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A note about the rate-and-state-dependent friction model in a thermodynamical framework of the Biot-type equation

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A note about the rate-and-state-dependent friction model in a thermodynamical framework of the Biot-type equation

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Abstract: The conventional, phenomenological rate-and-state-dependent friction model of Dieterich-Ruina's type is discussed and slightly modified so that, after introducing an artificial internal variable (formally in a position like effective temperature) on the fault, it is driven by a free and a dissipative energies. In contrast to the original model, it thus allows for a formulation in the framework of rational thermodynamics, including the energy balance, and for rigorous numerical analysis. This also suggests an analogous rate-and-state-dependent plastic bulk model using damage/temperature as the state variable controlling the plastic yield stress.

Keywords: Frictional contact, Dieterich-Ruina model, effective temperature, thermodynamics, damage, plasticity.

1 Introduction – abstract Biot-type structure

Mechanical models in general (and those used in geophysics in particular) are (or should be) typically believed to be governed by energies and, most often, in a way that the conservative and the dissipative parts are separated. In the isothermal variant, the systems have a lucid abstract structure

$$\mathscr{M}'\ddot{q}+\partial_{\dot{q}}\mathscr{R}(q;\dot{q})+\partial\mathscr{E}(q)=\mathscr{F}(t,q) \tag{1}$$

with a kinetic energy $\mathscr{M}$, a (pseudo)potential of dissipative forces $\mathscr{R}(q;\cdot)$, a stored energy $\mathscr{E}$, and external forcing $\mathscr{F}$ as a function of the state q. This state typically involves, beside of the displacement, also some internal parameters like the creep and the plastic strains, damage, etc., complying with the concept of *generalized standard materials with internal parameters* [22]. The internal parameters vary in time and space in general, and thus conventionally (although not consistently e.g. with [22]) we should and will rather speak about internal variables. The inertial term $\mathscr{M}'\ddot{q}$ acts typically only on displacement, not on the internal variables and, disregarding this term, (1) is (after a series of works, cf. e.g. [6]) called a *Biot* (or Biot-type) *equation*. Actually, as indicated in the title, the main emphasis of this article holds with $\mathscr{M}\equiv 0$, as well. In (1), we use the notation $\dot{q}=\frac{\mathrm{d}q}{\mathrm{d}t}$ and $\ddot{q}=\frac{\mathrm{d}^2q}{\mathrm{d}t^2}$, and $(\cdot)'$ denotes the differential while "∂" denotes a generalized differential (typically a convex subdifferential) of functionals which can be nondifferentiable typically because they describe some unilateral, unidirectional, or activated phenomena – then (1) is an inclusion rather than an equation, cf. e.g. [47] for a brief survey of the corresponding mathematical formalism and tools.

This energy-governed structure (1) allows to control the energetics: indeed, testing (1) by $\dot{q}$, we arrive (at least formally) to the energy balance on a time interval $[0,t]$:

$$\underbrace{\mathscr{M}(\dot{q}(t))+\mathscr{E}(q(t))}_{\substack{\text{kinetic + stored energy}\\ \text{at time } t}}+\underbrace{\int_0^t\Xi(q(t);\dot{q}(t))\,\mathrm{d}t}_{\substack{\text{dissipated energy over}\\ \text{the time interval }[0,t]}}=\underbrace{\mathscr{M}(\dot{q}(0))+\mathscr{E}(q(0))}_{\substack{\text{kinetic+stored energy}\\ \text{at time 0}}}+\underbrace{\int_0^t\big\langle\mathscr{F}(t,q),\dot{q}\big\rangle\,\mathrm{d}t}_{\substack{\text{work done by loading}\\ \text{over time interval }[0,t]}} \tag{2a}$$

with the dissipation rate

$$\Xi\big(q(t);\dot{q}(t)\big)=\big\langle\partial_{\dot{q}}\mathscr{R}(q;\dot{q}(t)),\dot{q}(t)\big\rangle; \tag{2b}$$

in both (2a) and (2b), the notation $\langle\cdot,\cdot\rangle$ stands for a scalar product (or, more precisely, a duality pairing between relevant linear spaces). More precisely but without any details, (2a) is usually obtained from the sub-differential formulation rather as an inequality only, while the equality in (2a) needs some data qualification.

Moreover, this structure allows for numerically stable time discretisation, and for rigorous analysis as far as convergence and existence of solutions to (1) concerns. Also, it allows for an extension

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for anisothermal situations which is simultaneously thermodynamical consistent, i.e. comply with the total-energy conservation, nonnegativity of temperature, and the Clausius-Duhem entropy inequality, cf. Section 4 below. In particular, the dissipation rate $\Xi = \Xi(q;\dot{q})$ should be non-negative even locally for subsystems.

The structure (1) with $\Xi \geq 0$ is largely ignored in usual geophysical models and cannot be explicitly identified. Typically, no energetics like (2a) is computationally verified, neither it is even explicitly formulated. Thus, likely, a possibility to devise numerically stable or even convergent computational algorithms is lost. Numerical algorithms thus often exhibit mesh dependency, indicating their non-convergence or even lack of solutions of the particular model so that there is nothing to converge to.

The goal of this contribution is to revisit from this perspective the conventional rate-and-state-dependent friction model of the Dieterich-Ruina's type [14, 50], presented briefly in Section 2, which is used widely in modelling of stick-slip motion of lithospheric faults, see e.g. in [9, 16, 21, 26, 29, 32], or also [17] for a survey. Except Section 6, we carry out the discussion by using a usual ansatz of a so-called single degree-of-freedom slider, i.e. in terms of ordinary (instead of) partial differential equations. In Section 3, we then discuss a variant of this model which uses a given friction (= a so-called Tresca friction) and difficulties with putting the rate-and-state model into the Biot-equation context. In view of these difficulties, we attempt to modify a bit the original friction model to fit with the abstract Biot-equation context at least if augmented in a full thermodynamical concept which we abstractly formulate in Section 4. Then we use it in Section 5 to modify the rate-and-state model in a way that the evolution of the ageing variable is governed by energies while imitating arbitrarily precisely the response of the conventional rate-and-state-dependent friction model. Eventually, in Section 6, we devise a bulk analog of the friction model combining conventional concepts of Prandtl-Reuss plasticity with rate-and-state-dependent plastic yield stress, gradient damage with healing, and thermodynamically consistent heat transfer with presumably the capacity to produce narrow shear bands (= faults) with slip response like in the rate-and-state-dependent friction. This yields a hint how to transfer the sound conventional rate-and-state-dependent friction interfacial model to the conventional thermo-plastic model with temperature-dependent damage with healing.

2 Rate-and-state-dependent friction model

The original dry-friction law by Dieterich and Ruina [14, 50] uses the friction coefficient (also called *sliding resistance*) $\mu = \mu(v,\theta)$ in the form

$$\mu(v,\theta) = \mu_0 + a\ln\frac{v}{v_{\rm ref}} + b\ln\frac{v_{\rm ref}\theta}{d_{\rm c}} \tag{3}$$

and then balances the normal stress $\sigma_{\rm n} = \vec{n}^\top\sigma\vec{n}$ with the tangential stress $\sigma_{\rm t} = \sigma\vec{n} - \sigma_{\rm n}\vec{n}$ standardly as

$$\sigma_{\rm t} = \sigma_{\rm t}(v,\theta) = \sigma_{\rm n}\,\mu(v,\theta); \tag{4}$$

here σ is the stress tensor and $\vec{n} = \vec{n}(x)$ the unit normal vector to a *contact interface* (= a *fault*) at a point x, and

$$v = |\dot{u}| \tag{5}$$

with u denoting the difference of displacements on two sides of the contact interface while θ is an internal variable (or possibly a vector of internal variables) on the interface. This internal variable θ (being interpreted as an *ageing* parameter) is governed by a specific flow rule typically of the form of an ordinary differential equation at each spot of the fault, say:

$$\dot{\theta} = f_0(\theta) - f_1(\theta)v \tag{6}$$

with some continuous nonnegative functions f_0 and f_1. More specifically, $f_0(\theta) = 1$ and $f_1(\theta) = \theta/d_{\rm c}$ with $d_{\rm c} > 0$ was considered e.g. in [4, 5, 8, 13, 26, 44]; then for the static case $v = 0$, the ageing variable θ

grows linearly in time and has indeed the meaning of an "age" as a time elapsed from the time when the fault ruptured in the past. Alternatively, one can consider some modified flow rule $f_0(\theta) = (1-\theta/\theta_\infty)^+$ and $f_1(\theta) = \theta/d_c$, cf. [40], and then θ stays bounded and asymptotically approaches θ_∞ in the steady state, which suggests to interpret θ rather as a certain hardening or "gradual locking" of the fault in the "calm" steady state $v = 0$.

The parameters a, b, μ_0, and $v_{\rm ref}$ in (3) are given. If $a-b > 0$, we speak about velocity strengthening while, if $a-b < 0$, we speak about *velocity weakening* – the latter case may lead to instabilities and is used for earthquake modelling.

The difference of displacements (as three-dimensional vectors) occurring in (5) is governed by specific evolution partial differential equation in the bulk material, involving possibly also some other internal variables as the creep or various other inelastic strains. Here, however, for our purposes and for simplicity, we will rather (use what is in geophysical modelling literature often called *single degree-of-freedom slider* and) reduce it to an ordinary differential equation for a single scalar variable u:

$$\rho\ddot{u} + \sigma_t(v,\theta)\,\mathrm{sign}\,\dot{u} + \mathbb{C}u = \mathbb{C}u_{\rm D}(t) \tag{7}$$

with v from (5), and with a mass-like density $\rho > 0$, an elasticity coefficient $\mathbb{C} > 0$, and a prescribed displacement $u_{\rm D}$ varying in time; needless to say, the physical units are considered appropriate so that the physical dimension of all terms in (7) matches. In fact, "sign" in (7) denotes the set-valued signum (i.e. $\mathrm{sign}(\pm v) = \pm 1$ for $v > 0$ and $\mathrm{sign}(0) = [-1,1]$) so that, more precisely, (7) should be written as an inclusion rather than an equation. The state q considered in (1) is then the couple (u,θ).

An obvious undesired attribute of (3) is, as already noted in [17, p.108], that, "as v or θ approach zero, eqn. (3) yields unacceptably small (or negative) values of sliding resistance" μ. The energy-dissipation rate related to the friction (3) is $\sigma_t v = \mu_0\sigma_n v + a\sigma_n \ln(v/v_{\rm ref})v + b\ln(v_{\rm ref}\theta/d_c)v$. In particular, as always $\sigma_n > 0$ due to big lithostatic pressure, if $a \neq 0$, the energy-dissipation rate $\sigma_t v$ may become negative, which means that some energy is artificially pumped into the interface – this might be apparently advantageous to nucleate numerically the sliding in such a model but it violates the Clausius-Duhem inequality and is thus not physically relevant, and, after all, it also would likely destroy rigorous mathematical analysis of such a model, if any.

Such a physically nonrealistic model has been (and still is) used in dozens of articles (cf. e.g. [8,9,11, 13,21,26,32,34,37,42,44] or e.g. [25,36] for a modified but indefinite formula like (3)), relying that in specific applications the solutions might not slide into these physically wrong regimes violating Clausius-Duhem entropy inequality. Nevertheless, a regularization leading to $\mu > 0$ and thus to a physically correct non-negative dissipation is used, too, typically as [15], cf. e.g. also [40]:

$$\mu = \mu(\theta,v) = \mu_0 + \alpha(v) + \beta(\theta) \quad \text{with} \ \ \alpha(v) = a\ln\Big(\frac{v}{v_{\rm ref}}+1\Big) \ \ \text{and} \ \ \beta(\theta) = b\ln\Big(\frac{v_{\rm ref}\theta}{d_c}+1\Big). \tag{8}$$

In what follows, we will therefore have in mind rather (8) than (3). The *frictional dissipation rate* is then

$$\xi(\theta;v) = \sigma_n\mu\big(\theta,v\big)v = \sigma_n\big(\mu_0+\alpha(v)+\beta(\theta)\big)v \geq 0. \tag{9}$$

On top of it, it has been known from the beginning of this rate-and-state model that it does not fit well some experiments [49] and (rather speculative) modifications e.g. by using several ageing variables (which naturally opens a space for fitting more experiments) have been devised, cf. [50]. This indicates that the standard phenomenological rate-and-state friction model suffers, beside the conceptual drawbacks, also some weak modeling points and its slight modification is not any "crime" or even it is desired.

3 Given friction - the Tresca model

The model (4)–(8) was designed to fit with laboratory experiments, cf. e.g. [14,17,34,36]. Its usage for real earthquake modelling has some limitations. In particular, the friction-type model (4)–(8) itself with very big compressive normal stress σ_n would obviously lead to enormous tangential stresses σ_t needed to trigger deep earthquakes. Therefore, it seems well acceptable (or even more realistic) to take σ_n in

(4) as a prescribed parameter $\sigma_n^{\rm eff}$ (not even necessarily related with the lithostatic pressure and thus here not necessarily related with the evolution of u); this is sometimes called a given friction or also a *Tresca friction* model and, in geophysics, it is related with presence of fluids in pores and a concept of a so-called *effective normal stress*. This is in a certain analogy with the plastic/damage bulk model (cf. [48] or also (31) below) where a plastic strain is usually considered as a deviatoric tensor so that the spherical part of the stress tensor (= the lithostatic pressure) does not directly influence the plastification processes at all. For a comparison with the rate-and-state friction on an interface (fault) with a damage-plasticity on a narrow stripe we refer also to [32], although an analog of the flow rule (6) for the ageing variable, playing in the bulk the role of damage, is not explicit there.

Therefore, in what follows we focus on the Tresca model with

$$\sigma_n = \sigma_n^{\rm eff} = \text{ fixed effective normal stress} \tag{10}$$

in (4) and with μ from (8). In geophysical models, $\sigma_n^{\rm eff}$ is considered rather low (say 30-100 MPa), which also causes a relatively small frictional heat generated during stick-slip motion of the fault – which is known as a so-called heat-flux paradox.

Moreover, the ambiguity of the dynamics (6) in literature (or even (29) below) indicates that this model, even in its physically relevant variant (4)–(8), is very phenomenological and rather speculative.

The interpretation of θ as a hardening or a reinforcement or a “quality of locking” of the fault indicates that θ should *contribute to the stored energy* $\mathscr{E}$, cf. the Φ-terms in (23a) below. Then, vice versa, such contribution should give a driving force for the evolution of θ through the Biot structure (1).

Yet, it seems difficult or rather impossible to put the model (4)–(8) directly into the framework of the materials with internal variables and in particular the isothermal Biot equation from Section 1. This is likely the reason that the energetics of the model (4)–(8) in the sense (2) has never been scrutinized in the literature; cf. e.g. [51] where (as a positive exception) the energetics of the rate-and-state-dependent friction has been performed but eventually only the stress equilibrium like (7) without explicit involvement of the flow rule for the internal variable (like here (6)) has been balanced.

For (an unsuccessful) example, one might have an idea to re-write (6) in terms of a rescaled ageing variable $\Theta = g(\theta)$ with g a primitive function to $1/f_1$. Then, by taking (5) into account, one gets

$$\dot\Theta + |\dot u| = \frac{f_0(\theta)}{f_1(\theta)} = \frac{f_0\circ g^{-1}(\Theta)}{f_1\circ g^{-1}(\Theta)} =: F(\Theta)\,. \tag{11}$$

Introducing further a (presumably small) “nonlinear coefficient” $\varepsilon = \varepsilon(\Theta,\cdot) > 0$ monotonically dependent on the second argument and the new variable $w = \Theta + u$, (11) transforms to

$$\varepsilon(w{-}u,\dot w) = E(w{-}u) \quad \text{ with } \quad E(\Theta) := \varepsilon(\Theta, F(\Theta)) \tag{12}$$

provided that u is monotone (non-decreasing) so that $|\dot u| = \dot u$. The natural requirement that $\varepsilon(\Theta,\dot w)\dot w \geq 0$ and that E is nondecreasing and $E(0) = 0$ so that its potential, let us denote it by $\mathfrak{E}$, is convex with its minimum at 0, can be satisfied. The stored energy which would generate the driving force $\partial_u\mathscr{E}$ for (7) and $\partial_w\mathscr{E}$ for (12) is then $\mathscr{E}(u,w) = \mathfrak{E}(w-u) + \frac{1}{2}\mathbb{C}u^2$ while $\mathscr{F}(t) = \mathbb{C}u_{\rm D}(t)$ in (1), the dissipation rate is $\Xi\big(u,w;\dot u,\dot w\big) = \xi(g^{-1}(w{-}u);v) + \varepsilon(w{-}u,\dot w)\dot w$ with ξ from (9), and the kinetic energy is $\mathscr{M}(\dot u,\dot w) = \frac{\varrho}{2}|\dot u|^2$. Of course, Ξ determines also the corresponding (pseudo) potential of dissipative forces $\mathscr{R}(u,w;\dot u,\dot w)$. Such a presentation of the model (4)–(8) in the Biot framework brings however a (presumably small) modification of the original equilibrium law on the interface due to the contribution from $\mathscr{E}(u,w)$, namely as a certain *back-force* (as an analog of back-stress in plasticity with hardening) so that the force equilibrium (4) is rather

$$\sigma_t + \mathfrak{E}'(w-u) = \sigma_n^{\rm eff}\mu. \tag{13}$$

As $\varepsilon > 0$ can be chosen small, also $\mathfrak{E}'$ can be considered small, and thus this “regularization” might not have an essential influence on the response of this model provided $|\dot u| = \dot u$. Although $|\dot u| = \dot u$ might be possibly expected during a typical stick-slip motion in a one-dimensional model of a fault without inertia, the general multidimensional situation does not comply with this assumption.

Another example of an unsuccessful attempt of a direct interpretation of such model is in [52], facing difficulties with non-negativity of the dissipation rate.

Both these attempts meet the structural problem that (6) with $v = |\dot{u}|$ and thus also (11) are not monotonically dependent on the rates, which is standardly considered as an ultimate requirement. Other objection might be an inconsistency with the philosophical standpoint that instantaneous variation of "controllable" variables (i.e. here the displacement u) should not cause any instantaneous variation of internal variables (i.e. here the ageing θ), as articulated in [35, Sect. A.1.3.3]. This last point here would mean exactly $f_1 = 0$ in (6) but it would completely degenerate the model which would then become of no interest at all.

Of course, one can easily consider both evolution problems (6) and (7) to have their own stored-energy potential, cf. [41], but this (rather trivial and formal) structure does not have any deeper physical meaning.

4 Biot equation in thermodynamic context

In view of the above mentioned difficulties, our goal now is to find some (small) modification of the model (4)–(8) that would exhibit the structure of the Biot-type equation (1) and allow for a reasonable interpretation. A noteworthy "side effect" will be a possibility for a rigorous mathematical analysis and design of numerically stable and convergent algorithmic strategies, cf. (26) below. To this goal, we first augment the abstract general Biot-type model (1) to a full thermodynamical framework, however.

If the stored (or now rather *free*) *energy* $\mathscr{E} = \mathscr{E}(q,T)$ and the dissipation potential $\mathscr{R}(q,T;\dot{q})$ depend also on *temperature* T, the Biot-type equation (1) should be augmented to a full thermodynamical system

$$\mathscr{M}'\ddot{q} + \partial_{\dot{q}}\mathscr{R}(q,T;\dot{q}) + \partial_q\mathscr{E}(q,T) \ni \mathscr{F}(t,q), \tag{14a}$$

$$c_{\rm v}(q,T)\dot{T} + \mathrm{j} = \big(\partial_{\dot{q}}\mathscr{R}(q,T;\dot{q}) + T\partial_{qT}^2\mathscr{E}(q,T)\big)\dot{q} \qquad \text{with } \ c_{\rm v} = -T\partial_{TT}^2\mathscr{E}(q,T), \tag{14b}$$

where j is the abstract "heat-flux production" energy rate. Actually, the heat equation (14b) arises from the *entropy equation*

$$T\dot{s} + \mathrm{j} = \text{dissipation rate}, \tag{15}$$

where the *entropy* is defined by the Gibbs relation as

$$s = s(q,T) = -\partial_T\mathscr{E}(q,T). \tag{16}$$

From (15), we indeed obtain the heat equation (14b) just by the chain-rule

$$\dot{s} = -(\partial_T\mathscr{E}(q,T))^{\bullet} = -\partial_{qT}^2\mathscr{E}(q,T)\dot{q} - \partial_{TT}^2\mathscr{E}(q,T)\dot{T}.$$

Furthermore, defining the *internal energy* $\mathscr{W} = \mathscr{W}(q,T)$ by

$$\mathscr{W}(q,T) = \mathscr{E}(q,T) + s(q,T)T, \tag{17}$$

and using (16), we have

$$\dot{\mathscr{W}} = \partial_q\mathscr{E}\dot{q} + \partial_T\mathscr{E}\dot{T} + \dot{s}T + s\dot{T} = \partial_q\mathscr{E}\dot{q} + \dot{s}T$$

so that, by substituting (15) we obtain the *energy balance* as

$$\underbrace{\mathscr{M}(\dot{q}(t)) + \mathscr{W}\big(q(t),T(t)\big)}_{\substack{\text{kinetic + internal energy}\\ \text{at time } t}} + \underbrace{\int_0^t \mathrm{j}\,\mathrm{d}t}_{\substack{\text{heat flux over the}\\ \text{time interval } [0,t]}} = \underbrace{\mathscr{M}(\dot{q}(0)) + \mathscr{W}\big(q(0),T(0)\big)}_{\substack{\text{kinetic+internal energy}\\ \text{at time } 0}} + \underbrace{\int_0^t \big\langle \mathscr{F}(t,q),\dot{q}\big\rangle\,\mathrm{d}t}_{\substack{\text{work by loading over}\\ \text{the time interval } [0,t]}}. \tag{18}$$

5 Model (4)–(8) in thermodynamic context

The anisothermal framework (14) suggests as a first approximation to introduce, in addition to the mechanical state variables $q = (u, \theta)$, an auxiliary internal variable, let us denote it by T, formally in a position of an *effective temperature* which will be (approximately) dependent on the slip velocity magnitude v such that $T \sim Kv$ with some coefficient K. The interpretation of this variable is the temperature of a virtual layer on the contact interface between two adjacent bulk domains which have its own temperature (here considered constant). This two-temperature idea has occasionally been used in physical and engineering literature, cf. [10] and references therein, and recently in mathematical literature, too, cf. Bonetti *et al.* 2009,2011.

In geophysical literature, the heat produced during frictional sliding is believed “to produce significant changes in temperature, thus the change of strength of faults during seismic slip will be a function of ... also temperature”, cf. [11, p.7260]. The usage of an (effective) interfacial temperature occurs in [12, Chap.2], cf. also [13], following ideas from [28], claiming that the “effective temperature describes the configurational disorder in the material”, cf. [12, p.29], and that the “effective temperature is different from the thermal temperature, but it evolves in a similar manner” and “unlike state variables in friction laws such as Dieterich-Ruina, which are governed by an ordinary differential equation, effective temperature follows a partial differential equation” as (30) below, cf. [12, p.32].

Such effective temperature can also be linked with the plastic strain rate like in (31) below, which in some idealization (and localization) can be understood as an interfacial plastic rate which is actually (after a regularization) more or less just the magnitude of the slip velocity v, cf. [48, Remark 3.1].

Here, in a certain approximation, one can borrow an idea of “an external constant temperature reservoir” from [5, p.483]. In literature, cf. e.g. [8, 9, 37], such an “interfacial temperature” is known (or considered) indeed to vary in many hundreds of degrees due to a very fast slip during earthquakes even exceeding 1000°C, cf. [7, 42], while obviously the bulk temperature cannot vary substantially in this relatively short moment, so there is even some experimental motivation of this scenario. Sometimes, a microscopical explanation is by so-called “flash heating” of asperities in the fault core, cf. e.g. [4, 42]. The heat source during the slip (earthquake) exhibit an “extreme localization to a a zone $< 1-5$ mm wide within a finely granulated fault core”, see [43].

The modelling assumption which we adopt is that coefficient K is large so that the effective temperature T, which later in this section (as the simplest option) will be set approximately to Kv, ranges such big values so that the constant temperature of an external bulk can be considered just 0. (Here K is considered constant, yet it may be made depend e.g. on on v and θ, yielding a more general dependence of T on Kv and also on θ, cf. (28) below.)

The heat production rate due to the friction (= the *frictional heat*) is just $\xi = \xi(\theta; v)$ from (9). Considering the heat-transfer coefficient $\kappa = \kappa(\theta, T)$ between the contact interfacial layer and the adjacent bulk and the heat capacity of this interfacial layer $c_\text{v} = c_\text{v}(\theta, T) > 0$, the heat equation (14b) is then

$$c_\text{v}(\theta,T)\dot{T} + \kappa(\theta,T)T = \underbrace{\xi(\theta;v) + \varepsilon(\theta)|\dot{\theta}|^2}_{\substack{\text{dissipation rate } \Xi \\ \text{from (23b) below}}} + \underbrace{T\,\phi_1(\theta)\,\dot{\theta}}_{\substack{\text{adiabatic heat } T\partial^2_{qT}\mathscr{E}(q,T))\dot{q} \\ \text{determined by (23a) below}}} \tag{19}$$

with ε and ϕ_1 from (21) below.

Rather as an artificial but simple example, we further adopt a very special choice and, for a fixed K, consider the heat-transfer coefficient as

$$\kappa(\theta,T) = \frac{1}{K}\sigma_\text{n}^\text{eff}\mu\Big(\theta, \frac{T}{K}\Big) \tag{20}$$

with μ from (8); a more general ansatz is discussed in the paragraph around (28)–(29) below. The philosophy of the ansatz (20) relies on that the heat capacity c_v of this (infinitesimally thin) layer is naturally to be considered very small, as well as the coefficients $\varepsilon > 0$ and $\phi_1 > 0$ occurring in (19) are small, and then (19) with (9) is approximately $\kappa(\theta,T)T \doteq \xi(\theta;v) = \sigma_\text{n}^\text{eff}\mu(\theta,v)v$ so that, from $\kappa(\theta,T)T \doteq \sigma_\text{n}^\text{eff}\mu(\theta,\frac{T}{K})\frac{T}{K}$, we have approximately $T \doteq Kv$ provided $v \mapsto \mu(\theta,v)v$ is monotonically increasing and thus the inverse of this function does exist. Note that this monotonicity does not need monotonicity of $\mu(\theta,\cdot)$ itself, and thus does not exclude slip weakening.

Having in mind the above mentioned scenario of suppression of the adiabatic effect and the influence of the heat capacity and conductivity of the interfacial layer, we can rely on $T \sim Kv$ and we now replace v in (6) by T and write the ageing flow rule in the form

$$\varepsilon(\theta)\dot{\theta} = \phi_0(\theta) - \phi_1(\theta)T \tag{21a}$$

$$\text{with} \quad \phi_0(\theta) = \varepsilon(\theta) f_0(\theta) \tag{21b}$$

$$\text{and} \quad \phi_1(\theta) = \varepsilon(\theta) f_1(\theta)/K \tag{21c}$$

with some $\varepsilon(\theta) > 0$ to be chosen arbitrarily (and presumably small). Note that the last term in (19) is approximately

$$T\phi_1(\theta)\dot{\theta} \doteq Kv\frac{\varepsilon(\theta) f_1(\theta)}{K}\dot{\theta} = v\varepsilon(\theta) f_1(\theta)\dot{\theta} \tag{22}$$

so that, by choosing ε small, both terms $\varepsilon(\theta)|\dot{\theta}|^2$ and $T\phi_1(\theta)\dot{\theta}$ in the right hand side of (19) can indeed be made negligible provided the slip velocity v and the ageing rate $\dot{\theta}$ stay bounded. This scaling suppresses the influence of the ageing flow rule (21) to the overall energetics but anyhow does not totally ignore it, in contrast to the standing rate-and-state geophysical models.

Let us still assume, for simplicity, that the heat capacity $c_{\rm v}(\theta,T) = c_{\rm v}(T)$ depends only on temperature T. The overall free energy, the dissipation rate, and kinetic energy are

$$\mathscr{E}(u,\theta,T) = \frac{1}{2}\mathbb{C}u^2 + \Phi_1(\theta)T - \Phi_0(\theta) + \Psi(T), \tag{23a}$$

$$\Xi(\theta;\dot{u},\dot{\theta}) = \xi(\theta;|\dot{u}|) + \varepsilon(\theta)|\dot{\theta}|^2, \tag{23b}$$

$$\mathscr{M}(\dot{u}) = \frac{\rho}{2}|\dot{u}|^2, \tag{23c}$$

and $\mathscr{F} = \mathbb{C}u_{\rm D}(t)$. In (23a), Φ_0 and Φ_1 are primitive functions to ϕ_0 and ϕ_1 from (21), respectively. Note that, as $\phi_0 > 0$, the contribution $-\Phi_0(\cdot)$ into the interfacial energy is a decreasing function so that, if the ageing variable θ increases, this energy decreases. The function Ψ is determined (up to an unimportant affine term) from $\Psi''(T) = c_{\rm v}(T)/T$.

The advantageous feature of this setting is that the entropy $s(\theta,T) = -\Phi_1(\theta) - \Psi'(T)$ separates the mechanical and the thermal variables, so that the resulted heat capacity $c_{\rm v}(\theta,T) = -T\Psi''(T)$ is indeed only temperature dependent. Yet, a general coupling $\Phi(\theta,T)$ is also possible, and modern techniques are developed for treating general situations even in the distributed-parameter setting like (30) below, cf. e.g. [45, Sect. 13.9].

The resulted system (4)–(5)–(7)–(19)–(21) is to be solved for the initial conditions at time $t = 0$:

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0, \quad T(0) = T_0 \tag{24}$$

with $(u_0, v_0, \theta_0, T_0)$ prescribed. The abstract energy balance (18) is now obtained by testing (7) by $\dot{u}$, (21) by $\dot{\theta}$, and (19) by 1. Denoting by $C_{\rm v}$ the primitive function to the heat capacity $c_{\rm v}$, one thus gets

$$\begin{aligned}\frac{\rho}{2}|\dot{u}(t)|^2 + C_{\rm v}(T(t)) + \frac{1}{2}\mathbb{C}u(t)^2 - \Phi_0(\theta(t)) &= \frac{\rho}{2}|v_0|^2 \\ &+ C_{\rm v}(T_0) + \frac{1}{2}\mathbb{C}u_0^2 - \Phi_0(\theta_0) + \int_0^t \mathbb{C}u_{\rm D}\dot{u} + \kappa(\theta,T)T\,{\rm d}t.\end{aligned} \tag{25}$$

The asymptotics for $\mathbb{C} \to \infty$ leads to $u \sim u_{\rm D}$ and (19)–(21) decouples from the rest of the system, and allows directly for a "canonical" experiment with jumping piece-wise constant velocity of the slip $u \doteq u_{\rm D} = u_{\rm D}(t)$ to obtain a typical jumping response of the sliding resistance $\mu = \mu(\theta,T)$ with $T \doteq Kv = K\dot{u}_{\rm D}$ like this one presented in literature, cf. e.g. [14, 17, 32, 34, 50], cf. also Fig. 1 below. Philosophically, letting the sliding resistance dependent also on temperature in a way that it increases when temperature increasing is compatible with some computational and experimental studies, cf. e.g. [8, 37].

Considering an equidistant time partition with the time step $\tau > 0$, and denoting by u_τ^k the approximate value of $u(k\tau)$, and similarly for T_τ^k etc., a semi-implicit *time discretisation* of the system (4)-(5)-(7)-(19)-(21) can be done by the formula:

$$\rho\frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2} + \mathbb{C}u_\tau^k + \sigma_{t,\tau}^k \operatorname{sign}(u_\tau^k - u_\tau^{k-1}) = \mathbb{C}u_{\rm D}(k\tau), \tag{26a}$$

$$\varepsilon(\theta_\tau^{k-1})\frac{\theta_\tau^k - \theta_\tau^{k-1}}{\tau} = \phi_0(\theta_\tau^k) - \phi_1(\theta_\tau^k)T_\tau^k, \tag{26b}$$

$$c_{\rm v}(T_\tau^k)\frac{T_\tau^k - T_\tau^{k-1}}{\tau} + \kappa(\theta_\tau^k, T_\tau^{k-1})T_\tau^k = \xi(\theta_\tau^k; v_\tau^k) + \varepsilon(\theta_\tau^{k-1})\Big|\frac{\theta_\tau^k - \theta_\tau^{k-1}}{\tau}\Big|^2 + T_\tau^k\phi_1(\theta_\tau^k)\frac{\theta_\tau^k - \theta_\tau^{k-1}}{\tau}, \tag{26c}$$

$$\sigma_{t,\tau}^k = \sigma_{\rm n}^{\rm eff}\mu(\theta_\tau^k, T_\tau^{k-1}), \qquad v_\tau^k = \Big|\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\Big|. \tag{26d}$$

The system (26) is to be solved recursively for $k = 1, 2, \ldots$, starting from the initial conditions $u_\tau^0 = u_0$, $u_\tau^{-1} = u_\tau^0 - \tau v_0$, $\theta_\tau^0 = \theta_0$, and $T_\tau^0 = T_0$, cf. (24). Existence of a solution to this fully coupled system is usually to be proved by a fixed-point argument and the computational implementation is by an iterative numerical procedure. Assuming naturally $T_0 \geq 0$, we have $T_\tau^k \geq 0$ for all k. Other natural assumptions $\theta_0 \geq 0$ and $f_1(0) = 0$ yields that $\theta_\tau^k \geq 0$ for all k. Yet, it should be emphasized that usage of (26) for computational purposes is practically limited to quasi-static case (i.e. $\rho = 0$) due to typically unacceptably large numerical attenuation of elastic waves in this theoretical discretisation scheme, i.e. the conservation of energy of the type (18) with a reasonably good accuracy if also inertial effects are counted requires unacceptably small time step $\tau > 0$. Threrefore, more sophisticated time-integration strategies are to be used, although usually their theoretical convergence in general cases is not studied. Examples of efficient methods used in seismic-wave modelling are an explicit acceleration Newmark scheme [26] or an arbitrarily high-order derivative (ADER) time integration method [19, 27] combined with various spatial discretisation in the distributed-parameter cases, e.g. discontinuous Galerkin method, as also in [39], or finite differences as e.g. in [44], etc.

The analog of the energy balance (25) at least as an inequality can be obtained by testing (26a,b,c) respectively by $u_\tau^k - u_\tau^{k-1}$, $\theta_\tau^k - \theta_\tau^{k-1}$, and τ. By summation, we can enjoy cancellation of the dissipative terms $\pm\varepsilon(\theta_\tau^{k-1})|\frac{\theta_\tau^k - \theta_\tau^{k-1}}{\tau}|^2$ as well as the adiabatic terms $\pm T_\tau^k\phi_1(\theta_\tau^k)\frac{\theta_\tau^k - \theta_\tau^{k-1}}{\tau}$. Assuming ϕ_1 nondecreasing (so that Φ_1 is convex), ϕ_0 nonincreasing (so that Φ_0 concave), and $c_{\rm v}$ nondecreasing (so that $C_{\rm v}$ is convex), we thus obtain an analog of (25), namely

$$\frac{\rho}{2}|v_\tau^k|^2 + C_{\rm v}(T_\tau^k) + \frac{1}{2}\mathbb{C}(u_\tau^k)^2 - \Phi_0(\theta_\tau^k) = \frac{\rho}{2}|v_0|^2 + C_{\rm v}(T_0) + \frac{1}{2}\mathbb{C}u_0^2 - \Phi_0(\theta_0) + \sum_{l=1}^k \mathbb{C}u_{\rm D}(l\tau)\frac{u_\tau^l - u_\tau^{l-1}}{\tau} + \kappa(\theta_\tau^l, T_\tau^l)T_\tau^l\,. \tag{27}$$

Without going into technical (but standard) details, let us only mention that, by using the discrete Gronwall inequality, (27) yields a-priori estimates of both the stored and the internal energies uniformly in time, and then by modifying the above test also of the dissipated energy. This makes a rigorous base for numerical stability of the algorithm (26) and also the convergence for $\tau \to 0$ in such a way that it allows for an extension on the distributed-parameter variants as outlined in Section 6 below.

The above interpretation allows also for various generalizations: e.g. one can consider another ansatz than (20), e.g. just $\kappa > 0$ a (small) constant. Then the effective temperature T would, instead of Kv, be (in a nonlinear way) dependent approximately on both v and also on θ. Vice versa, v would depend nonlinearly on θ and T, and then the flow rule (21) would take a more general form, say

$$\varepsilon(\theta)\dot{\theta} = \phi(\theta, T). \tag{28}$$

Then, in (23a), T could be coupled with θ in a general nonlinear manner, so that the entropy would not separate temperature and mechanical variables, and the heat capacity would depend also on that mechanical variables and the treatment of such models would be more complicated, cf. [45, Sect. 13.9]. Such nonlinear variant of (21) suggests to consider directly a nonlinear variant of (6). Examples for it might be

$$\dot{\theta} = -\frac{v\theta}{d_c}\ln\Big(\frac{v\theta}{d_c}\Big) \quad \text{or} \quad \dot{\theta} = 1-\Big(\frac{v\theta}{2d_c}\Big)^2 \tag{29}$$

devised in [50] or [40], cf. also for [34] for comparison with (6). For some others, see e.g. [51]. Sometimes even more general dynamics e.g. involving $\dot{v}$ can be found in literature, cf. [36, Formula (20)]. Another generalization might consist in $\dot{\theta}$ acting nonlinearly in (21a) similarly as used in engineering models for the bulk damage, cf. (31c,d) below; this can also capture activation phenomena for ageing.

6 Towards distributed-parameter models

The "full" distributed-parameter variant of the above presented model should consider $T = T(t,x)$ and $v = v(t,x)$ with x ranging the 2-dimensional interface (the fault), and the heat equation (19) is then

$$c_v(\theta,T)\dot{T} + \kappa(\theta,T)T - \mathrm{div}_S(\kappa_s\nabla_S T) = \xi(\theta;v) + \varepsilon(\theta)|\dot{\theta}|^2 + T\phi_1(\theta)\dot{\theta}, \tag{30}$$

where "div_S" and ∇_S denotes the surface divergence and gradient operators, respectively. Assuming that, likewise c_v, also the heat conductivity $\kappa_s > 0$ of the (infinitesimally thin) interface is very small, we can again rely on the scenario $T \doteq Kv$. As for $u = u(t,x)$, x ranges the 3-dimensional bulk around the fault. Then (7) should be replaced by the force equilibrium in the bulk $\rho\ddot{u} - \mathrm{div}\,\sigma = f$ with the visco-elastic stress $\sigma = \mathbb{D}e(\dot{u}) + \mathbb{C}e(u)$ with the small-strain tensor $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$, the visco- and the elastic-moduli tensors $\mathbb{D}$ and $\mathbb{C}$, and f the gravity force. It is to be accompanied by the boundary conditions $u = u_D$ and the Coulomb friction law (in the Tresca simplification) on the fault $[\![u]\!]_n = 0$ and $[\![\dot{u}]\!]_t = 0$ if $|\sigma_t| < \sigma_n^{\text{eff}}\mu(\theta,T)$ and $\sigma_t = \sigma_n^{\text{eff}}\mu(\theta,T/K)[\![\dot{u}]\!]_t/|[\![\dot{u}]\!]_t|$, where $[\![u]\!]_n = [\![u]\!]\cdot\vec{n}$, $[\![u]\!]_t = [\![u]\!] - [\![u]\!]_n\vec{n}$, and $\sigma_t = \sigma\vec{n} - (\vec{n}^\top\sigma\vec{n})\vec{n}$ with the unit normal $\vec{n}$ to the interface, while (5) should be replaced by $v = |[\![\dot{u}]\!]|$ where $[\![\cdot]\!]$ denotes the difference of the traces from both sides of the interface. In fact, some ("enough dissipating" variant of) Maxwellien-type rheology can rather be used for the stress σ, e.g. the Jeffreys' one, cf. [33, 48]. The Tresca variant represents certain simplification allowing, in particular, for an analysis of the model. Another variant is to allow for a real non-constant normal stress $\sigma_n^{\text{eff}} = |\vec{n}^\top\sigma\vec{n}|$ but make an adhesive-contact regularization like in Bonetti *et al.* 2012.

The semi-implicit discretisation of the type (26) combined with a spatial discretisation can yield a numerically stable and convergent computational scheme. The analysis of the parabolic heat equation (30) needs a rather sophisticated estimation of the temperature gradient by using a "nonlinear test" by $1-(1+T)^{-\delta}$ with $\delta > 0$ and an interpolation by means of the Gagliardo-Nirenberg inequality; cf. e.g. [45, Ch. 9] for rather complicated details.

Eventually, one can take inspiration from the above thermodynamically consistent modification of the rate-and-state friction and formulate a corresponding *bulk model*, using a very conventional concepts. More specifically, one can replace the concept of frictional contact by the concept of *plasticity* without hardening which, in the rate-independent variant typically give rise to infinitesimally thin *shear bands* which then yield the same effects as interfaces (= faults). The frictional stress $\sigma_n^{\text{eff}}\mu(\theta,T)$ then becomes a *rate-and-state-dependent plastic yield stress*, while the ageing variable θ stands in a position of *damage* which allows also for *healing*, cf. also [48, Remark 3.2]. Using a deviatoric (i.e. $\mathrm{tr}\,\pi \equiv 0$) plastic strain π as another internal variable so that the mechanical state will be $q = (u,\theta,\pi)$, such model reads as

$$\rho\ddot{u} - \mathrm{div}\,\sigma = f \quad \text{with} \quad \sigma = \mathbb{D}(e(\dot{u}) - \dot{\pi}) + \mathbb{C}(e(u) - \pi), \tag{31a}$$

$$\partial_{\dot{\pi}} r(\theta,T;\dot{\pi}) = \mathrm{dev}\,\sigma, \tag{31b}$$

$$\varepsilon(\theta,T;\dot{\theta}) = \phi_0(\theta) - \phi_1(\theta)T + \gamma\Delta\theta, \tag{31c}$$

$$c_v(T)\dot{T} - \mathrm{div}(\kappa(\theta,T)\nabla T) = \xi(\theta,T;\dot{u},\dot{\pi}) + \Big(\varepsilon(\theta,T;\dot{\theta}) + T\phi_1(\theta)\Big)\dot{\theta}, \tag{31d}$$

with $\mathrm{dev}\,\sigma = \sigma - \frac{1}{3}(\mathrm{tr}\,\sigma)\mathbb{I}$ denoting the deviatoric part of the stress tensor σ, with the dissipation rate

$$\xi(\theta,T;\dot{u},\dot{\pi}) = \mathbb{D}(e(\dot{u}) - \dot{\pi}):(e(\dot{u}) - \dot{\pi}) + \partial_{\dot{\pi}} r(\theta,T;\dot{\pi}):\dot{\pi}. \tag{31e}$$

The notation “ : ” in (31e) means scalar product of two 3×3-tensors, summing them over two indexes. The coefficient $\gamma>0$ in (31c) measures, roughly speaking, influence of the damage at a material point on/from its vicinity and, in this way, determines the length-scale of the damage (i.e. the width of the typical damage zone of a fault, usually 1-100 m). This is a so-called *gradient damage* theory (cf. e.g. [3, 20, 38]), used in geophysical models e.g. in [33]. Bigger γ will lead to wider damage zone and vice versa, and a particular value of γ is a vital part of phenomenology related, within the framework of this model, to a particular fault modelled.

Like in (7), $\partial_{\dot\pi} r(\theta,T;\cdot)$ is typically set-valued due to non-smoothness of $r(\theta,T;\cdot)$ at 0, describing an activated character of the plastification process, cf. (33) below. Thus (31b) is to be rather an inclusion than the equation.

The energy balance of the type (25) can be obtained by multiplying the particular equations in (31) respectively by $\dot u$, $\dot\pi$, $\dot\theta$, and 1, integrating over the considered spatial domain (let us denote it by Ω) and using Green formula and specific boundary conditions (here not specified, however). This leads to the energy balance of the type (18) with the free energy

$$\mathscr{E}(u,\pi,\theta,T)=\int_\Omega\frac12\mathbb{C}(e(u)-\pi){:}(e(u)-\pi)+\Phi_1(\theta)T-\Phi_0(\theta)+\Psi(T)+\frac{\gamma}{2}|\nabla\theta|^2\mathrm{d}x \tag{32a}$$

cf. (23a) for Φ’s and Ψ, so that the internal energy (17) is then

$$\begin{aligned}\mathscr{W}(u,\pi,\theta,T)&=\mathscr{E}(u,\pi,\theta,T)+\int_\Omega s(q,T)T\,\mathrm{d}x\\&=\int_\Omega\frac12\mathbb{C}(e(u)-\pi){:}(e(u)-\pi)-\Phi_0(\theta)+C_{\rm v}(T)+\frac{\gamma}{2}|\nabla\theta|^2\mathrm{d}x,\end{aligned} \tag{32b}$$

and the kinetic energy is now $\mathscr{M}(\dot u)=\int_\Omega\frac{\rho}{2}|\dot u|^2\mathrm{d}x$, so that, up to external heat/mechanical sources, (18) leads to *conservation of the total energy* in the form

$$\mathscr{M}(\dot u)+\mathscr{W}(u,\pi,\theta,T)=\int_\Omega\frac{\rho}{2}|\dot u|^2+\frac12\mathbb{C}(e(u)-\pi){:}(e(u)-\pi)-\Phi_0(\theta)+C_{\rm v}(T)+\frac{\gamma}{2}|\nabla\theta|^2\mathrm{d}x.$$

Let us note the analog of (31c) and (30) with (21a) and (31d), respectively. Note also that, likewise $\partial_{\dot\pi} r$ and $c_{\rm v}$, also $\varepsilon=\varepsilon(\theta,T;\dot\theta)$ in (31c,d) depends on temperature T; such type of dependence is used in geophysical modelling e.g. in [31, 32]. Moreover, instead of a mere positive coefficient, ε is considered in (31) as possibly a nonlinear operator acting on on $\dot\theta$, which allows for modelling activation phenomena as typically desirable (and usual) in damage mechanics. On the other hand, in contrast to the damage-controlled plastification flow rule $c\dot\pi=(\dot\theta)^+\mathrm{dev}\,\sigma$ with c so-called effective damage-related viscosity frequently used in geophysical models (cf. e.g. [23, 30, 32, 33]) with only limited validity bacause plastification obviously stops when damage is completed, our flow rule (31b) follows the standard and widely used concept of plasticity at small strains, cf. e.g. [24, 35] or also [1, 46]. Such plastic model combined with damage was tested computationally (in a variant of the single degree-of-freedom slider) to demonstrate a capacity to simulate stick-slip type motion and reoccurring earthquakes in [48] under external loading with a constant rate even already in an isothermal variant. Further tests of the full model (31) to demonstrate ability to develop narrow shear bands (= faults) and imitating the typical response of the rate-and-state friction model on them is desirable and expected in future work. Of course, computational implementation of narrow shear bends with not a-prioiri known position (as e.g. in nucleation of a new fault) is not easy; cf. [2] for a 2-dimensional perfect thermo-plasticity without damage, however.

Imitating the frictional force $\sigma_{\rm t}$ considered before as $\sigma_{\rm n}^{\rm eff}(\mu_0+\alpha(|[\![\dot u]\!]_{\rm t}|)+\beta(\theta))[\![\dot u]\!]_{\rm t}/|[\![\dot u]\!]_{\rm t}|$ or, nearly equivalently, temperature-dependent (formally similarly as in [5, 11]) as $\sigma_{\rm n}^{\rm eff}(\mu_0+\alpha(T/K)+\beta(\theta))[\![\dot u]\!]_{\rm t}/|[\![\dot u]\!]_{\rm t}|$, the (pseudo) potential of dissipative forces in (31b) can combine both options and be considered as

$$r(\theta,T;\dot\pi)=\underbrace{\sigma_{\rm n}^{\rm eff}\big(\mu_0+\alpha_0(|\dot\pi|)+\alpha_1(T)+\beta(\theta)\big)}_{\substack{\text{rate-and-state-dependent}\\ \text{plastic yield stress}}}\,|\dot\pi| \tag{33}$$

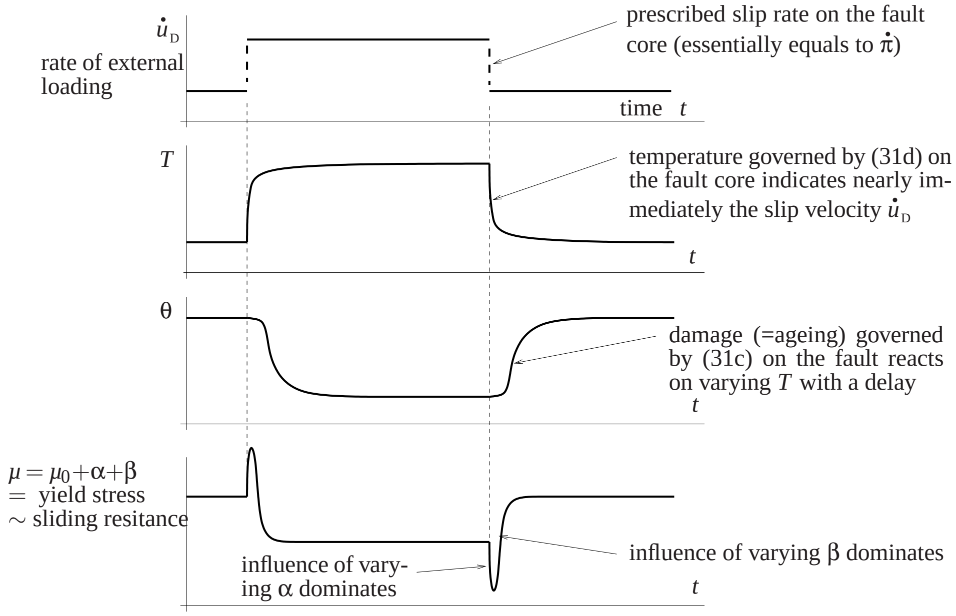

Fig. 1. A schematic illustration of the response of the displacement-driven experiment of the single-degree-of-freedom-slider type on the jumping velocity of the loading, assuming rigid rock blocks sliding by prescribed u_D, imitating the standard laboratory experiment as e.g. in [5,11,14,17,18,25,32,50,51] showing a typical nonmonotone response of the sliding resistance μ.

with suitable α_0, α_1, $\beta > 0$, and a fixed $\sigma_n^{\rm eff}$. The dependence of the coefficient α_0 on $|\dot{\pi}|$ makes the flow rule (31b) rate dependent and determines the length-scale of the core of the fault (usually much narrower than the damage zone) related to slide velocity. If $\alpha_0 = 0$, we would get perfect plasticity; the mathematical difficulties can be seen from [46] and would require here also $\alpha_1 = \beta = 0$ to justify rigorously such model. Therefore, one should consider $\alpha_0 > 0$ and then we get the rate-dependent plasticity with the temperature-dependent yield-stress $\sigma_n^{\rm eff}(\mu_0+\alpha_1(T))$, like that one mathematically and computationally scrutinized in [1], here additionally damage dependent like in [48], combined with the gradient-damage model. The combination of plasticity and damage is indeed a very classical concept, cf. e.g. [35, Ch. 10]. For conceptually similar plasticity-damage model and its scaling and comparison with the rate-and-state-dependent friction see also [30, 32]. If also the elastic response were influenced by damage, i.e. if $\mathbb{C} = \mathbb{C}(\theta)$, then the term $\frac{1}{2}\mathbb{C}'(\theta)(e(u)-\pi)^2$ would contribute to the driving force in (31c), cf. [48]. Furthermore, a stored energy which is non-quadratic and even non-convex in terms of the elastic strain $e(u)-\pi$ as proposed and used in [31] and then also e.g. in [23, 30] might be considered but rigorous mathematical treatment would then require to include the concept of non-simple materials, cf. e.g. [45, Sect. 13.9].

Let us very schematically explain the desired functioning of the model (31) on the displacement-driven experiment for which the state-and-rate friction model has been originally devised in [14, 49]. It consists of two sliding rock blocks which are considered as nearly rigid (i.e. $\mathbb{C}$ very large) so that the displacement of the controlled boundary is right transferred on the flat contact boundary (i.e. in our model (31) the thin fault core) and the bulk model (31) then imitates the one-degree-of-freedom slider from Section 5. An important phenomenon is that the heat conductivity κ is influenced by damage θ, cf. (20), reflecting in a rough way the natural expectation that more damaged rock conducts heat harder (and, vice versa, less damaged rock is a better heat conductor) so that varying the heat production due to θ does not substantially influence T which stays approximately reflecting the sliding velocity $v \sim \dot{u}_D$ (in the simplest scenario as $T \sim Kv$). The delay in the response of θ, important for getting qualitatively the desired nonmonotone response of the yield stress μ which is in the position of sliding resistance in this experiment, is due to the "inertia coefficient" $\varepsilon(\theta,\cdot)$ in (31c) and also due to the diffusion $\gamma\Delta\theta$ which causes a certain delay by damaging or healing also a bit wider vicinity of the narrow fault core.

7 Conclusion

The modification (or, one can say a regularization) of the conventional rate-and-state-dependent friction model in order to comply with standard framework of rational thermodynamics and to allow for formulation of an energy balance and a rigorous numerical analysis has been addressed.

A simplified lumped-parameter setting admitting the interpretation as a single degree-of-freedom slider (leading to a system of ordinary differential equations instead of partial differential equations) with a given normal force (called the Tresca friction) has been used to explain an essence of the model in a lucid way. Some physically inconsistent attributes of this friction model, as also sometimes presented in literature, have been discussed. Lacking attempts in literature to identify thermodynamical driving forces governing the model and to derive an explicit energetics were pointed out, together with presentation of some unsuccessful attempts documenting the difficulties, although the clear proof of impossibility of thermodynamically consistent formulation has not been (and does not seem easy to be) casted.

Due to these difficulties in particular in isothermal setting, an auxiliary internal variable (perhaps only formally) in a position of an interfacial effective temperature T governed by the heat-transfer equation (19) has been introduced. For some coefficients in this equation chosen small, we could use that, from (19) with (9), the heat flux $\kappa(\theta,T)T$ is approximately the frictional heat production rate $\xi(\theta;v) = \sigma_{\rm n}^{\rm eff}\mu(\theta,v)v$, and for this heat production $\sigma_{\rm n}^{\rm eff}\mu(\theta,v)v$ depending monotonically on the velocity magnitude v, we could therefore consider approximately $T\sim Kv$. On such conditions, we can use such auxiliary temperature T instead of v in the flow rule for ageing variable. With this (small) modification, we recover (approximately) the original Dieterich-Ruina's type model (4)–(8) in the mentioned Tresca variant, i.e. with $\sigma_{\rm n}^{\rm eff}$ given, but now in the rational-thermodynamical context. A numerically stable semi-implicit time discretisation has been devised.

The corresponding partial-differential-equation variant of this frictional contact can serve to devise a bulk model combining standard perfect, rate-independent (so-called Prandtl-Reuss) plasticity and gradient damage (which itself is very standard concept in engineering, cf. e.g. [3] for a survey) allowing for healing, which itself can already model re-occurring earthquakes, as shown in [48], and which is still completed here in a thermodynamically consistent way by a heat-transfer equation to allow for modelling the typical response of the sliding resistance as schematically depicted on Figure 1. While influence of the damage (representing an analog of ageing) on the elastic response is not pronounced (like in the isothermal tests in [48]) or, more precisely, here even completely suppressed, its influence on the dissipation through the yield stress for plastification is essential and at this point the bulk model advantageously transfers the phenomenology from the rate-and-state friction model. One can perhaps interprete this yield-stress decay within the rising temperature and decaying damage/ageing (at least partly) as a melting of the fault core, which is the effect well advocated in the literature, cf. e.g. [4, 7, 42].

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