A generalization of the Darcy-Forchheimer equation involving an implicit, pressure-dependent relation between the drag force and the velocity

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Abstract

We study mathematical properties of steady flows described by the system of equations generalizing the classical porous media models of Darcy's and Forchheimer's. The considered generalizations are outlined by implicit relations between the drag force and the velocity, that are in addition parametrized by the pressure. We analyze such drag force–velocity relations which are described through a maximal monotone graph varying continuously with the pressure. Large-data existence of a solution to this system is established, whereupon we show that under certain assumptions on data, the pressure satisfies a maximum or minimum principle, even if the drag coefficient depends on the pressure exponentially.

Keywords

Darcy-Forchheimer equation, pressure dependent material coefficient, implicit constitutive theory, maximal monotone graph, existence theory, maximum/minimum principle

1 Introduction

1.1 Setting

Our aim is to develop a mathematical theory for steady, isochoric flows through a saturated porous medium described as the problem of finding a triplet $(\boldsymbol{m}, \boldsymbol{v}, p) : \Omega \to \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ solving

$$\left.\begin{aligned} \nabla p + \boldsymbol{m} &= \boldsymbol{f} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega, \\ \boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) &= \boldsymbol{0} && \text{in } \Omega, \\ (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_1, \\ p - p_0 &= 0 && \text{on } \Gamma_2. \end{aligned}\right\} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$ is supposed to be a Lipschitz domain with an outer normal $\boldsymbol{n}$ and $\Gamma_{1,2} \subset \partial\Omega$ are relatively open parts of the boundary such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$. The reader may know Γ_1 as the *exterior boundary* and Γ_2 as the *accessible boundary*, see [2]. A velocity field $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^d$ is given to dictate the normal component of $\boldsymbol{v}$ on Γ_1, as well as $p_0 : \Omega \to \mathbb{R}$, prescribing the boundary pressure on Γ_2. Known external body forces are contained in $\boldsymbol{f} : \Omega \to \mathbb{R}^d$. Throughout the paper there will often

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appear a real number r, always satisfying $1 < r < \infty$, and we define $r' := r/(r-1)$. Traces and normal traces are not denoted differently from the original functions, i.e. we write, for example, $p_0 \in W^{1,r'}(\Omega)$ as well as $p_0 \in L^\infty(\Gamma_2)$.

The quantity $\boldsymbol{h} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ in $(1)_3$ is a given continuous function and we will make the following identification:

$$\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) = \mathbf{0} \quad \Longleftrightarrow \quad (\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A}, \tag{2}$$

where $\mathcal{A}$ denotes a *maximal monotone r-graph* with respect to $\boldsymbol{m}$ and $\boldsymbol{v}$ that is in addition parametrized by p. This means $\mathcal{A} \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ satisfies each of the conditions listed below:

(A1) *inclusion of the origin*

$$\forall p \in \mathbb{R} : (\mathbf{0}, \mathbf{0}, p) \in \mathcal{A},$$

(A2) *monotonicity*

$$\forall (\boldsymbol{m}_1, \boldsymbol{v}_1, p), (\boldsymbol{m}_2, \boldsymbol{v}_2, p) \in \mathcal{A} : (\boldsymbol{m}_1 - \boldsymbol{m}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) \geq 0,$$

(A3) *maximality*

$$(\boldsymbol{m}', \boldsymbol{v}', p) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R},$$
$$\forall (\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A} : (\boldsymbol{m}' - \boldsymbol{m}) \cdot (\boldsymbol{v}' - \boldsymbol{v}) \geq 0 \Rightarrow (\boldsymbol{m}', \boldsymbol{v}', p) \in \mathcal{A},$$

(A4) *(r, r')-coercivity for $\boldsymbol{v}$ and $\boldsymbol{m}$*

$$\exists\, c_1 > 0, c_2 \geq 0\ \forall (\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A} : \boldsymbol{m} \cdot \boldsymbol{v} \geq c_1(|\boldsymbol{v}|^r + |\boldsymbol{m}|^{r'}) - c_2,$$

(A5) *existence of a Carathéodory selection*, i.e. $\boldsymbol{m}^* : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ such that

(i) $\boldsymbol{m}^*(\cdot, p) : \mathbb{R}^d \to \mathbb{R}^d$ is measurable for every $p \in \mathbb{R}$,
(ii) $\boldsymbol{m}^*(\boldsymbol{v}, \cdot) : \mathbb{R} \to \mathbb{R}^d$ is continuous for a.e. $\boldsymbol{v} \in \mathbb{R}^d$,
(iii) $\forall (\boldsymbol{v}, p) \in \mathbb{R}^d \times \mathbb{R} : (\boldsymbol{m}^*(\boldsymbol{v}, p), \boldsymbol{v}, p) \in \mathcal{A}$,
(iv) $\exists\, c > 0\, \forall (\boldsymbol{v}, p) \in \mathbb{R}^d \times \mathbb{R} : |\boldsymbol{m}^*(\boldsymbol{v}, p)| \leq c(1 + |\boldsymbol{v}|^{r-1})$.

1.2 Motivation and examples

The problem (1) describes steady (slow) flows of fluids through porous media (see for example Nield and Bejan [25]). It can be also viewed as a special case in the hierarchical development of the theory of interacting continua (as presented in Rajagopal [28]), where we ignore the viscous effects within the fluid but take into account only the drag due to the flow which is a consequence of the friction at the solid pores as the fluid flows. This leads to the relation between $\boldsymbol{m}$, representing the interaction force (linear momentum) between a fluid and a rigid solid, and the velocity of the fluid $\boldsymbol{v}$. Since $\boldsymbol{v}$ is also the relative velocity between the solid and the liquid, it is frame-indifferent. Taking the simplest case $\boldsymbol{m} = \alpha \boldsymbol{v}$ for certain $\alpha > 0$, one obtains a well known *Darcy's law* for an isotropic medium. Its linearity in the seeping velocity $\boldsymbol{v}$ does not relate well to reality for other than *sufficiently small* velocities [25, 31] and one is driven to a non-linear extension of the form $\boldsymbol{m} = \alpha(|\boldsymbol{v}|)\boldsymbol{v}$, known as *(Darcy-)Forchheimer's equation* if α is an affine function. Moving on to $\boldsymbol{m} = \alpha(p, |\boldsymbol{v}|)\boldsymbol{v}$ as a means of capturing a pressure-related viscosity [18, 32] yields a generalized *Darcy-Forchheimer's model.* As Rajagopal [27] argued, it turns out that not even such setting is always satisfactory in mathematical modelling and one is driven to relate $\boldsymbol{m}$, $\boldsymbol{v}$ and p implicitly, hence $(1)_3$.

Apart from Darcy's or Darcy-Forchheimer's models, which are somewhat uninteresting in regard to our setting emphasizing p-dependent interactions, a prime example satisfying **(A1)**–**(A5)** that the reader might have in mind is $\mathcal{A}$ with $\boldsymbol{m}$ given as e.g.

$$\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{v}, p) = \alpha(p)|\boldsymbol{v}|^{r-2}\boldsymbol{v}, \tag{3}$$

with $r > 1$ and $\alpha \in \mathcal{C}(\mathbb{R})$, satisfying also $0 < \inf_{\mathbb{R}} \alpha \leq \sup_{\mathbb{R}} \alpha < \infty$. Another simple example falling within this category is

$$|\boldsymbol{m}| \leq \sigma(p) \Leftrightarrow \boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad |\boldsymbol{m}| > \sigma(p) \Leftrightarrow \boldsymbol{m} = \sigma(p)\frac{\boldsymbol{v}}{|\boldsymbol{v}|} + \gamma(p)|\boldsymbol{v}|^{r-2}\boldsymbol{v}, \tag{4}$$

with $\sigma(p)$ and $\gamma(p)$ having the same properties as $\alpha(p)$ above. This situation resembles Herschel-Bulkley responses between the Cauchy stress and the velocity gradient in the constitutive theory of non-Newtonian fluids, or Bingham responses in the special case $r = 2$. Note that the relation (4) can be rewritten equivalently as

$$\big(\gamma(p)\big)^{\frac{1}{r-1}}\boldsymbol{v} = \big((\boldsymbol{m} - \sigma(p))_+\big)^{\frac{1}{r-1}}\frac{\boldsymbol{m}}{|\boldsymbol{m}|},$$

which corresponds to $\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) = \boldsymbol{0}$ with

$$\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, p) = \big(\gamma(p)\big)^{\frac{1}{r-1}}\boldsymbol{v} - \big((\boldsymbol{m} - \sigma(p))_+\big)^{\frac{1}{r-1}}\frac{\boldsymbol{m}}{|\boldsymbol{m}|}.$$

Here, for $z \in \mathbb{R}$ we use $z_+ := \max\{z, 0\}$ to denote its positive part. See Bulíček et al. [27] for an analogon thereof in the case of Bingham fluids.

The two given examples, with α, σ and γ bounded from above, pale into insignificance in the face of interactions of the form

$$\boldsymbol{m}(\boldsymbol{v}, p) = \alpha_1 \exp(\alpha_2 p)\boldsymbol{v}, \qquad \alpha_{1,2} > 0 \tag{5}$$

that actually lie at the centre of our attention here. Let us recall that even for simple incompressible fluids, it is known that the viscosity changes significantly at high pressures. In fact, Barus' experimental study (see [3]) led him to the conclusion that the viscosity changes with the pressure exponentially (similarly as the coefficient relating $\boldsymbol{m}$ and $\boldsymbol{v}$ in (5)). For flows of fluid through rigid media, the internal fluid friction is frequently neglected as the friction between the fluid and solid is dominant. If such flows take place at high pressures, then one needs to involve the (exponential) dependence of the coefficient relating $\boldsymbol{m}$ and $\boldsymbol{v}$ on the pressure; see Nakshatrala and Rajagopal [24] for more details. Even if the coefficient α_2 in (5) is very small ($\alpha_2 \sim 10^{-5}$, see [3]), it is evident that a choice like (5) is beyond the purview of **(A4)** and **(A5)**$_{\text{(iv)}}$. Luckily enough, this case and those akin can also be included under certain circumstances into the existence theory developed in this paper; see Sect. 5.

We may also take a perturbation of (5) in a form

$$\boldsymbol{m}(\boldsymbol{v}, p) = \max\{\alpha_1, \alpha_1 \exp(\alpha_2 p)\}\boldsymbol{v}, \tag{6}$$

for existence theory of which we will be able to slightly slacken our hypotheses, see Remark 7. The reason is that inserting this choice into (3) with $r = 2$, the condition $\inf_{\mathbb{R}} \alpha > 0$ is met trivially.

1.3 Results

Within the setting of **(A1)**–**(A5)** we are able to establish the existence of a solution to the problem (1) fulfilling the first three equations pointwise (almost everywhere) in Ω; see Theorem 3 below. Although this theorem does not include the models of our main interest such as (5), it provides a tool how (5) can be analyzed, together with a maximum/minimum principle that is well-known for Darcy's model but is newly discovered for cases like (5) in this paper. The maximum/minimum principle is presented in Theorem 5 and its combination with Theorem 3 then culminates in Theorem 5, where the existence of a solution to situations such as (5) or, under less stringent hypotheses (6), is established.

It is worth pointing out a remarkable difference between the results presented here and the results concerning those generalizations of incompressible Stokes and Navier-Stokes equations, stationary and evolutionary, in which the viscosity grows more than linearly with the pressure. While here for (1) with (5) we develop, under certain assumptions, large data existence theory, no such mathematical theory is available for the systems such as

$$\nabla p - \operatorname{div}[2\nu(p,\cdot)\boldsymbol{D}\boldsymbol{v}] + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) = \boldsymbol{f}, \qquad \boldsymbol{D} := \frac{1}{2}(\nabla + \nabla^T), \tag{7}$$

if ν depends on p exponentially. With exception of studies concerning flows in special geometries (see [16], [17], [26], [29], [33], [34], [36]), we are aware of merely a few, rather preliminary studies concerning flows in general domains (see [15, 14] and [30]). We remark that in [21] and subsequent studies [12], [7], [8], [6] (that also includes a detailed summary of the available theory), the authors have been able to identify the class of the viscosities depending on the pressure and $|\boldsymbol{D}\boldsymbol{v}|^2$ and to develop large data mathematical theory for relevant boundary and initial boundary value problems. This subclass, however, does not allow to include (5). Remarkably enough, there is no maximum principle to eq. (7), not even if the equation were stripped of the inertial term $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})$.

There is abundance of available literature on qualitative analysis of Darcy-Forchheimer's equations, or their generalizations like Brinkman-Forchheimer's equations when a diffusive term is added. With the exception of investigating regularity, authors address the evolutionary case right away, see e.g. [2] for the compressible case and [37] for the incompressible one, and papers cited therein. In [35] existence of an attractor for these equations is studied. Regularity of the (unique) solution to Darcy-Forchheimer's equations is examined in [10].

In defiance of a cornucopia of sources, they are all confined to the case where $\boldsymbol{h}$ in $(1)_1$ does not depend on the pressure. The p-dependent and implicitly related situation of Darcy-Forchheimer equations analyzed within the current paper seems to have remained, at least to the best of authors' knowledge, almost a terra incognita so far.

1.4 Further comments

Behold even at this early phase that **(A4)** hints at setting the stage for working in Lebesgue spaces. It is therefore natural to ask why not plunge ourselves directly into general Orlicz spaces in the vein of Bulíček et al. [4, 5] instead. Even though such an extension should not require much additional effort, we chose the Lebesgue setting for the sake of simplicity, as it allows us to accentuate the ideas concerning p-dependence of the graph $\mathcal{A}$ and the maximum and minimum principles.

As far as **(A5)** is concerned, a general question of existence of a measurable selection for the case of $\boldsymbol{h}$ being independent of p is confirmed e.g. in Chiado' Piat et al. [9, Theorem 1.4]. In our setting, we want in addition the selection being continuous with respect to p and also bounded in that variable in the sense of $\mathbf{(A5)}_{\mathrm{(iv)}}$. Note that similarly tame behavior is expected in **(A4)** by requiring uniformity in p.

It is not particularly difficult to show that a maximal monotone graph (independent of p) can be rotated so as to form a graph of a 1-Lipschitz function (see [1, 11, 23]). This observation is likely to lead to another feasible way of approaching the existence theory for (1), devoid of any need for selections. The path is not followed in our paper save this remark.

Drawing this introduction to its end, in the following brief Sect. 2 we deal with a couple of useful mathematical properties to be invoked later on. We then devote an entire Sect. 3 to formulate and prove an existence theorem of solutions to the problem (1) provided **(A1)**–**(A5)** are all satisfied. The penultimate Sect. 4 is somewhat autonomous and serves to state and justify a maximum and a minimum principle for the pressure in (1). It will prove invaluable in the last Sect. 5, where it authorizes us to somewhat weaken **(A4)** and $\mathbf{(A5)}_{\mathrm{(iv)}}$, wherein effect it shows existence for situations like (5), supposing certain other hypotheses are satisfied indeed.

2 Preliminaries

For $\delta > 0$ denote

$$\omega_\delta(\boldsymbol{x}, t) := \delta^{-(d+1)} \omega\left(\frac{\boldsymbol{x}}{\delta}, \frac{t}{\delta}\right),$$

where ω is the usual mollification kernel on $\mathbb{R}^{d+1}$. With its help we define the *regularized selection*

$$\boldsymbol{m}_\delta(\boldsymbol{x}, t) := \int_{\mathbb{R}^d \times \mathbb{R}} \boldsymbol{m}^*(\boldsymbol{x} - \boldsymbol{y}, t - s)\, \omega_\delta(\boldsymbol{y}, s)\, d\boldsymbol{y}\, ds.$$

Lemma 1 *The selection $\boldsymbol{m}^*$ and its regularization $\boldsymbol{m}_\delta$ enjoy the following properties, which will be made use of later:*

*(i) (r, r')-coercivity **(A4)** holds for $\boldsymbol{m}_\delta$. The constants may be different but independent of $0 < \delta < 1$.*

*(ii) The property **(A3)** is actually tantamount to apparently a weaker one*

$$(\boldsymbol{m}' - \boldsymbol{m}^*(\boldsymbol{v}, p)) \cdot (\boldsymbol{v}' - \boldsymbol{v}) \geq 0 \text{ for a.e. } \boldsymbol{v} \in \mathbb{R}^d \Rightarrow (\boldsymbol{m}', \boldsymbol{v}', p) \in \mathcal{A}.$$

Proof. For a proof of (i), we see that $\boldsymbol{m}^*$ is evidently (r, r')-coercive and hence we compute:

$$\begin{aligned}
\boldsymbol{m}_\delta(\boldsymbol{x}, t) \cdot \boldsymbol{x} &= \int_{\mathbb{R}^d \times \mathbb{R}} \boldsymbol{m}^*(\boldsymbol{x} - \boldsymbol{y}, t - s) \cdot (\boldsymbol{x} - \boldsymbol{y})\, \omega_\delta(\boldsymbol{y}, s)\, d\boldsymbol{y}\, ds \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}} \boldsymbol{m}^*(\boldsymbol{x} - \boldsymbol{y}, t - s) \cdot \boldsymbol{y}\, \omega_\delta(\boldsymbol{y}, s)\, d\boldsymbol{y}\, ds \\
&\geq \int_{\mathbb{R}^d \times \mathbb{R}} \left[c_1 \left(|\boldsymbol{x} - \boldsymbol{y}|^r + |\boldsymbol{m}^*(\boldsymbol{x} - \boldsymbol{y}, t - s)|^{r'}\right) - c_2\right] \omega_\delta(\boldsymbol{y}, s)\, d\boldsymbol{y}\, ds \\
&\quad - \int_{\mathbb{R}^d \times \mathbb{R}} \left(\frac{c_1}{2} |\boldsymbol{m}^*(\boldsymbol{x} - \boldsymbol{y}, t - s)|^{r'} + c_3 |\boldsymbol{y}|^r\right) \omega_\delta(\boldsymbol{y}, s)\, d\boldsymbol{y}\, ds \\
&\geq c_4(|\boldsymbol{x}|^r + |\boldsymbol{m}_\delta(\boldsymbol{x}, t)|^{r'}) - c_5.
\end{aligned}$$

First we employed Young's inequality and then Jensen's inequality was invoked. Note that neither c_4 nor c_5 depend on $\delta > 0$ as long as δ is bounded.

Towards showing (ii), let

$$\mathcal{A}_p = \{(\boldsymbol{m}, \boldsymbol{v}) \in \mathbb{R}^d \times \mathbb{R}^d \mid (\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A}\}$$

and $(\boldsymbol{m}', \boldsymbol{v}', p) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ such that we have

$$(\boldsymbol{m}' - \boldsymbol{m}^*(\boldsymbol{v}, p)) \cdot (\boldsymbol{v}' - \boldsymbol{v}) \geq 0 \quad \text{for a.e. } \boldsymbol{v} \in \mathbb{R}^d. \tag{8}$$

The aim is to attest $(\boldsymbol{m}' - \boldsymbol{m}) \cdot (\boldsymbol{v}' - \boldsymbol{v}) \geq 0$ for every $(\boldsymbol{m}, \boldsymbol{v}) \in \mathcal{A}_p$: Let $(\boldsymbol{m}, \boldsymbol{v}) \in \mathcal{A}_p$ be arbitrary. It is trivial to show that the set

$$M_{\boldsymbol{v}} = \{\widehat{\boldsymbol{m}} \in \mathbb{R}^d \mid (\widehat{\boldsymbol{m}}, \boldsymbol{v}) \in \mathcal{A}_p\}$$

is convex and closed. Note that $M_{\boldsymbol{v}}$ is also non-empty and bounded, for else one could find $\boldsymbol{u} \in \mathbb{R}^d$ such that $\boldsymbol{m}^*(\boldsymbol{u}, p) = \infty$, contradicting $\textbf{(A5)}_{\text{(iv)}}$ (but see Remark 2). Therefore we may express $\boldsymbol{m} = \lambda \boldsymbol{m}_1 + (1 - \lambda)\boldsymbol{m}_2$ for some $0 \leq \lambda \leq 1$ and $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \partial M_{\boldsymbol{v}}$.

Now, $\mathcal{A}_p$ can be seen as a d-dimensional Lipschitz manifold in $\mathbb{R}^d \times \mathbb{R}^d$ without a boundary [1], whence if $\widetilde{\boldsymbol{m}} \in \partial M_{\boldsymbol{v}}$, there exists $\{(\boldsymbol{m}_n, \boldsymbol{v}_n)\} \subset \mathcal{A}_p$, $\boldsymbol{v}_n \neq \boldsymbol{v}$, such that $(\boldsymbol{m}_n, \boldsymbol{v}_n) \to (\widetilde{\boldsymbol{m}}, \boldsymbol{v})$, as $n \to \infty$. Of course, otherwise the point $(\widetilde{\boldsymbol{m}}, \boldsymbol{v})$ would be a boundary point of $\mathcal{A}_p$. Finally, let $\boldsymbol{v}_n$ be chosen so that the set $\{\widehat{\boldsymbol{m}} \in \mathbb{R}^d \mid (\widehat{\boldsymbol{m}}, \boldsymbol{v}_n) \in \mathcal{A}_p\}$ is a singleton for every n, i.e. $\boldsymbol{m}^*(\boldsymbol{v}_n, p) = \boldsymbol{m}_n$, and (8) holds for all $\boldsymbol{v}_n$. It is achievable, since the set of all $\widehat{\boldsymbol{v}} \in \mathbb{R}^d$ such that $M_{\widehat{\boldsymbol{v}}}$ contains more than one element has Hausdorff dimension equal to $d-1$ [1, Remark 2.3].

Thus we find $\{\boldsymbol{v}_1^n\}, \{\boldsymbol{v}_2^n\} \subset \mathbb{R}^d$, for which $\boldsymbol{v}_i^n \to \boldsymbol{v}$ and $\boldsymbol{m}^*(\boldsymbol{v}_i^n, p) \to \boldsymbol{m}_i$ as $n \to \infty$, for $i = 1, 2$. Given that both $\{\boldsymbol{v}_1^n\}$ and $\{\boldsymbol{v}_2^n\}$ satisfy (8), the goal $(\boldsymbol{m}' - \boldsymbol{m}) \cdot (\boldsymbol{v}' - \boldsymbol{v}) \geq 0$ follows from passing to limit $n \to \infty$, multiplying by λ and $1 - \lambda$, respectively, and finally summing up. □

Remark 2 A third useful property of $\boldsymbol{m}^*$ is its local boundedness in the sense that $|\boldsymbol{m}^*(\cdot, p)|$ is bounded on bounded domains for every $p \in \mathbb{R}$. This is trivial due to **(A5)**$_{(\mathrm{iv})}$, yet it would hold even without this requirement. See e.g. [19, Theorem 2], which can be applied to address the question.

3 Principal existence theorem

Before formulation of the main result, notation for several function spaces that will often be used shall be introduced. First, for Lebesgue and Sobolev spaces we use the standard notation. To handle the Dirichlet data for the pressure, we define, for $q \in (1, \infty)$,

$$W^{1,q}_{\Gamma_2}(\Omega) := \left\{ u \in W^{1,q}(\Omega) \mid u = 0 \text{ on } \Gamma_2 \right\}.$$

In case of $\Gamma_2 = \emptyset$, we make a natural modification

$$W^{1,q}_{\Gamma_2}(\Omega) := \left\{ u \in W^{1,q}(\Omega) \;\Big|\; \int_\Omega u = 0 \right\}.$$

Note that in either instance, $W^{1,q}_{\Gamma_2}(\Omega)$ is a closed subspace of $W^{1,q}(\Omega)$. Next, since we will deal with solenoidal functions with a prescribed normal trace on a part of the boundary, we denote

$$L^q_{\mathrm{div}}(\Omega) := \left\{ \boldsymbol{\varphi} \in L^q(\Omega)^d \;\Big|\; \operatorname{div} \boldsymbol{\varphi} = 0 \right\}.$$

The condition on zero divergence is meant in the sense of distributions. As the zero distribution is regular, we can legally say in particular $\operatorname{div} \boldsymbol{\varphi} = 0$ a.e. in Ω for any $\boldsymbol{\varphi} \in L^q_{\mathrm{div}}(\Omega)$. It is well known (see [13, chapter III.2]) that one can talk about normal traces (remember Ω is Lipschitz) of elements of $L^q_{\mathrm{div}}(\Omega)$, seeing them as elements of $\left(W^{\frac{1}{q}, q'}(\partial\Omega)\right)^*$. Understanding $\boldsymbol{\varphi} \cdot \boldsymbol{n}$ on Γ_1 in this generalized sense, we can also introduce

$$L^q_{\mathrm{div}, \Gamma_1}(\Omega) := \left\{ \boldsymbol{\varphi} \in L^q_{\mathrm{div}}(\Omega) \mid \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_1 \right\}.$$

To conclude, for $K > 0$ we define a cutoff function $T_K : \mathbb{R} \to \mathbb{R}$ as

$$T_K(x) := \begin{cases} -K & \text{for } x \leq -K, \\ x & \text{for } -K < x < K, \\ K & \text{for } x \geq K. \end{cases} \tag{9}$$

Here and there we will silently use trivial $|T_K(x)| \leq |x|$ for every $x \in \mathbb{R}$. When applying the truncator T_K to vectors, we consider the component-wise truncation, i.e. for $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we set $T_K(\boldsymbol{x}) := (T_K(x_1), \ldots, T_K(x_d))$.

Having finalized indispensable preparations, the promised existence theorem can be formulated:

Theorem 3 *Let Ω be a Lipschitz domain and $r \in (1, \infty)$ be given. Assume $\boldsymbol{f} \in L^{r'}(\Omega)^d$, $\boldsymbol{v}_0 \in L^r_{\mathrm{div}}(\Omega)$ and $p_0 \in W^{1,r'}(\Omega)$. Moreover, assume that $\mathcal{A}$ is a maximal monotone r-graph in the sense of* **(A1)–(A5)**. *Then there exists a triplet $(\boldsymbol{m}, \boldsymbol{v}, p) \in L^{r'}(\Omega)^d \times L^r_{\mathrm{div}}(\Omega) \times W^{1,r'}(\Omega)$ solving (1), i.e. $(1)_1$–$(1)_3$*

are satisfied a.e. in Ω *and*

$$\boldsymbol{v}-\boldsymbol{v}_0 \in L^r_{\mathrm{div},\Gamma_1}(\Omega),$$
$$p-p_0 \in W^{1,r'}_{\Gamma_2}(\Omega).$$

Proof. The proof of the theorem constitutes the remainder of this section. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset L^r(\Omega)^d \cap L^\infty(\Omega)^d$ and $\{q_i\}_{i\in\mathbb{N}} \subset W^{1,r'}_{\Gamma_2}(\Omega)$ be linearly independent, with linear spans dense in $L^r(\Omega)^d$ and $W^{1,r'}_{\Gamma_2}(\Omega)$, respectively.

To begin with, we deduce existence of solutions to an approximate problem, i.e. for $n \in \mathbb{N}$ and $\varepsilon, \delta > 0$ to find

$$\boldsymbol{v}^{\varepsilon,\delta}_n(x) = T_n(\boldsymbol{v}_0)(x) + \sum_{i=1}^{n} a^{\varepsilon,\delta,i}_n \boldsymbol{w}_i(x), \tag{10}$$

$$p^{\varepsilon,\delta}_n(x) = p_0(x) + \sum_{i=1}^{n} b^{\varepsilon,\delta,i}_n q_i(x), \tag{11}$$

satisfying

$$\int_\Omega \nabla p^{\varepsilon,\delta}_n \cdot \boldsymbol{w}_i + \int_\Omega \boldsymbol{m}_\delta(\boldsymbol{v}^{\varepsilon,\delta}_n, p^{\varepsilon,\delta}_n)\cdot \boldsymbol{w}_i = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}_i,\ i=1,\dots,n, \tag{12}$$

$$\varepsilon \int_\Omega |\nabla(p^{\varepsilon,\delta}_n - p_0)|^{r'-2}\nabla(p^{\varepsilon,\delta}_n - p_0)\cdot\nabla q_i = \int_\Omega (\boldsymbol{v}^{\varepsilon,\delta}_n - T_n(\boldsymbol{v}_0))\cdot\nabla q_i,\ i=1,\dots,n. \tag{13}$$

Replacing solenoidality of the velocity field with the eq. (13) is a so-called *quasi-compressible* approximation (see [12] and [22, p. 416]), which facilitates construction of the pressure. Note that, at this point at least informally, the limit $\varepsilon \to 0_+$ should produce a divergence-free velocity.

The aim of δ-regularization is to obtain a solution to (12) and (13). This is actually the first approximation parameter to be dropped due to a limiting process. Since it will require boundedness of $\{\boldsymbol{v}^{\varepsilon,\delta}_n\}_\delta$ in $L^\infty(\Omega)^d$, we need to truncate $\boldsymbol{v}_0$ as seen in (10).

Towards showing existence of $\{a^{\varepsilon,\delta,i}_n\}_{i=1}^n$ and $\{b^{\varepsilon,\delta,i}_n\}_{i=1}^n$, we employ the following standard corollary of Brouwer's fixed point theorem, whose justification follows from lines to come and will not be discussed in detail.

Lemma 4 [20, Lemme 4.3] *Let* $\boldsymbol{F}:\mathbb{R}^d\to\mathbb{R}^d$ *be a continuous function satisfying* $\boldsymbol{F}(\boldsymbol{\xi})\cdot\boldsymbol{\xi}\ge 0$ *if* $|\boldsymbol{\xi}|=\varrho$ *for certain* $\varrho>0$*. Then there exists* $\boldsymbol{\xi}_0\in\mathbb{R}^d$*,* $|\boldsymbol{\xi}_0|\le\varrho$*, for which* $\boldsymbol{F}(\boldsymbol{\xi}_0)=\boldsymbol{0}$*.*

Multiplying eq. $(12)_i$ by $a^{\varepsilon,\delta,i}_n$ and eq. $(13)_i$ by $b^{\varepsilon,\delta,i}_n$ and summing the resultant $2n$ equalities, we obtain

$$\varepsilon\|\nabla(p^{\varepsilon,\delta}_n - p_0)\|^{r'}_{r'} + \int_\Omega \boldsymbol{m}_\delta(\boldsymbol{v}^{\varepsilon,\delta}_n, p^{\varepsilon,\delta}_n)\cdot(\boldsymbol{v}^{\varepsilon,\delta}_n - \boldsymbol{v}_0) = \int_\Omega (\boldsymbol{f}-\nabla p_0)\cdot(\boldsymbol{v}^{\varepsilon,\delta}_n - T_n(\boldsymbol{v}_0)). \tag{14}$$

As we may assume $\delta < 1$, recalling Lemma 1 for (r,r')-coercivity of $\boldsymbol{m}_\delta$, Hölder's and Young's inequalities, eq. (14) is processed into

$$\varepsilon\|\nabla(p^{\varepsilon,\delta}_n - p_0)\|^{r'}_{r'} + \|\boldsymbol{m}_\delta(\boldsymbol{v}^{\varepsilon,\delta}_n, p^{\varepsilon,\delta}_n)\|^{r'}_{r'} + \|\boldsymbol{v}^{\varepsilon,\delta}_n\|^r_r \le C(\|\boldsymbol{f}-\nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r). \tag{15}$$

In particular, the constant C is independent of δ, n or ε. The *energy inequality* (15) will serve us as the starting point for taking the limits $\delta\to 0_+$, $n\to\infty$ and $\varepsilon\to 0_+$, in this order.

3.1 δ-limit

In order to accomplish the first limit passage, we start with observation that (15) entails

$$\sup\left\{|a_n^{\varepsilon,\delta,i}|, |b_n^{\varepsilon,\delta,i}| \mid 0<\delta<1,\ i=1,\dots,n\right\} < C(n,\varepsilon).$$

We may hence assume

$$\begin{aligned} a_n^{\varepsilon,\delta,i} &\to a_n^{\varepsilon,i},\\ b_n^{\varepsilon,\delta,i} &\to b_n^{\varepsilon,i}, \end{aligned} \tag{16}$$

as $\delta \to 0_+$, for each $i = 1,\dots,n$. This result allows us to observe also the following convergences:

$$\begin{aligned} \boldsymbol{v}_n^{\varepsilon,\delta} &\to \boldsymbol{v}_n^{\varepsilon} && \text{in } L^\infty(\Omega)^d,\\ p_n^{\varepsilon,\delta} - p_0 &\to p_n^{\varepsilon} - p_0 && \text{in } W^{1,r'}_{\Gamma_2}(\Omega),\\ p_n^{\varepsilon,\delta} &\to p_n^{\varepsilon} && \text{a.e. in } \Omega,\\ |\nabla(p_n^{\varepsilon,\delta}-p_0)|^{r'-2}\nabla(p_n^{\varepsilon,\delta}-p_0) &\to |\nabla(p_n^{\varepsilon}-p_0)|^{r'-2}\nabla(p_n^{\varepsilon}-p_0) && \text{in } L^r(\Omega)^d,\\ \boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) &\rightharpoonup \boldsymbol{m}_n^{\varepsilon} && \text{in } L^{r'}(\Omega)^d. \end{aligned} \tag{17}$$

For $(17)_1$ and $(17)_2$ we used (10), (11) and (16); the limits $(17)_3$ and $(17)_4$ are justified by $(17)_2$, and the last passage utilized ineq. (15) and reflexivity of $L^{r'}(\Omega)^d$. The subscript in $\boldsymbol{m}_n^{\varepsilon}$ does not correspond to mollification any longer, it is used merely to follow the same notation as p_n^{ε} and $\boldsymbol{v}_n^{\varepsilon}$.

As for what equations the limit quantities satisfy, (17) makes passing to limit $\delta \to 0_+$ in equations (12) and (13) easy and we obtain

$$\begin{aligned} \int_\Omega \nabla p_n^{\varepsilon}\cdot \boldsymbol{w}_i + \int_\Omega \boldsymbol{m}_n^{\varepsilon}\cdot\boldsymbol{w}_i &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}_i,\ i=1,\dots,n,\\ \varepsilon\int_\Omega |\nabla(p_n^{\varepsilon}-p_0)|^{r'-2}\,\nabla(p_n^{\varepsilon}-p_0)\cdot\nabla q_i &= \int_\Omega (\boldsymbol{v}_n^{\varepsilon} - T_n(\boldsymbol{v}_0))\cdot\nabla q_i,\ i=1,\dots,n. \end{aligned} \tag{18}$$

Before proceeding to the second passage, we will yet show the limit functions now lie in the graph, i.e. $(\boldsymbol{m}_n^{\varepsilon}, \boldsymbol{v}_n^{\varepsilon}, p_n^{\varepsilon}) \in \mathcal{A}$ a.e. in Ω. This objective can be achieved by means of the maximality property **(A3)**, specifically by its version from Lemma 1. In the given situation, we have to verify

$$(\boldsymbol{m}_n^{\varepsilon} - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon}))\cdot(\boldsymbol{v}_n^{\varepsilon}-\boldsymbol{u}) \ge 0 \quad \text{a.e. in } \Omega \text{ for a.e. } \boldsymbol{u}\in\mathbb{R}^d, \tag{19}$$

in order of which it suffices to check

$$\liminf_{\delta\to 0_+}\Big(\boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon})\Big)\cdot(\boldsymbol{v}_n^{\varepsilon,\delta}-\boldsymbol{u}) \ge 0 \quad \text{a.e. in } \Omega \text{ for a.e. } \boldsymbol{u}\in\mathbb{R}^d. \tag{20}$$

Indeed it does: Let us consider only $\boldsymbol{u}\in\mathbb{R}^d$ at which $\boldsymbol{m}^*(\boldsymbol{u},\cdot)$ is continuous. For an arbitrary measurable $E\subset\mathbb{R}^d$ of non-zero Lebesgue measure, (20) implies

$$\liminf_{\delta\to 0_+}\int_E \Big(\boldsymbol{m}_\delta(\boldsymbol{v}_n^{\varepsilon,\delta}, p_n^{\varepsilon,\delta}) - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon,\delta})\Big)\cdot(\boldsymbol{v}_n^{\varepsilon,\delta}-\boldsymbol{u}) \ge 0.$$

Due to convergences (17) and properties of $\boldsymbol{m}^*$, we pass to the limit

$$\int_E (\boldsymbol{m}_n^{\varepsilon} - \boldsymbol{m}^*(\boldsymbol{u}, p_n^{\varepsilon}))\cdot(\boldsymbol{v}_n^{\varepsilon}-\boldsymbol{u}) \ge 0.$$

Arbitrary nature of E yields (19). Moving on to the proof of (20), monotonicity implies

$$\int_{\mathbb{R}^d\times\mathbb{R}} (\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\hat{\boldsymbol{u}} - \boldsymbol{u})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt \geq 0,$$

which holds a.e. in Ω. We reshuffle the relation into

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}} &(\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \boldsymbol{u})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt \\ &\geq \int_{\mathbb{R}^d\times\mathbb{R}} (\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt. \end{aligned} \tag{21}$$

Limit passage in (21) is manageable, for firstly we have

$$\lim_{\delta\to 0_+} \int_{\mathbb{R}^d\times\mathbb{R}} \boldsymbol{m}^*(\boldsymbol{u},t)\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt = \boldsymbol{m}^*(\boldsymbol{u}, p_n^\varepsilon), \tag{22}$$

due to continuity and boundedness of $\boldsymbol{m}^*(\boldsymbol{u},\cdot)$ and pointwise convergence of $\{p_n^{\varepsilon,\delta}\}_\delta$. Secondly, $\{\boldsymbol{v}_n^{\varepsilon,\delta}\}_\delta$ is bounded in $L^\infty(\Omega)^d$, and in conjunction with $\mathbf{(A5)}_{(\mathrm{iv})}$ we observe

$$\begin{aligned}\Big|\int_{\mathbb{R}^d\times\mathbb{R}} &(\boldsymbol{m}^*(\hat{\boldsymbol{u}},t) - \boldsymbol{m}^*(\boldsymbol{u},t)) \cdot (\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}})\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt\Big| \\ &\leq C(\boldsymbol{u}, \|\boldsymbol{v}_n^{\varepsilon,\delta}\|_\infty) \underbrace{\Big(\int_{\mathbb{R}^d\times\mathbb{R}} |\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}|^r\, \omega_\delta(\boldsymbol{v}_n^{\varepsilon,\delta} - \hat{\boldsymbol{u}}, p_n^{\varepsilon,\delta} - t)\, d\hat{\boldsymbol{u}}\, dt\Big)^{1/r}}_{\to 0 \text{ a.e. in } \Omega \text{ as } \delta \to 0_+}. \end{aligned} \tag{23}$$

Applying (22) and (23) on (21), we obtain (20), i.e. $(\boldsymbol{m}_n^\varepsilon, \boldsymbol{v}_n^\varepsilon, p_n^\varepsilon) \in \mathcal{A}$ a.e. in Ω.

3.2 n-limit

The weak lower semicontinuity of norms applied on (15) produces a second level of that energy inequality, meaning

$$\varepsilon\, \|\nabla(p_n^\varepsilon - p_0)\|_{r'}^{r'} + \|\boldsymbol{m}_n^\varepsilon\|_{r'}^{r'} + \|\boldsymbol{v}_n^\varepsilon\|_r^r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r), \tag{24}$$

whence we may pass to the limit $n \to \infty$, assuming

$$\begin{aligned} \boldsymbol{v}_n^\varepsilon &\rightharpoonup \boldsymbol{v}^\varepsilon && \text{in } L^r(\Omega)^d, \\ p_n^\varepsilon - p_0 &\rightharpoonup p^\varepsilon - p_0 && \text{in } W^{1,r'}_{\Gamma_2}(\Omega), \\ |\nabla(p_n^\varepsilon - p_0)|^{r'-2}\, \nabla(p_n^\varepsilon - p_0) &\rightharpoonup \boldsymbol{\chi} && \text{in } L^r(\Omega)^d, \\ \boldsymbol{m}_n^\varepsilon &\rightharpoonup \boldsymbol{m}^\varepsilon && \text{in } L^{r'}(\Omega)^d, \\ T_n(\boldsymbol{v}_0) &\to \boldsymbol{v}_0 && \text{in } L^r(\Omega)^d. \end{aligned} \tag{25}$$

The last result is an easy consequence of Chebyshev's inequality. Convergences (25) let us pass to the limit in eq. (18). Using the density property of $\{\boldsymbol{w}_i\}$ in $L^r(\Omega)^d$ and $\{q_i\}$ in $W^{1,r'}_{\Gamma_2}(\Omega)$, we obtain furthermore

$$\begin{aligned} \int_\Omega \nabla p^\varepsilon \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{w} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}, \quad \forall \boldsymbol{w} \in L^r(\Omega)^d, \\ \varepsilon \int_\Omega \boldsymbol{\chi} \cdot \nabla q &= \int_\Omega (\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0) \cdot \nabla q, \quad \forall q \in W^{1,r'}_{\Gamma_2}(\Omega). \end{aligned} \tag{26}$$

Like previously, we have to check $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon) \in \mathcal{A}$ a.e. in Ω. Also, weak convergence prevented us from inferring identity of the weak limit $\boldsymbol{\chi}$ and it is necessary yet to verify $\boldsymbol{\chi} = |\nabla(p^\varepsilon - p_0)|^{r'-2} \nabla(p^\varepsilon - p_0)$. We will use the standard monotone operator theory, namely the Minty's method.

From (26) we deduce

$$\varepsilon \int_\Omega \boldsymbol{\chi} \cdot \nabla(p^\varepsilon - p_0) + \int_\Omega \boldsymbol{m}^\varepsilon \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{v}_0), \tag{27}$$

while (18) implies similarly

$$\varepsilon \int_\Omega |\nabla(p_n^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot (\boldsymbol{v}_n^\varepsilon - T_n(\boldsymbol{v}_0)) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v}_n^\varepsilon - T_n(\boldsymbol{v}_0)), \tag{28}$$

for every $n \in \mathbb{N}$. Using (25), comparing (27) with (28) yields

$$\lim_{n\to\infty} \varepsilon \int_\Omega |\nabla(p_n^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon = \varepsilon \int_\Omega \boldsymbol{\chi} \cdot \nabla(p^\varepsilon - p_0) + \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon. \tag{29}$$

This will be our first foothold. Next, recall monotonicity of the p-Laplace operator and $(\boldsymbol{m}_n^\varepsilon, \boldsymbol{v}_n^\varepsilon, p_n^\varepsilon) \in \mathcal{A}$ a.e. in Ω. Hence we know that for all $q \in W^{1,r'}_{\Gamma_2}(\Omega)$,

$$\begin{aligned} 0 \le \varepsilon \int_\Omega \big(|\nabla(p_n^\varepsilon - p_0)|^{r'-2} \nabla(p_n^\varepsilon - p_0) - |\nabla q|^{r'-2} \nabla q\big) \cdot \nabla(p_n^\varepsilon - p_0 - q) \\ + \int_\Omega \big(\boldsymbol{m}_n^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon)\big) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon). \end{aligned} \tag{30}$$

As we are allowed to assume $p_n^\varepsilon \to p^\varepsilon$ from $(25)_2$, properties **(A5)** yield

$$\boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon) \to \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p^\varepsilon) \quad \text{in } L^{r'}(\Omega) \text{ as } \boldsymbol{n} \to \infty,$$

which we on top of that mingle with $(25)_1$ and observe

$$\boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon) \to 0 \quad \text{in } L^1(\Omega) \text{ as } \boldsymbol{n} \to \infty. \tag{31}$$

Combining (25), (29), (31) and taking the limit $n \to \infty$ in the monotonicity relation (30) gives rise to

$$0 \le \int_\Omega \big(\boldsymbol{\chi} - |\nabla q|^{r'-2} \nabla q\big) \cdot \nabla(p^\varepsilon - p_0 - q), \quad \forall q \in W^{1,r'}_{\Gamma_2}(\Omega). \tag{32}$$

Setting $q = p^\varepsilon - p_0 \pm t\varphi$ with $t > 0$ and $\varphi \in W^{1,r'}_{\Gamma_2}(\Omega)$, we divide (32) by t and then perform $t \to 0_+$. Arbitrary nature of φ yields

$$\boldsymbol{\chi} = |\nabla(p^\varepsilon - p_0)|^{r'-2} \nabla(p^\varepsilon - p_0). \tag{33}$$

Identity (29) can now be rewritten

$$\lim_{n\to\infty} \varepsilon \int_\Omega |\nabla(p_n^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon = \varepsilon \int_\Omega |\nabla(p^\varepsilon - p_0)|^{r'} + \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon. \tag{34}$$

By weak lower semicontinutity of a norm, (34) indicates

$$\|\nabla(p^\varepsilon - p_0)\|_{r'} \le \liminf_{n\to\infty} \|\nabla(p_n^\varepsilon - p_0)\|_{r'} \Rightarrow \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \ge \limsup_{n\to\infty} \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon. \tag{35}$$

This is actually sufficient for $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon) \in \mathcal{A}$ a.e. in Ω. We will derive it again from the maximality property reformulated in Lemma 1.

On the one hand, we have $\big(\boldsymbol{m}_n^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon)\big) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon) \geq 0$ a.e. in Ω. However, (35) leads us to

$$0 \leq \limsup_{n\to\infty} \int_\Omega \big(\boldsymbol{m}_n^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon)\big) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon)$$
$$= \limsup_{n\to\infty} \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon - \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon) \leq 0,$$

due to (31) and (35). Therefore $\big(\boldsymbol{m}_n^\varepsilon - \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon)\big) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon) \to 0$ in $L^1(\Omega)$ for $n \to \infty$. Since a strong convergence implies the weak one, for all $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$ a.e. in Ω, we have

$$\lim_{n\to\infty} \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}_n^\varepsilon \,\varphi \,dx = \lim_{n\to\infty} \int_\Omega \boldsymbol{m}_n^\varepsilon \cdot \boldsymbol{v}^\varepsilon \,\varphi + \boldsymbol{m}^*(\boldsymbol{v}^\varepsilon, p_n^\varepsilon) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{v}^\varepsilon)\,\varphi\,dx = \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \,\varphi\,dx. \tag{36}$$

The last equality made again use of (31). Now, we take an arbitrary $\boldsymbol{u} \in \mathbb{R}^d$ and use monotonicity to write

$$\int_\Omega \big(\boldsymbol{m}_n^\varepsilon - \boldsymbol{m}^*(\boldsymbol{u}, p_n^\varepsilon)\big) \cdot (\boldsymbol{v}_n^\varepsilon - \boldsymbol{u})\,\varphi\,dx \geq 0, \quad \forall n \in \mathbb{N}. \tag{37}$$

Owing to **(A5)**, (25) and (36), it is possible to take the limit $n \to \infty$ and infer

$$\int_\Omega \big(\boldsymbol{m}^\varepsilon - \boldsymbol{m}^*(\boldsymbol{u}, p^\varepsilon)\big) \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u})\,\varphi\,dx \geq 0,$$

yielding $(\boldsymbol{m}^\varepsilon - \boldsymbol{m}^*(\boldsymbol{u}, p^\varepsilon)) \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u}) \geq 0$ a.e. in Ω, which further begets $(\boldsymbol{m}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon) \in \mathcal{A}$ a.e. in Ω by Lemma 1.

3.3 ε-limit

In the spirit of the previous limit, (25) and the weak lower semicontinuity of a norm applied on (24) produce the third energy inequality

$$\varepsilon \,\|\nabla(p^\varepsilon - p_0)\|_{r'}^{r'} + \|\boldsymbol{m}^\varepsilon\|_{r'}^{r'} + \|\boldsymbol{v}^\varepsilon\|_r^r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r). \tag{38}$$

The first term actually does not pose much of a problem, for eq. $(26)_1$ implies a pointwise identity

$$\nabla p^\varepsilon = \boldsymbol{f} - \boldsymbol{m}^\varepsilon \quad \text{a.e. in } \Omega,$$

whence there follows optimization of (38), namely

$$\|\nabla(p^\varepsilon - p_0)\|_{r'} + \|\boldsymbol{m}^\varepsilon\|_{r'} + \|\boldsymbol{v}^\varepsilon\|_r \leq C(\|\boldsymbol{f} - \nabla p_0\|_{r'}, \|\boldsymbol{v}_0\|_r). \tag{39}$$

Like twice before already, we can find subsequences

$$\begin{aligned} \boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{in } L^r(\Omega)^d,\\ p^\varepsilon - p_0 &\rightharpoonup p - p_0 && \text{in } W^{1,r'}_{\Gamma_2}(\Omega),\\ \varepsilon\,|\nabla(p_n^\varepsilon - p_0)|^{r'-2}\,\nabla(p_n^\varepsilon - p_0) &\rightharpoonup \boldsymbol{0} && \text{in } L^r(\Omega)^d,\\ \boldsymbol{m}^\varepsilon &\rightharpoonup \boldsymbol{m} && \text{in } L^{r'}(\Omega)^d, \end{aligned} \tag{40}$$

for $\varepsilon \to 0_+$. The limit quantities satisfy

$$\begin{aligned} \int_\Omega \nabla p \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{m} \cdot \boldsymbol{w} &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}, \quad \forall \boldsymbol{w} \in L^r(\Omega)^d,\\ 0 &= \int_\Omega (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \nabla q, \quad \forall q \in W^{1,r'}_{\Gamma_2}(\Omega), \end{aligned} \tag{41}$$

that is

$$\begin{aligned} \nabla p + \boldsymbol{m} = \boldsymbol{f} &\quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 &\quad \text{in } \Omega, \\ (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \boldsymbol{n} = 0 &\quad \text{on } \Gamma_1, \\ p - p_0 = 0 &\quad \text{on } \Gamma_2. \end{aligned}$$

In order to reach (1), the sole remaining step is showing $(\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A}$ a.e. in Ω. Let us first take (27) with $\boldsymbol{\chi}$ already identified from (33), recall $|\nabla(p^\varepsilon - p_0)|^{r'}$ is bounded in $L^1(\Omega)$ and pass to the limit $\varepsilon \to 0_+$:

$$\lim_{\varepsilon \to 0_+} \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon - \int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}_0 = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v} - \boldsymbol{v}_0).$$

The limit equation (41) yields, on the other hand

$$\int_\Omega \boldsymbol{m} \cdot (\boldsymbol{v} - \boldsymbol{v}_0) = \int_\Omega (\boldsymbol{f} - \nabla p_0) \cdot (\boldsymbol{v} - \boldsymbol{v}_0),$$

whereby we infer

$$\lim_{\varepsilon \to 0_+} \int_\Omega \boldsymbol{m}^\varepsilon \cdot \boldsymbol{v}^\varepsilon = \int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}.$$

The rest would follow along the same lines as what came after (35). Of course, by (40) we may again tacitly assume $p^\varepsilon \to p$ a.e. in Ω. Thus justification of $(\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A}$ a.e. in Ω is complete and with it, the proof of Theorem 3. □

4 Maximum and minimum principle

What ensues is an observation that in the case of conservative forces and pure inflow, or pure outflow over Γ_1, one obtains a minimum or a maximum principle, respectively, for the pressure. Note that this result can be relatively easily obtained for the primordial Darcy's model, i.e. $\boldsymbol{m} = \alpha \boldsymbol{v}$, for some $\alpha > 0$ where, after formal application of the divergence operator, one ends up with an elliptic problem $\Delta p = \operatorname{div} f$. The property of maximum and minimum principle thus endured extensions at least up to ours.

We start with introducing an additional assumption on the graph, namely

(A6) *strict monotonicity at the origin*

$$\forall (\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A} : \boldsymbol{m} \cdot \boldsymbol{v} = 0 \Rightarrow \boldsymbol{m} = \boldsymbol{0}.$$

Note that this condition follows trivially from **(A4)** provided $c_2 = 0$.

Theorem 5 *Let assumptions of Theorem 3 be in force and Ω be additionally connected. Let **(A6)** hold and $\boldsymbol{f} = \nabla g$ for some $g \in W^{1,r'}(\Omega)$. Then*

(i) $\boldsymbol{v}_0 \cdot \boldsymbol{n} \geq 0$ on Γ_1 implies $p - g \leq \operatorname{ess\,sup}_{\Gamma_2}(p_0 - g)$ a.e. in Ω.

(ii) $\boldsymbol{v}_0 \cdot \boldsymbol{n} \leq 0$ on Γ_1 implies $p - g \geq \operatorname{ess\,inf}_{\Gamma_2}(p_0 - g)$ a.e. in Ω.

In particular, if $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on Γ_1, Γ_2 is non-trivial in the sense $|\Gamma_2|_{d-1} > 0$, $p_0 \in L^\infty(\Gamma_2)$ and $g \in L^\infty(\Omega) \cap W^{1,r'}(\Omega)$, then $p \in L^\infty(\Omega)$.

Proof. We will concentrate on the maximum principle only, its minimum counterpart would be verified completely analogously.

Without loss of generality assume $\operatorname{ess\,sup}_{\Gamma_2}(p_0 - g) < \infty$. The proof hinges on a proper choice of a test function in the weak formulation (41) of the problem (1). Define a truncation operator

$$T(x) = \begin{cases} 0 & \text{for } x \le 0, \\ x & \text{for } 0 < x \le 1, \\ 1 & \text{for } x > 1, \end{cases}$$

and a test function

$$\boldsymbol{w} = T(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g))\boldsymbol{v}.$$

Abbreviating $T := T(p - g - \operatorname{ess\,sup}_{\Gamma_2}(p_0 - g))$ when necessary, we arrive at

$$\int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}\, T\, dx = -\int_\Omega \nabla(p - g) \cdot \boldsymbol{v}\, T\, dx. \tag{42}$$

On the one hand, the right-hand side of (42) can be rewritten as

$$\begin{aligned} -\int_\Omega \nabla(p - g) \cdot \boldsymbol{v}\, T\, dx &= -\int_\Omega \nabla(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) \cdot \boldsymbol{v}\, T\, dx \\ &= -\int_\Omega \nabla H(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) \cdot \boldsymbol{v}\, dx, \end{aligned}$$

where $H(x) = \int_0^x T(s)\, ds \ge 0$. Then the integration by parts and $\boldsymbol{v}_0 \cdot \boldsymbol{n} \ge 0$ on Γ_1 yield

$$-\int_\Omega \nabla H(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) \cdot \boldsymbol{v}\, dx \;=\; -\int_{\Gamma_1 \cup \Gamma_2} H(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g))\boldsymbol{v} \cdot \boldsymbol{n}\, dS \;\le\; 0.$$

Eq. (42) hence gives $\int_\Omega \boldsymbol{m} \cdot \boldsymbol{v}\, T\, dx \le 0$. On the other hand, **(A1)** and **(A2)** imply $\boldsymbol{m} \cdot \boldsymbol{v} \ge 0$ a.e. in Ω and therefore

$$\boldsymbol{m} \cdot \boldsymbol{v}\, T(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 - g)) = 0 \quad \text{a.e. in } \Omega.$$

Denoting $V = \{x \in \Omega \mid (p - g)(x) > \operatorname{ess\,sup}_{\Gamma_2}(p_0 + g)\}$, **(A6)** entails $\boldsymbol{m} = 0$ a.e. in V. Fom $(1)_1$ we deduce $\nabla(p - g) = 0$ a.e. in V, so that

$$\nabla\Big[\big(p - g - \operatorname*{ess\,sup}_{\Gamma_2}(p_0 + g)\big)_+\Big] = \mathbf{0} \quad \text{a.e. in } \Omega.$$

Therefore $\big(p - g - \operatorname{ess\,sup}_{\Gamma_2}(p_0 + g)\big)_+ \equiv C$ for some constant C due to connectedness of Ω. However, this constant must be zero, since $p - g$ is a Sobolev function. Therefore $p - g \le \operatorname{ess\,sup}_{\Gamma_2}(p_0 - g)$ a.e. in Ω. □

5 Extended existence theorem

The primal benefit of Theorem 5 is that, at certain price, we can significantly slacken the draconian restrictions imposed by $\mathbf{(A5)}_{(\mathrm{iv})}$, as well as **(A4)**, by allowing the constants c and c_1 to be actually functions of the pressure. Thus we can vastly extend the class of admissible interactions $\boldsymbol{m}$ and cover some physically relevant cases. More precisely, let us consider there exist $\alpha, \beta \in \mathcal{C}(R)$ strictly positive everywhere on $\mathbb{R}$, such that

(A4*) $\exists c_2 \geq 0\ \forall (\boldsymbol{m}, \boldsymbol{v}, p) \in \mathcal{A} : \boldsymbol{m} \cdot \boldsymbol{v} \geq \alpha(p)(|\boldsymbol{v}|^r + |\boldsymbol{m}|^{r'}) - c_2,$

(A5)$_{(\mathrm{iv}^*)}$ $\forall (\boldsymbol{v}, p) \in \mathbb{R}^d \times \mathbb{R} : |\boldsymbol{m}^*(\boldsymbol{v}, p)| \leq \beta(p)(1 + |\boldsymbol{v}|^{r-1}).$

Theorem 6 *Let Ω be a connected Lipschitz domain and $r \in (1, \infty)$. Assume $\boldsymbol{f} = \nabla g$ for some $g \in L^\infty(\Omega) \cap W^{1,r'}(\Omega)$, $\boldsymbol{v}_0 \equiv \boldsymbol{0}$, $|\Gamma_2|_{d-1} > 0$ and $p_0 \in W^{1,r'}(\Omega) \cap L^\infty(\Gamma_2)$. Moreover, assume that $\mathcal{A}$ is a maximal monotone r-graph in the sense of* **(A1)–(A6)**, *with **(A4)** and **(A5)**$_{(iv)}$ replaced by **(A4*)** and **(A5)**$_{(iv^*)}$, respectively. Then the existence result of Theorem 3 still holds.*

Proof. Take $K := \|g\|_{\infty,\Omega} + \|p_0 - g\|_{\infty,\Gamma_2}$ and recall (9) for the definition of T_K. The truncated problem

$$\begin{aligned} \nabla p + \boldsymbol{m} &= \nabla g && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega, \\ \boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, T_K(p)) &= \boldsymbol{0} && \text{in } \Omega, \\ (\boldsymbol{v} - \boldsymbol{v}_0) \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_1, \\ p - p_0 &= 0 && \text{on } \Gamma_2, \end{aligned} \tag{43}$$

is amenable to Theorem 3. Indeed, setting $c_1 := \min_{[-K,K]} \alpha$ and $c := \max_{[-K,K]} \beta$, we have $c_1 > 0$ and $0 < c < \infty$. Taking $\widehat{\boldsymbol{m}}^*(\boldsymbol{v}, p) := \boldsymbol{m}^*(\boldsymbol{v}, T_K(p))$ as a selection to be used, invoking the above mentioned theorem is just. Now Theorem 5 yields $\|p\|_\infty \leq K$, which implies $p = T_K(p)$ a.e. in Ω and we are done, as problems (1) and (43) coincide. □

Remark 7 We conclude this paper with an easy observation stemming from the foregoing proof. Namely, if $\inf_{\mathbb{R}_+} \alpha > 0$ and $\sup_{\mathbb{R}_+} \beta < \infty$, there is no need for the maximum principle anymore and instead of $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on Γ_1, mere $\boldsymbol{v}_0 \cdot \boldsymbol{n} \leq 0$ on Γ_1 would suffice to ensure validity of the still indispensable minimum principle. Indeed, in $(43)_3$ we could just as well take

$$\boldsymbol{h}(\boldsymbol{m}, \boldsymbol{v}, \max\{T_K(p), p\}) = 0$$

and $\widehat{\boldsymbol{m}}^*(\boldsymbol{v}, p) := \boldsymbol{m}^*(\boldsymbol{v}, \max\{T_K(p), p\})$ for the selection. Vice versa, we need only the maximum principle, i.e. $\boldsymbol{v}_0 \cdot \boldsymbol{n} \geq 0$ on Γ_1, provided $\inf_{\mathbb{R}_-} \alpha > 0$ and $\sup_{\mathbb{R}_-} \beta < \infty$. The drag coefficient (6) is a prime example of such a situation.

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