + \nabla_x \Phi_\varepsilon$$

where the symbol $\mathbf{H}$ denotes the *Helmholtz projector* onto the space of solenoidal (divergence-free) functions. Of course, strictly speaking, the decomposition itself depends on ε as we have assumed that the motion takes place in Ω_ε. For simplicity of presentation, we shall assume that $\Omega_\varepsilon = \Omega$ and we also omit the effect of the potential force $\varrho_\varepsilon \nabla_x F$.

Since the limit velocity field $\mathbf{U}$ is expected to be solenoidal (see (6.17)), the gradient component $\nabla_x \Phi_\varepsilon$, where Φ_ε is termed *acoustic potential*, should "disappear" in the course of the limit passage $\varepsilon \to 0$. In order to describe the behavior of $\nabla_x \Phi_\varepsilon$, we derive the so-called *Lighthill's acoustic analogy*, see Lighthill [42], [43], [44].

The problem of the strong (a.a. pointwise) convergence of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is intimately related to propagation and attenuation of acoustic waves represented by the functional Φ_ε. The poinwise convergence is not expected if the fluid is confined to *bounded* domains with acoustically hard boundary (the complete slip-boundary conditions), where large amplitude rapidly oscillating waves are generated in the limit $\varepsilon \to 0$ (see, for instance, Lions and Masmoudi [46], or Schochet [62]). Here, we focus on the case where the target domain Ω be *unbounded* and the dispersion of the acoustic waves takes place. More specifically, the two closely related properties must be satisfied:

- the point spectrum of the associated wave operator is empty;
- the *local* acoustic energy decays in time.

We remark that problems related to propagation of acoustic waves in R^3 were studied by Desjardins and Grenier [12] and the effect of the boundary layer created by the no-slip conditions (ignored in the present paper) was examined by Desjardins at al. [13].

To derive the acoustic equation governing the behavior of Φ_ε, we begin by introducing a "time lifting" Σ_ε of the entropy production σ_ε. Note that, in the weak formulation, σ_ε must by interepreted as a non-negative measure. Accordingly, we take

$$< \Sigma_\varepsilon; \varphi > = < \sigma_\varepsilon; I[\varphi] >,$$

where we have set

$$< \Sigma_\varepsilon; \varphi > = < \sigma_\varepsilon; I[\varphi] >, \ I[\varphi](t,x) = \int_0^t \varphi(z,x) \ \mathrm{d}z \text{ for any } \varphi \in L^1(0,T; C(\overline{\Omega})). \tag{6.30}$$

Following the original idea of Lighthill [44], we rewrite the Navier-Stokes-Fourier system in the form:

$$\varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon, \tag{6.31}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon \Big(\mathrm{div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_\varepsilon \Big), \tag{6.32}$$

supplemented with the homogeneous boundary conditions

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{6.33}$$

where we set

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + \frac{A}{\varepsilon \omega} \Sigma_\varepsilon, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon, \tag{6.34}$$

$$\mathbf{F}^1_\varepsilon = \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) \mathbf{u}_\varepsilon + \frac{A}{\omega} \frac{\kappa \nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon}, \ \mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{6.35}$$

and

$$F^3_\varepsilon = \omega \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2} \right) + A \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2} \right) - \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2} \right). \tag{6.36}$$

The constants A and ω has to be chosen so that

$$A \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \omega + A \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}. \tag{6.37}$$

Note that the *wave speed* ω is strictly positive as a direct consequence of *hypothesis of thermodynamic stability.*

Our goal is to show how the strong (pointwise a.a.) convergence of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, claimed in (6.16), can be deduced from (6.31 - 6.33). Since the velocity field enjoy certain compactness in the space variable (cf. (6.29), it is enough to show that

$$\Big[t \mapsto \int_\Omega \mathbf{V}_\varepsilon(t, \cdot) \cdot \mathbf{w} \ \mathrm{d}x \Big] \to \Big[t \mapsto \int_\Omega \mathbf{V}(t, \cdot) \cdot \mathbf{w} \ \mathrm{d}x \Big] \text{ in } L^1(0,T) \tag{6.38}$$

for any fixed $\mathbf{w} \in C^\infty_c(\Omega; R^3)$, where $\mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$. Moreover, since our problem has been reduced to showing (6.38), we may assume, with help of a simple approximation, that all quantities appearing in the acoustic equations are smooth. This relaxation of the original problem as well the smoothing procedure are discussed in detail in [26].

Thus our task may be reduced to the following problem:

Show that the family

$$\left[t \mapsto \int_\Omega \mathbf{V}_\varepsilon(t,\cdot) \cdot \mathbf{w} \ {\rm d}x \right] \text{ is precompact in } L^1(0,T) \tag{6.39}$$

for any $\mathbf{w} \in C^\infty_c(\Omega; R^3)$, on condition that

$$\varepsilon \partial_t Z_\varepsilon + {\rm div}_x \mathbf{V}_\varepsilon = \varepsilon {\rm div}_x \mathbf{F}^1_\varepsilon \text{ in } (0,T) \times \Omega, \tag{6.40}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon {\rm div}_x \mathbb{F}^2_\varepsilon \text{ in } (0,T) \times \Omega, \tag{6.41}$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{6.42}$$

$$Z_\varepsilon(0,\cdot) = Z_{0,\varepsilon}, \ \mathbf{V}_\varepsilon(0,\cdot) = \mathbf{V}_{0,\varepsilon} \text{ in } \Omega, \tag{6.43}$$

where

$$Z_{0,\varepsilon} \in C^\infty_c(\Omega), \ \{Z_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega), \ \mathbf{V}_{0,\varepsilon} \in C^\infty_c(\Omega; R^3), \ \{\mathbf{V}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3),$$

and with the functions

$$\mathbf{F}^1_\varepsilon \in C^\infty_c((0,T) \times \Omega; R^3), \ \mathbb{F}^2_\varepsilon \in C^\infty_c((0,T) \times \Omega; R^{3\times 3}),$$

$$\{\mathbf{F}^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T; L^2(\Omega; R^3)), \ \{\mathbb{F}^2_{\varepsilon,0}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T; L^2(\Omega; R^{3\times 3})). \tag{6.44}$$

We first focus on compactness of the solenoidal part of $\mathbf{V}_\varepsilon$. To this end, consider a function $\psi \in W^{1,2} \cap W^{1,\infty}(\Omega; R^3)$, ${\rm div}_x \psi = 0$, $\psi \cdot \mathbf{n}|_{\partial\Omega} = 0$. Multiplying equation (6.41) on ψ and integrating by parts, we obtain

$$\frac{\rm d}{{\rm d}t} \int_\Omega \mathbf{V}_\varepsilon \cdot \psi \ {\rm d}x = - \int_\Omega \mathbb{F}^2_\varepsilon : \nabla_x \psi \ {\rm d}x, \ \int_\Omega \mathbf{V}_\varepsilon(0,\cdot) \cdot \psi \ {\rm d}x = \int_\Omega \mathbf{V}_{0,\varepsilon} \cdot \psi \ {\rm d}x.$$

In particular, we deduce that the family

$$\left[t \mapsto \int_\Omega \mathbf{V}_\varepsilon \cdot \psi \ {\rm d}x \right] \text{ is precompact in } C[0,T]. \tag{6.45}$$

We note that this step is strongly conditioned by our choice of the complete slip boundary condition.

To perform the analysis of acoustic waves (the gradient component of $\mathbf{V}_\varepsilon$), we rewrite system (6.40), (6.41) in terms of an abstract self-adjoint operator (the *Neumann Laplacean* Δ_N):

$$\Delta_N[v] = \Delta v,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0,\ v(x) \to 0 \text{ as } |x| \to \infty,$$

with

$$\mathcal{D}(\Delta_N) = \{w \in L^2(\Omega) \mid w \in W^{2,2}(\Omega),\ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

If Ω is a regular (unbounded) domain in R^3, exterior to a compact set, it can be shown that $-\Delta_N$ is a self-adjoint, non-negative operator in $L^2(\Omega)$, with an absolutely continuous spectrum $[0,\infty)$. Moreover, Δ_N satisfies the *limiting absorption principle*:

$$\sup_{\lambda \in C, 0<\alpha\leq \mathrm{Re}[\lambda]\leq\beta<\infty,\ \mathrm{Im}[\lambda]\neq 0} \left\| \mathcal{V} \circ (-\Delta_N - \lambda)^{-1} \circ \mathcal{V} \right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \leq c_{\alpha,\beta}, \tag{6.46}$$

where

$$\mathcal{V}(x) = (1 + |x|^2)^{-\frac{s}{2}},\ s > 1$$

see Leis [40]. Similar properties may be shown on other types of unbounded domains like a half-space or the entire physical space R^3.

For the *acoustic potential*

$$\Phi_\varepsilon = \Delta_N^{-1}[\mathrm{div}_x \mathbf{V}_\varepsilon], \tag{6.47}$$

the system of equations (6.40), (6.41) reads

$$\varepsilon\partial_t Z_\varepsilon + \Delta_N \Phi_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon,\ \varepsilon\partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \Delta_N^{-1} \mathrm{div}_x \mathrm{div}_x \mathbb{F}^2_\varepsilon. \tag{6.48}$$

The *acoustic potential* Φ_ε may be therefore expressed by means of the standard *Duhamel formula*:

$$\Phi_\varepsilon(t,\cdot) \tag{6.49}$$

$$= \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\Delta_N[h^1_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[h^2_\varepsilon] \pm \mathrm{i}\left(\Delta_N[h^3_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[h^4_\varepsilon]\right)\right]$$

$$+ \int_0^t \exp\left(\pm \mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\Delta_N[H^1_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[H^2_\varepsilon] \pm \mathrm{i}\left(\Delta_N[H^3_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[H^4_\varepsilon]\right)\right]\ \mathrm{d}s,$$

for certain functions

$$\{h^i_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega),\ \{H^i_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2((0,T)\times\Omega). \tag{6.50}$$

Now, there are several possibilities how to show strong convergence of the acoustic potential. Here, we first revoke the space-time decay estimates for the group $\exp(\mathrm{i}t\sqrt{-\Delta_N})$ obtained by Kato [34].

Theorem 6.1 [Reed and Simon [60, Theorem XIII.25 and Corollary]]

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty. \tag{6.51}$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Thus the desired conclusion (6.39) follows by applying Theorem 6.1 in the situation

$$X = L^2(\Omega),\ H = \sqrt{-\Delta_N},\ A[v] = \varphi G(-\Delta_N)[v],\ v \in X,$$

with

$$G \in C_c^\infty(0,\infty),\ \varphi \in C_c^\infty(\Omega) \text{ given functions.}$$

Kato's result is applicable provided the domain Ω and the associated Neumann Laplacean obey the limiting absorption principle (6.46). More general domains may be handled by means of the celebrated RAGE theorem, see Cycon et al. [9, Theorem 5.8]:

Theorem 6.2 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto the space of continuity H_c of A, specifically,*

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span}\{ w \in H \mid w \text{ an eigenvector of } A \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA) \ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \text{ as } \tau \to \infty. \tag{6.52}$$

RAGE theorem is optimal in the sense that represents both necessary and sufficient condition for the local pointwise converegence of the acoustic waves, namely, the absence of eigenvalues of the Neumann Laplacean in the domain Ω. Kato's result and RAGE theorem may be viewed as two extremal cases of the abstract theory of the acoustic waves propagation described in terms of the associated spectral measures, see [19] for details.

6.2 High Reynolds - low Mach number limit: From compressible viscous to incompressible inviscid fluid flows

In many real world applications, in particular in meteorology, the fluids are asymptotically incompressible, and, at the same time, the transport coefficients - the viscosity and the heat conductivity - are small. This the situation when the Mach number is small but Reynolds and Peclet numbers are high. The corresponding scaled system, in the absence of external forces, reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{6.53}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{6.54}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \varepsilon^b \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_{\mathbf{x}} \vartheta)}{\vartheta} \right) = \sigma_\varepsilon, \tag{6.55}$$

supplemented with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \ \mathrm{d}x = 0, \tag{6.56}$$

where the entropy production rate σ_ε satisfies

$$\sigma_\varepsilon \geq \frac{1}{\vartheta} \left(\varepsilon^{a+2} \mathbb{S} : \nabla_x \mathbf{u} + \varepsilon^b \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \right) \geq 0, \tag{6.57}$$

and where a, b are positive exponents specified below.

The main difficulty of the asymptotic limit $\varepsilon \to 0$, besides the oscillations due to the acoustic velocity component, is the lack of bounds on the velocity and temperature gradient when $\varepsilon \to 0$. In particular, the *uniform* bounds (6.27), (6.28) are no longer available and must be replaced by certain stability relations.

The limit (target) problem can be (formally) identified quite easily as the incompressible Euler system:

$$\mathrm{div}_x \mathbf{v} = 0, \tag{6.58}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{6.59}$$

supplemented with a transport equation for the temperature deviation T,

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0. \tag{6.60}$$

Similarly to the preceding section, the function $\mathbf{v}$ is the limit velocity while $T \approx \frac{\vartheta - \overline{\vartheta}}{\varepsilon}$. Note that the system (6.58 - 6.60) can be obtained as a hydrodynamic limit of the Boltzmann equation, see

Golse [28].

To avoid problems related to the boundary conditions, and, at the same time, to guarantee the dispersive estimates for the acoustic equation, we consider the problem in the whole space $\Omega = R^3$ and prescribe only the “far field” boundary conditions:

$$\varrho \to \overline{\varrho} > 0,\ \vartheta \to \overline{\vartheta},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty.$$

Furthermore, we consider the initial data in the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{6.61}$$

The inviscid system (6.58 - 6.60) is known to possess a smooth solution at least on a short time interval that may depend on the size of the initial data, see, for instance, Kato and Lai [35]. Consequently, the solutions $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of the scaled Navier-Stokes-Fourier system are expected to converge to solutions of (6.58 - 6.60) on the interval of existence of the latter. We report the following result, [22, Theorem 3.1]:

Theorem 6.3 *Let the thermodynamic functions p, e, and s as well as the transport coefficients μ and κ comply with the hypotheses of Theorem 3.2, with $\alpha = 1$. Let*

$$b > 0, \ 0 < a < \frac{10}{3}. \tag{6.62}$$

Furthermore, take the initial data (6.61) in such a way that

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(R^3), \ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0, \ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(R^3),$$

and

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(R^3;R^3), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^3;R^3),$$

where

$$\varrho^{(1)}_0, \ \vartheta^{(1)}_0 \in W^{1,2} \cap W^{1,\infty}(R^3), \ \mathbf{H}[\mathbf{u}_0] = \mathbf{v}_0 \in W^{2,k}(R^3;R^3) \text{ for a certain } k > \frac{5}{2}.$$

Let $T_{\max} \in (0,\infty]$ denote the maximal life-span of the regular solution $\mathbf{v}$ to the Euler system (6.58), (6.59) satisfying $\mathbf{v}(0,\cdot) = \mathbf{v}_0$. Finally, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ be a dissipative solution of the Navier-Stokes-Fourier system in $(0,T) \times R^3$, $T < T_{\max}$ introduced in Section 4.2.1.

Then

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| \ \varrho_\varepsilon(t,\cdot) - \overline{\varrho} \ \|_{L^2+L^{5/3}(R^3)} \le \varepsilon c,$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \text{ in } L^\infty_{\rm loc}((0,T];L^2_{\rm loc}(R^3;R^3)) \text{ and weakly-(*) in } L^\infty(0,T;L^2(R^3;R^3)),$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \text{ in } L^\infty_{\rm loc}((0,T];L^q_{\rm loc}(R^3;R^3)) \text{ and weakly-(*) in } L^\infty(0,T;L^2 + L^q(R^3)), \ 1 \le q < 2,$$

where $\mathbf{v}$, T is the unique solution of the Euler-Boussinesq system (6.58 - 6.60), with the initial data

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0], \ T_0 = \overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}_0 - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\varrho^{(1)}_0.$$

The proof of Theorem 6.3 represents another application and illustrate the strength of the relative entropy inequality (4.9). Results of this type for a simpler compressible Navier-Stokes system (without temperature) were obtained by Masmoudi [52], [53]. The reader will have noticed that Theorem 6.3 applies to the *dissipative* solutions, meaning solutions belonging to the same class as the weak solutions but satisfying merely the relative entropy inequality (4.9).

The complete proof of Theorem 6.3, carried over in [22], is rather involved, however, the leading idea is relatively simple and consists in taking

$$\mathbf{U} = \nabla_x \Phi_\varepsilon + \mathbf{v},\ r = \overline{\varrho} + \varepsilon R_\varepsilon,\ \Theta = \overline{\vartheta} + \varepsilon T_\varepsilon$$

as the test functions in the relative entropy inequality (4.9). Here, $\mathbf{v}$ is the solution to the incompressible Euler system, while R_ε, T_ε, and Φ_ε solve the *acoustic equation*:

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0,$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0,$$

$$\alpha = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho},\ \beta = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta},\ \omega = \overline{\varrho} \left(\alpha + \frac{\beta^2}{\delta} \right).$$

The functions R_ε, T_ε are not uniquely determined; whence we introduce the *transport equation*

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0, \tag{6.63}$$

with

$$\delta = \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}.$$

Note that equation (6.63) is nothing other than a convenient linearization of the entropy balance. The resulting system of equations is now well-posed.

6.3 Rotating fluids

Rotating fluid systems appear in numerous applications of fluid mechanics, in particular in models of atmospheric and geophysical flows, see the monograph [7]. Earth's rotation, together with the influence of gravity and the fact that atmospheric Mach number is typically very small, give rise to a large variety of singular limit problems, where some of these characteristic numbers become large or tend to zero, see Klein [37], [38].

In certain situations, it is convenient to study rotating fluids in the coordinate system attached to the fluid reference domain. Accordingly, the Coriolis and centrifugal forces will appear as a new contribution in the momentum equation. Neglecting, for the sake of simplicity, the influence of the temperature on the fluid motion we arrive at the following scaled *Navier-Stokes system* describing the time evolution of the fluid density $\varrho = \varrho(t,x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{6.64}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}(\mathbf{b} \times \varrho \mathbf{u}) + \frac{1}{\varepsilon^{2m}} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x G. \tag{6.65}$$

The term $1/\varepsilon(\mathbf{b} \times \varrho \mathbf{u})$ is the *Coriolis force*, where the small paramater ε corresponds to the *Rossby number* and

$$\mathbf{b} = [0,0,1]$$

is the (vertical) rotation axis. Accordingly,

$$\frac{1}{\varepsilon^2} \varrho \nabla_x G, \ G = |(x_1, x_2)|^2$$

is the associated *centrifugal force*. Similarly to the preceding part, ε^m is the Mach number and m a positive exponent to be fixed below.

We consider a very simple geometry of the underlying physical space $\Omega \subset \mathbb{R}^3$, namely Ω is an infinite slab,

$$\Omega = \mathbb{R}^2 \times (0,1).$$

Moreover, to eliminate entirely the effect of the boundary on the motion, we prescribe the *complete slip* boundary conditions for the velocity field

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{6.66}$$

keeping in mind that the more standard no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0$$

would drive the fluid to the rest in the asymptotic limit, namely $\mathbf{u} \to 0$ for $\varepsilon \to 0$. On the other hand, the *Navier slip condition*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \beta \mathbf{u}_{\tan} + [\mathbb{S}\mathbf{n}]_{\tan}|_{\partial\Omega} = 0, \ \beta > 0, \tag{6.67}$$

would give rise to a friction term in the limit system known as *Ekman's pumping*, see [20]. The same effect is being produced in the anisotropic case where "vertical" and "horizontal" viscosities are of different order, see Masmoudi [49], [50], [51].

As shown in Chemin at al. [7], *incompressible* rotating fluids stabilize to a 2D motion described by the vertical averages of the velocity provided the Rossby number ε tends to zero. In addition, the stabilizing effect of rotation has been exploited by many authors, see e.g. Babin, Mahalov and Nicolaenko [1], [2]. On the other hand, as we have seen in the previous discussion, compressible fluid flows in the low Mach number regime behave like the incompressible ones, see the pioneering paper by Klainerman and Majda [36].

Thus, at least for $m >> 1$, solutions of the scaled system (6.64), (6.65) are expected to be rapidly driven to incompressibility and then to stabilize to a purely horizontal motion as $\varepsilon \to 0$. Such a scenarion has been rigorously confirmed in [20], the result of which we now reproduce. One of the main stumbling blocks in the study of this *multiscale* asymptotic limit is the action of the centrifugal force that becomes large for $|x| \to \infty$ thus interfering with the acoustic waves at the far field.

6.3.1 Hypotheses and main results

The initial data are *ill-prepared* with respect to the acoustic scaling, namely,

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},$$

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon^m r_{0,\varepsilon}, \tag{6.68}$$

where $\tilde{\varrho}_\varepsilon$ is a solution of the associated *static problem*:

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} \tilde{\varrho}_\varepsilon \nabla_x G \text{ in } \Omega.$$

Moreover, we assume that the pressure $p = p(\varrho)$ satisfies

$$p \in C^1[0,\infty) \cap C^2(0,\infty),\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = c > 0 \tag{6.69}$$

for a certain $\gamma > 3/2$, and we normalize

$$P(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} G,\ \text{where } P(\varrho) = \int_1^\varrho \frac{p'(z)}{z}\,\mathrm{d}z. \tag{6.70}$$

Finally, we suppose that the initial data satisfy

$$\left\{ \tilde{\varrho}_\varepsilon^{\frac{\gamma-2}{2}} r_{0,\varepsilon} \right\}_{\varepsilon>0} \text{ is bounded in } L^2(\Omega),\ \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2 \cap L^\infty(\Omega),$$
$$\{\sqrt{\tilde{\varrho}_\varepsilon}\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(\Omega;R^3). \tag{6.71}$$

Before formulating the main result, we introduce the vertical averages

$$\langle v \rangle (x_h) = \frac{1}{|\mathcal{T}^1|} \int_{\mathcal{T}^1} v(x_h, x_3) \ \mathrm{d}x_3,$$

where $x_h = [x_1, x_2]$ denotes the "horizontal" component. We report the following result, see [20, Theorem 1]:

Theorem 6.4 *Let the pressure p satisfy hypotheses (6.69), with $\gamma > 3/2$. Let ϱ_ε, $\mathbf{u}_\varepsilon$ be a finite energy weak solution of the Navier-Stokes system in $(0,T) \times \Omega$, emanating from the initial data (6.68), (6.71). In addition, suppose that*

$$m > 10$$

and that

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \ \textit{weakly in} \ L^2(\Omega; R^3).$$

Then

$$\operatorname{ess}\sup_{t \in (0,T)} \|\varrho_\varepsilon - 1\|_{(L^2 + L^\gamma)(K)} \leq \varepsilon^m c(K) \ \textit{for any compact} \ K \subset \Omega,$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \textit{weakly in} \ L^2(0,T; W^{1,2}(\Omega; R^3)),$$

where $\mathbf{U} = [\mathbf{U}_h(x_h), 0]$ is the unique solution to the 2D incompressible Navier-Stokes system

$$\mathrm{div}_h \mathbf{U}_h = 0, \tag{6.72}$$

$$\partial_t \mathbf{U}_h + \mathrm{div}_h(\mathbf{U}_h \otimes \mathbf{U}_h) + \nabla_h \Pi = \mu \Delta_h \mathbf{U}_h, \tag{6.73}$$

with the initial data

$$\mathbf{U}_h(0, \cdot) = \Big[\mathbf{H}\Big[[\langle \mathbf{U}_0 \rangle_h, 0]\Big]\Big]_h$$

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