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Maximally-dissipative local solutions to rate-independent systems and application to damage and delamination problems

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Preprint no. 2013-021

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Maximally-dissipative local solutions to rate-independent systems and application to damage and delamination problems[1]

Tomáš Roubíček[2]

Abstract: The system of two inclusions $\partial_u\mathcal{E}(t,u(t),z(t))\ni 0$ and $\partial\mathcal{R}(\dot z)+\partial_z\mathcal{E}(t,u(t),z(t))\ni 0$ with the dissipation potential $\mathcal{R}$ degree-1 homogeneous and with the stored energy $\mathcal{E}(t,\cdot,\cdot)$ separately convex is considered. An approximation by a semi-implicit time discretisation is shown to converge to specific local ($\sim$weak) solutions obeying maximimal-dissipation principle in a certain sense. Applications of such (in fact, force-driven) solutions are illustrated on specific examples from continuum mechanics at small strains involving inelastic processes in a bulk or on a surface, namely damage and delamination.

Keywords: rate-independent processes, weak solutions, maximum-dissipation principle, semi-implicit time discretisation, weak or strong convergence, applications in continuum mechanics, inelastic processes, uncomplete damage, adhesive contact.

AMS Subj. Class. 35K86, 35M86, 35Q74, 49S05, 65M12, 74R20, 74S30.

1. Introduction. *Rate-independent systems* are certain idealization of real (typically mechanical, magnetical, electrical, etc.) systems when various rate-dependent mechanism (like inertia, viscosity, heat, etc.) are neglected and only some activated mechanism remain. Such idealization is often relevant and useful for both theoretical and computational reasons. Wide mathematical theory of such systems have been developed especially during the past decade or two, cf. [8, 12, 26, 29–31] and references therein.

The mentioned simplification by neglecting rate-dependent mechanisms to dissipate or otherwise transfer the (typically mechanical, magnetical, electrical, etc.) energy causes that not only the formulation of the model of a real system in question is important but, except rather simple cases, also the *concept of its solution is a vital part of the modelling procedure*. The general concept relies on the weak or, in the theory of rate-independent processes, the local solutions (which are essentially equivalent to each other, as shown later here in Proposition 2.3). Within this broad class, there are several noteworthy concepts like globally stable (and energy conserving) local solutions [31] called also energetic solutions (or irreversible quasistatic evolution in [8, 9]), vanishing-viscosity solutions or parameterized solutions [10, 27], BV-solutions [28], approximable solutions [7, 17, 50], or semi-energetic solution [42]; cf. [26] for their comparison (except the last one) and still for other concepts more. It should be emphasized that, due to the mentioned forgotten rate-dependent dissipative mechanisms, the requirement of energy conservation itself (as adopted by the energetic solutions) need not be relevant any longer and, just opposite, may lead to nonphysical effects - typically too-early jumps of solutions if the system is governed by non-convex energies. In addition, the energetic solutions are based on global stability and a global-minimization principle applied to an incremental approximation, which may represent very serious computation difficulties in implementation of such concept, cf. e.g. [1]. The mentioned other concepts thus may be more relevant. However, their computation implementation is either not clear or also very difficult, cf. e.g. [19, 43] for the vanishing-viscosity-type concept.

The goal of this article is, at least in a bit special (although still quite well applicable) case with a *separately-convex stored energy* and a state-independent dissipation energy, to analyze the *semi-implicit time discretisation* of the fractional-step type. Such discretisation is often intuitively used as an *efficient approximation scheme* in engineering. After introducing the definitions of *local solutions* in Section 2 and identifying them essentially as conventional *weak solutions*, we use a motivating scalar example with explicitly known solutions and make an attempt to select a physically relevant *force-driven* solution by using the classical *maximum-dissipation principle* in Section 3. In particular, this scalar example clearly selects two extreme concepts of solution, driven by energy versus driven by force. In general, this is also recognized in engineering literature: let us quote a well-recognized article by D. Leguillon [23], saying that "the incremental form of the energy criterion gives a lower bound of admissible crack lengths. On the contrary, the stress criterion leads to an upper bound." The former concept surely correspond to the global minimization of the incremental energy arising by fully-implicit time discretisation, while the latter concept seem to be related (to some extent) rather with a local minimization. Here, on the 0-dimensional example(s) we will show its relation to the fractional-step semi-implicit discretisation, while the general multidimensional situation is to be validated a-posteriori case by case by using the integrated maximum-dissipation principle for the approximate solution, which is shown on that 0-dimensional example(s) to have ability to detect too-early jumps of solutions which are triggered under driving force not achieving the prescribed activation threshold (as it may happen e.g. in energy-driven solutions).

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[1]This research was performed within the grants 201/09/0917, 201/10/0357, and 13-18652S (GA ČR), together with the institutional support RVO: 61388998 (ČR).

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Then, in Section 4, on an abstract level we can prove the convergence of the mentioned semi-implicit approximation in time towards (a.e.) local solutions and, in particular, existence of such local solutions. Yet, it should be noted that satisfaction of the (integrated) maximum-dissipation principle in these local solutions is not proved and even existence of maximally-dissipative local solutions is not known in general. The usage of these abstract results is demonstrated on two specific examples from *continuum mechanics at small strains* involving *inelastic processes* on the surface or in the bulk, namely the delamination and the damage problems in Sections 5 and 6, respectively. For a still considerably many applications, we thus obtain very efficient numerical strategy relatively easy to be implemented which gives physically relevant solutions (cf. also the results and the discussion in [44]) and is supported by some rigorous mathematical analysis, although a lot of questions still remains rather open.

More specifically, on the abstract level, we assume the state of the rate-independent system evolving in time to be valued in a Banach space Q and, relying on the Cartesian structure $Q = U \times Z$ with U and Z reflexive separable Banach spaces, we consider the Gibbs-type stored-energy functional $\mathcal{E} : I \times U \times Z \to \mathbb{R} \cup \{\infty\}$ and the Rayleigh-type dissipation-energy functional $\mathcal{R} : X \to \mathbb{R} \cup \{\infty\}$ with $I = [0, T]$ a fixed time interval and $X \supset Z$ another Banach space. We consider the following evolution system of *inclusions of an abstract "elliptic/parabolic type"*:

$$\partial_u \mathcal{E}(t, u(t), z(t)) \ni 0, \tag{1.1a}$$

$$\partial \mathcal{R}(\dot{z}) + \partial_z \mathcal{E}(t, u(t), z(t)) \ni 0, \tag{1.1b}$$

on the time interval $I = [0, T]$ with using the notation ∂ or ∂_u and ∂_z for (partial) subdifferentials of convex functions and $\dot{z} := \frac{\mathrm{d}z}{\mathrm{d}t}$, and with prescribing the initial conditions

$$z(0) = z_0. \tag{1.1c}$$

In many nontrivial applications, $\mathcal{E}(t, \cdot, \cdot)$ is not convex but often both

$$\mathcal{E}(t, \cdot, z) : U \to \mathbb{R} \cup \{\infty\} \quad \text{is convex and} \tag{1.2a}$$

$$\mathcal{E}(t, u, \cdot) : Z \to \mathbb{R} \cup \{\infty\} \quad \text{is convex;} \tag{1.2b}$$

then we will speak about *separate convexity.* The former convexity is important for controlling jumps of u while the latter convexity is desirable to make the semi-stability condition (see (2.3) below) well motivated. Moreover, thorough this paper we assume that $\mathcal{R}$ is a so-called gauge, i.e.

$$\mathcal{R} : X \to \mathbb{R} \cup \{\infty\} \ \text{ is convex}, \qquad \mathcal{R} \geq 0, \qquad \forall a \geq 0 \ \ \forall z \in \mathrm{dom}\, \mathcal{R} : \ \ \mathcal{R}(az) = a\mathcal{R}(z), \tag{1.2c}$$

where $\mathrm{dom}\, \mathcal{R} := \{z \in Z;\ \mathcal{R}(z) < \infty\}$. The last property (i.e. *positive degree-1 homogeneity* of $\mathcal{R}$) makes the system (1.1a) invariant under monotone rescaling of time, i.e. *rate-independent.* It also implies $\mathcal{R}(0) = 0$. All the three convexity assumptions in (1.2) gives a sense to the three convex sub-differentials in (1.1).

Let us mention that a maximal dissipativity has been exploited by Stefanelli [48, Sect. 7] to devise a selection criterion of a similar nature: it selects local solutions that conserve energy out of their jumps and maximally dissipate during jumps. A drawback pointed out in [48] is, similarly as in the maximally-dissipative context devised here, that general existence results are not available at present, however. As shown in Remark 4.5 below, this concept is not entirely identical with our notion of maximal dissipative local solutions which cares rather about beginning of jumps than about their ending.

A comparison of various concepts of solutions, in particular energy versus stress-driven concepts, on a 2-dimensional example in crack mechanics with a prescribed crack path is also in [19, 33, 35]. In particular, [35] treats convergence towards local solutions, using local minimization of energy after time discretisation.

Also, let us emphasize that the application are not limited to those in Sections 5 and 6. With a certain modelling simplifications, some phase-transformation models may comply with the structure (1.2), cf. [6, 21, 32] and, as far as $\mathcal{E}$ is itself concerned, also [13, 14]. When damage model from Section 6 is combined with a certain healing term in $\mathcal{E}$, it gives Ambrosio-Tortorelli's functional used (in case of a unidirectional damage evolution) for a regularized fracture model (although mostly in scalar-valued antiplane-shear variant only), including the semi-implicit type time-discretisation, cf. e.g. in [2, 4, 22, 34], which, when iterated and converged, gives essentially the so-called alternating minimization algorithm popular in engineering calculations but without any mathematical analysis available in truly nonconvex cases. Counting a straightforward generalization in Remark 4.7 below, other problems as e.g. delamination with additional interfacial plasticity to distinguish between particular fracture modes [44] or the linearized plasticity combined with damage fit with the separate-convexity ansatz (1.2), too.

Eventually, let us also emphasize that numerically stiff problems (like (5.1) with a very big $\mathbb{K}$ to approach a brittle delamination or the mentioned Ambrosio-Tortorelli's functional approximating brittle cracks) computationally practically cannot be handled if one attempts to rely on global minimization and energetic solution concept which, moreover, gives completely different (and in some cases very obviously nonphysical) scenario of fracture. On the other hand, the semi-implicit discretisation devised in Section 4 yielding some sort (although still not fully characterized) local solutions works very efficiently, reliably, and robustly even for an extremely stiff problems, cf. [46] for the brittle delamination case. For a comparison of two mentioned computational strategies (and related solution concepts) in the context of delamination see also [51] or, for a special 2D prescribed-path crack, [35, Sect. 3.2] where the energetic and the maximally-dissipative solutions are labeled as (FM) or (G), referring to Fracture Mechanics or Griffith, respectively.

2. Local solutions. The concept of local solutions to rate-independent systems was introduced for a special crack problem in [50], as a general concept (under the name in "dissipative trajectory" in [48, Def.6.1]), and further generally investigated in [26]. Here, we additionally combine it with the concept of semi-stability from [39]. In fact, the notion of "semi-stability" was invented in [40] and then used also, e.g., in [22, Formula (5)] for a special dynamic fracture problem. It employs (1.2c), which implies $\partial\mathcal{R}(v)\subset\partial\mathcal{R}(0)$ for any v, so that for $v=\dot z$ from (1.1b) one gets

$$\partial\mathcal{R}(0)\ni\xi(t)\qquad\text{with (some) driving force}\quad \xi(t)\in-\partial_z\mathcal{E}(t,u(t),z(t)). \tag{2.1}$$

For $\mathcal{E}(t,u,\cdot)$ smooth, we have simply the equality $\xi=-\partial_z\mathcal{E}(t,u(t),z(t))$, but in general we will distinguish the (not-uniquely defined) *actual driving force* ξ and the (set-valued) *available driving force* $-\partial_z\mathcal{E}(t,u(t),z(t))$. The adjective "available" is sometimes used in fracture mechanics, referring to the energy release rate. From the convexity of $\mathcal{R}$ when taking into account that $\mathcal{R}(0)=0$, this inclusion is equivalent to

$$\mathcal{R}(v)-\langle\xi,v\rangle\geq\mathcal{R}(0)=0\qquad\text{for any } v\in Z. \tag{2.2}$$

Substituting $v=\tilde z-z(t)$ and using the convexity of $\mathcal{E}(t,u,\cdot)$, we obtain $0\leq\mathcal{R}(\tilde z{-}z(t))-\langle\xi,\tilde z{-}z(t)\rangle\leq\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}(\tilde z{-}z(t))-\mathcal{E}(t,u(t),z(t))$, i.e.

$$\mathcal{E}(t,u(t),z(t))\leq\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}(\tilde z{-}z(t))\qquad\text{for any }\tilde z\in Z, \tag{2.3}$$

which is naturally to be called *semi-stability* at time t (in contrast to a full stability which would vary also the u-variable, cf. [26, 29, 31]).

We will use the standard notation: For a Banach space V, $L^p(I;V)$ will denote the Bochner space of V-valued Bochner measurable functions $u:I\to V$ with its norm $\|u(\cdot)\|$ in $L^p(I)$, where $\|\cdot\|$ stands for the norm in V. Further, $W^{1,p}(I;V)$ denotes the Banach space of mappings $u:I\to V$ whose distributional time derivative is in $L^p(I;V)$, while $\mathrm{BV}(I;V)$ will denote the space of mappings $u:I\to V$ with a bounded variations, i.e. $\sup_{0\leq t_0<t_1<\ldots<t_{n-1}<t_N\leq T}\sum_{j=1}^N\|u(t_j){-}u(t_{j-1})\|<\infty$ where the supremum is taken over all finite partitions of the interval $I=[0,T]$. Further, $\mathfrak{M}(I;V)$ will denote that space of V-valued measures on I. Eventually, by $\mathrm{B}(I;V)$ we denote the space of bounded measurable (everywhere defined) mapping $I\to V$. The notation "$\to$", "$\rightharpoonup$", and "$\overset{*}{\rightharpoonup}$" stands for strong, weak, and weak* convergence, respectively.

Definition 2.1 (Local and a.e.-local solutions). *The pair (u,z) with $u\in\mathrm{B}(I;U)$ and $z\in\mathrm{B}(I;Z)\cap\mathrm{BV}(I;X)$ is called a local solution to (1.1) if, beside (1.1c), it holds that*

$$\forall_{\text{a.a.}}t\in I:\qquad \partial_u\mathcal{E}(t,u(t),z(t))\ni0, \tag{2.4a}$$

$$\forall_{\text{a.a.}}t\in I,\ \forall\tilde z\in Z:\qquad \mathcal{E}(t,u(t),z(t))\leq\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}(\tilde z-z(t)), \tag{2.4b}$$

$$\forall t_1,t_2\in I,\ t_1<t_2:\qquad \mathcal{E}(t_2,u(t_2),z(t_2))+\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2])\leq\mathcal{E}(t_1,u(t_1),z(t_1))+\int_{t_1}^{t_2}\mathcal{E}_t'(t,u(t),z(t))\,\mathrm{d}t \tag{2.4c}$$

where $\mathcal{E}_t'=\frac{\partial}{\partial t}\mathcal{E}$ and where $\mathrm{Diss}_{\mathcal{R}}(z;[r,s]):=\sup\sum_{j=1}^N\mathcal{R}(z(t_{j-1}){-}z(t_j))$ with the supremum taken over all finite partitions $r\leq t_0<t_1<\cdots<t_{N-1}<t_N\leq s$. Moreover, if (2.4c) holds only for a.a. $t_1,t_2\in I$ with $t_1<t_2$, then (u,z) is called an a.e.-local solution.

Definition 2.2 (Weak solution). *The pair (u,z) with $u\in \mathrm{B}(I;U)$ and $z\in \mathrm{B}(I;Z)\cap \mathrm{BV}(I;X)$ is called a weak solution to (1.1) if, beside (1.1c), there exists $\xi\in L^1(I;Z^*)$ such that it holds that*

$$\text{(2.5a)}\qquad \forall_{\text{a.a.}} t\in I: \qquad \partial_u\mathcal{E}(t,u(t),z(t))\ni 0,$$

$$\text{(2.5b)}\qquad \forall_{\text{a.a.}} t\in I: \qquad \partial_z\mathcal{E}(t,u(t),z(t))+\xi(t)\ni 0, \quad \textit{and}$$

$$\text{(2.5c)}\qquad \forall v\in Z\ \forall_{\text{a.a.}} t_1,t_2\in I,\ t_1<t_2: \qquad \mathcal{E}(t_2,u(t_2),z(t_2))+\mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2])\le \mathcal{E}(t_1,u(t_1),z(t_1)) + \int_{t_1}^{t_2}\Big(\mathcal{E}_t'(t,u(t),z(t))-\langle\xi,v\rangle+\mathcal{R}(v)\Big)\,\mathrm{d}t.$$

Let us see that Definition 2.2 indeed represents quite standard weak formulation of the flow rule (1.1b) which, assuming $\mathcal{E}$ smooth for a moment, means exactly $\mathcal{R}(\dot z)\le\langle\mathcal{E}_z'(t,u,z),v-\dot z\rangle+\mathcal{R}(v)$ for any $v\in Z$, and with the aim to substitute the troublesome term $\langle\mathcal{E}_z'(t,u,z),\dot z\rangle$ by integration over time interval $[t_1,t_2]$ and using the chain rule

$$\text{(2.6)}\qquad \mathcal{E}(t_2,u(t_2),z(t_2))=\int_{t_1}^{t_2}\big\langle\mathcal{E}_z'(t,u(t),z(t)),\dot z\big\rangle+\big\langle\mathcal{E}_u'(t,u(t),z(t)),\dot u\big\rangle+\mathcal{E}_t'(t,u(t),z(t))\,\mathrm{d}t+\mathcal{E}(t_1,u(t_1),z(t_1)),$$

it eventually yields (2.5c) when also (2.5a), i.e. $\mathcal{E}_u'(t,u(t),z(t))=0$, is taken into account. It should be emphasized that the terms $\langle\mathcal{E}_z'(t,u(t),z(t)),\dot z\rangle$ and $\langle\mathcal{E}_u'(t,u(t),z(t)),\dot u\rangle$ do not have any sense in general because $\dot z$ is a measure while $\mathcal{E}_z'(t,u,z)$ cannot be assumed continuous in time, and $\dot u$ does not have a meaning even as a measure at all. Hence, Definition 2.2 represents a very standard concept working even for rate dependent problems when (1.2c) is not satisfied. It is important to see that, in our rate-independent situation, Definition 2.1 is standard too. Even more, both definitions are essentially the same:

Proposition 2.3. *Let (1.2b,c) hold, and let* $\mathrm{dom}\,\mathcal{R}=Z$ *or* $\partial_z\mathcal{E}$ *is bounded in the sense that, for any $R\ge0$, there is $a_R\in L^1(I)$ such that $\|u\|\le R$ and $\|z\|\le R$ implies* $\sup_{f\in\partial_z\mathcal{E}(t,u,z)}\|f\|_{Z^*}\le a_R(t)$ *for a.a. $t\in I$. Then the a.e.-local solutions coincide with the weak solutions.*

Proof. Let us also note that (2.4b) means exactly that, choosing some driving force $\xi(t)\in-\partial_z\mathcal{E}(t,u(t),z(t))$, it holds $0\le\mathcal{R}(v)-\langle\xi(t),v\rangle$ for any $v\in Z$, cf. the argumentation used in (2.2). Adding it to (2.4c) reveals that any a.e.-local solution is also a weak solution. More precisely, if $\mathcal{E}(t,u,\cdot)$ is nonsmooth, we make a measurable selection of possible values of ξ. Here we also use that $\partial_z\mathcal{E}(t,u(t),z(t))$ is nonempty, which follows directly from our assumption that $\partial_z\mathcal{E}$ is bounded (reminding the standard convention that $\sup_{f\in\partial_z\mathcal{E}(t,u,z)}\|f\|_{Z^*}=+\infty$ if $\partial_z\mathcal{E}(t,u,z)=\emptyset$) or, in the case of $\mathrm{dom}\,\mathcal{R}=Z$, from (2.4b) which is at disposal because (u,z) is an a.e.-local solution. At this point, note that (2.4b) implies that $z(t)$ minimizes $\mathcal{E}(t,u(t),\cdot)+\mathcal{R}(\cdot-z(t))$ and thus

$$\text{(2.7)}\qquad 0\in\partial_z\big[\mathcal{E}(t,u(t),\cdot)+\mathcal{R}(\cdot-z(t))\big](z(t))=\partial_z\mathcal{E}(t,u(t),z(t))+\partial\mathcal{R}(0),$$

where one uses the qualification for the sum-rule that at least one of the summed functions (i.e. here $\mathcal{R}$) is continuous at some point in domains of all the summed functions, cf. [11, Ch.1, Prop.5.6].

The growth assumption on $\partial_z\mathcal{E}$ ensures $\xi\in L^1(I,Z^*)$, or, in case $\mathrm{dom}\,\mathcal{R}=Z$, we can use that also $\xi(t)\in\partial\mathcal{R}(0)$ as shown in (2.7) and that $\mathcal{R}(0)$ is now bounded so that even $\xi\in L^\infty(I,Z^*)$.

Conversely, putting $v=0$ into (2.5c), we obviously obtain (2.4c). By the degree-1 homogeneity of $\mathcal{R}$, we have

$$\int_{t_1}^{t_2}\mathcal{R}(k\tilde z)-\big\langle\xi(t),k\tilde z\big\rangle\,\mathrm{d}t=k\int_{t_1}^{t_2}\mathcal{R}(\tilde z)-\big\langle\xi(t),\tilde z\big\rangle\,\mathrm{d}t$$

for any $\tilde z\in Z$ and $k\in\mathbb{N}$, and putting $v=k\tilde z$ into (2.5c), and sending $k\to\infty$, we can see that $\int_{t_1}^{t_2}\mathcal{R}(\tilde z)-\langle\xi(t),\tilde z\rangle\,\mathrm{d}t\ge0$ because otherwise we would get a contradiction with (2.5c) for a sufficiently big k. This holds for all $v\in Z$ and for a.a. $0\le t_1<t_2\le T$. Assuming $\mathcal{R}(\tilde z(t))-\langle\xi(t),\tilde z(t)\rangle<0$ were hold for some t from a measurable set I_- of a positive measure and some $\tilde z(t)\in Z$ (which, in addition, can be taken in a measurable way as ξ is measurable), taking $\tilde z(t)=0$ for $t\in I\backslash I_-$, and using $\mathcal{R}(0)=0$, we would get $\int_{t_1}^{t_2}\mathcal{R}(\tilde z(t))-\langle\xi(t),\tilde z(t)\rangle\,\mathrm{d}t<0$ for some $[t_1,t_2]$, a contradiction. Thus, for a.a. $t\in I$ and all $\tilde z\in Z$, we get $\mathcal{R}(\tilde z)-\langle\xi(t),\tilde z\rangle\ge0$, which means $\partial\mathcal{R}(0)\ni\xi(t)$. By the argumentation (2.1)–(2.3), we obtain (2.4b); here (1.2b) was used. □

In fact, in the above proof, we needed only to ensure existence of a measurable integrable selection from the set-valued mapping $t\mapsto\partial\mathcal{R}(0)\cap\partial_z\mathcal{E}(t,u(t),z(t))$ for any $u\in B(I;U)$ and $z\in B(I;Z)$, which allows for finer conditions by combining qualification of $\mathcal{R}$ and $\partial_z\mathcal{R}$.

3. Maximally-dissipative local solutions. Let us start with a 0-dimensional example as in [42], essentially consisting from two linearly-responding elastic springs in series, one of them undergoing a damage. Thus u and z are just scalar variables, the whole problem has only 2 degrees of freedom, and everything can be made quite explicit. Let us make a simple experiment by considering the Dirichlet load starting from zero and growing in time with a constant speed $v_{\rm D} > 0$, i.e. $u_{\rm D}(t) = v_{\rm D}t$. We thus deal with the energies $\mathcal{E} : I\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ and $\mathcal{R} : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ given by

$$\mathcal{E}(t,u,z) = \begin{cases} \frac{1}{2}zKu^2 + \frac{1}{2}C|u-v_{\rm D}t|^2 & \text{if } 0 \le z, \\ +\infty & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathcal{R}(\dot z) = \alpha|\dot z| \tag{3.1}$$

with the "elastic" moduli $K > 0$ and $C > 0$ just scalars, and with $\alpha > 0$ a prescribed activation threshold for triggering damage. Our goal is to calculate the time when the damageable spring (=adhesive, cf. Section 5 below) breaks. We start with the initial condition $z(0) = z_0 \equiv 1$. Note that, if $z > 0$, the driving force $-\mathcal{E}'_z(t,u,z) = -\frac{1}{2}Ku^2$ is non-positive. Therefore z must be nonincreasing until reaching possibly its minimal value 0. This is reflected by the definition of local solutions: if $z(t_1) < z(t_2)$ for some $t_1 < t_2$, then the energy inequality (2.4c) on $[t_1, t_2]$ would be violated.

Analyzing the semi-stability condition (2.4b) for (3.1), i.e. $\frac{1}{2}K(\tilde z-z)u^2 + \alpha|z-\tilde z| \ge 0$ for all $\tilde z \in [0,1]$, we can see that the rupture time $t_{\rm LS}$ of a local solution will be at most the time when the elastic energy of the fully bonded adhesive reaches the activation threshold α, i.e. $\frac{1}{2}Ku^2 = \alpha$. This means, by calculating the equilibrium u for $z = 1$, that $\frac{1}{2}K(v_{\rm D}Ct_{\rm LS}/(K+C))^2 = \alpha$, from which we can see that the delamination happens at latest at the time, let us denote it by $t_{\rm MD}$, which can be calculated as

$$t_{\rm MD} = (K/C+1)\sqrt{2\alpha/K}/v_{\rm D}. \tag{3.2}$$

This time is characterized by the driving force for the delamination $-\partial_z\mathcal{E}(t,u,z)$ achieving the boundary of the "elastic" domain $\partial\mathcal{R}(0)$, cf. Fig. 1(lower row). Therefore, this "latest-time" scenario can be understood as a *force driven* one. The actual mechanical stress $\sigma = \partial_u\mathcal{E}(t,u(t),z(t))$ is then $\partial_u\mathcal{E}(t_{\rm MD}, u(t_{\rm MD}), 1) = CKv_{\rm D}t_{\rm MD}/(C+K)$. In fact, the semistability does not give any information before this time because obviously always $\frac{1}{2}K(\tilde z-z)u^2 + \alpha|z-\tilde z| \ge (\alpha-\frac{1}{2}Ku^2)|z-\tilde z| \ge 0$ provided $\frac{1}{2}Ku^2 < \alpha$. Therefore the rupture time $t_{\rm LS}$ is allowed even before but not earlier than at the time, let us denote it by $t_{\rm ES}$, when the globally stable (and thus energy-conserving) local solutions breaks because then the energy balance would be violated; the mentioned global stability means that $\mathcal{E}(t,u(t),z(t)) \le \mathcal{E}(t,\tilde u,\tilde z)+\mathcal{R}(\tilde z-z(t))$ holds for any $(\tilde u,\tilde z) \in U\times Z$. This shows the very low selectivity of the local-solution approach, as pointed out already in [26].

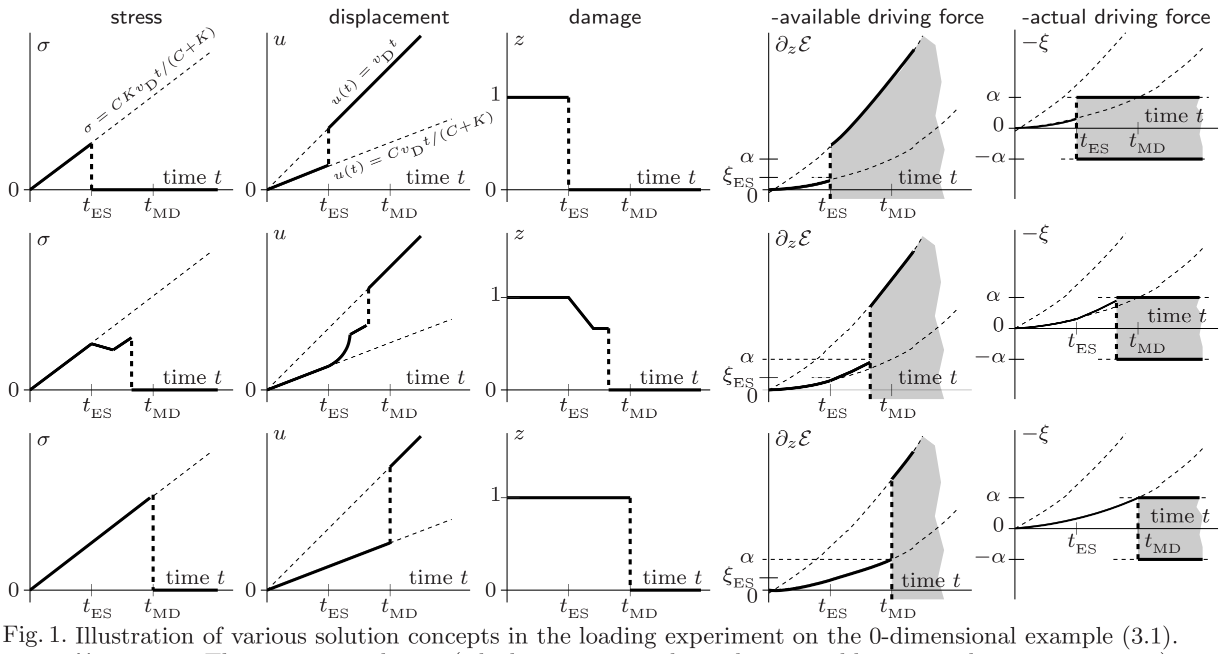

Fig. 1. Illustration of various solution concepts in the loading experiment on the 0-dimensional example (3.1).
Upper row: The energetic solution (which ruptures at the earliest possible time and preserves energy).
Middle row: Another example of a local solution (from many others).
Lower row: The maximally dissipative local solution (which ruptures at the latest time, when the driving force achieves the activation threshold α).

In [42], it was calculated that $t_{\rm ES} = \sqrt{2\alpha(1/C+1/K)}/v_{\rm D}$; therefore such "energetic solution" breaks already when the (negative) driving force

$$\xi_{\rm ES} = \alpha\frac{C}{K+C} \tag{3.3}$$

is less than α, cf. Fig. 1(upper row) and also Fig. 2(C). After the rupture, the driving force $-\partial_z\mathcal{E}(t,u(t),z(t))$ here becomes multivalued and even may jump up to $\frac12 K(v_{\rm D}t_{\rm ES})^2 = \alpha(K{+}C)/C > \alpha$ and continue growing in time as the outer load continues growing. Yet, the selection $\xi(t)$ should always belong to $\partial\mathcal{R}(0)$ which now means that is should stay within $[-\alpha,\alpha]$, cf. again Fig. 1(upper row). We can summarize

$$t_{\rm ES} \le \overset{\swarrow\text{arbitrary}}{t_{\rm LS}} \le t_{\rm MD}. \tag{3.4}$$

The vanishing-viscosity solution breaks also at $t_{\rm MD}$, no matter whether the viscosity is considered in u- or in z-variable, cf. [42]. The rupture time of BV-solutions (which are known to be essentially vanishing-viscosity solutions in specially qualified cases, e.g. finite-dimensional) is also $t_{\rm MD}$. It is interesting to have a look at the overall work of external forces: assuming the rupture at one time $t_{\rm LS}$, the undamaged structure is loaded by the hard-device with the velocity $v_{\rm D}$ under the force which is $Kv_{\rm D}Ct/(K{+}C)$, so that the power of the loading is $Kv_{\rm D}^2Ct/(K{+}C)$, and, integrating it over the time interval $[0,t_{\rm LS}]$, we can see that

$$\text{the work of the loading on the time interval } \big[0,t_{\rm LS}\big] = \frac{Kv_{\rm D}^2C}{2K{+}2C}t_{\rm LS}^2. \tag{3.5}$$

The minimal work done until the delamination is when $t_{\rm LS} = t_{\rm ES}$ and it is not surprising that it equals just to α. The maximal work is for $t_{\rm LS} = t_{\rm MD}$ and equals to $\alpha(1{+}K/C)$. For a general local solution which may rupture possibly gradually like on Fig. 1(middle row), the situation is similar. In any case, after all, the work of the external forces is dissipated. One can thus say that the *force-driven* solution which breaks at time $t_{\rm MD}$ is simultaneously *maximally-dissipative* among all *local solutions*.

Although there is some discussion in engineering literature whether force (or stresses) themselves can be responsible on activation of inelastic processes during fracture if there is not enough energy around a crack tip (with perhaps a certain conclusion that "both energy and stress criteria are necessary conditions for fracture but neither one nor the other are sufficient", cf. [23]), this particular example suggests to advocate only the force- (or stress-driven) local solutions as physically relevant in the considered macroscopical-type model.

This observation on the above very explicit example suggests to seek a connection to the classical maximum-dissipation principle. This principle relies on the degree-1 homogeneity of $\mathcal{R}$. Assuming $z \in W^{1,1}(I;Z)$ and using maximal-monotonicity of the subdifferential, the flow rule (1.1b) means just that $\langle\widetilde\xi-\xi, v-\dot z\rangle_{Z^*\times Z} \ge 0$ for any v and any $\widetilde\xi \in \partial\mathcal{R}(v)$ with the available driving force $\xi \in -\partial_z\mathcal{E}(t,u,z)$. In particular, for $v=0$, defining the convex set $K := \partial\mathcal{R}(0)$, one obtains

$$\big\langle\xi(t),\dot z(t)\big\rangle_{Z^*\times Z} = \max_{\widetilde\xi\in K}\big\langle\widetilde\xi,\dot z(t)\big\rangle_{Z^*\times Z} \qquad \text{with } \ \xi(t)\in-\partial_z\mathcal{E}(t,u(t),z(t)) \quad \text{for} \quad t\in I. \tag{3.6}$$

To derive (3.6), we have used that $\xi \in \partial\mathcal{R}(\dot z) \subset \partial\mathcal{R}(0) = K$ thanks to the degree-0 homogeneity of $\partial\mathcal{R}(\cdot)$, so that always $\langle\xi,\dot z\rangle \le \max_{\widetilde\xi\in K}\langle\widetilde\xi,\dot z\rangle$ and, as $\xi\in K$, the equality in (3.6) is indeed attained at least at $\widetilde\xi = \xi$. The identity (3.6) says that the dissipation due to the driving force ξ is maximal provided that the order-parameter rate $\dot z$ is kept fixed, while the vector of possible driving forces $\widetilde\xi$ varies freely over all admissible driving force from the "elastic" domain K. This just resembles the so-called Hill's *maximum-dissipation principle* [16]. Also it says that the rates are "orthogonal" to the "elastic domain" K, known as an orthogonality principle [52] generalizing Onsager's principle [36]. Of course, the adjective "orthogonality" generally refers only to duality between Z^* and Z, and historically arises from the special situation that Z is a Hilbert space identified with its own dual where orthogonality has a standard meaning. It is also the isothermal variant of the *maximal entropy production principle* [38]. See also [15, 24, 37, 49] for more discussion and details.

The above example showing that force-driven rupturing also needs maximal work of external forcing (and thus dissipates maximal energy) among all local solutions perhaps illuminates the essence of the maximum-dissipation principle. Interestingly, it does not mean that the energy dissipated by the inelastic processes is maximized, cf. Remark 4.5 below.

It is now the aim to select a suitable subclass of local solutions that could be considered as driven by force and not exhibiting tendency to too-early jumps like e.g. globally stable solutions in situations when $\mathcal{E}(t,\cdot,\cdot)$ is not convex and when such energetic solutions have questionable applicability. To this goal, being motivated from the above observations, we strengthen the definition of the (a.e.) local solutions by requiring the maximum-dissipation principle to be valid everywhere on I. Formally, this works in a simple way because, assuming for simplicity that $\mathcal{E}(t,u,\cdot)$ is smooth, (2.3) together with (3.6), which implies (or, if $\mathcal{E}(t,u,\cdot)$ is convex, is even equivalent to)

$$\text{(3.7a)} \qquad \mathcal{R}(v) + \big\langle \mathcal{E}'_z(t,u(t),z(t)), v\big\rangle \geq 0 \qquad \text{together with}$$

$$\text{(3.7b)} \qquad \big\langle -\mathcal{E}'_z(t,u(t),z(t)), \dot z(t)\big\rangle = \max_{\widetilde\xi\in K} \big\langle \widetilde\xi, \dot z(t)\big\rangle = \mathcal{R}(\dot z(t)) \qquad \text{for any } v\in Z,$$

This further implies just by summing (3.7a) and (3.7b) that

$$\text{(3.7c)} \qquad \mathcal{R}(v) + \big\langle \mathcal{E}'_z(t,u(t),z(t)), v-\dot z(t)\big\rangle \geq \mathcal{R}(\dot z(t)) \qquad \text{for any } v\in Z,$$

which just means $-\mathcal{E}'_z(t,u(t),z(t)) \in \partial\mathcal{R}(\dot z(t))$, cf. Fig. 2(D). Let us note that (2.3) was derived by assuming convexity of $\mathcal{E}(t,u,\cdot)$ but, in fact, it is involved in Definition 2.1 even without this qualification of $\mathcal{E}$.

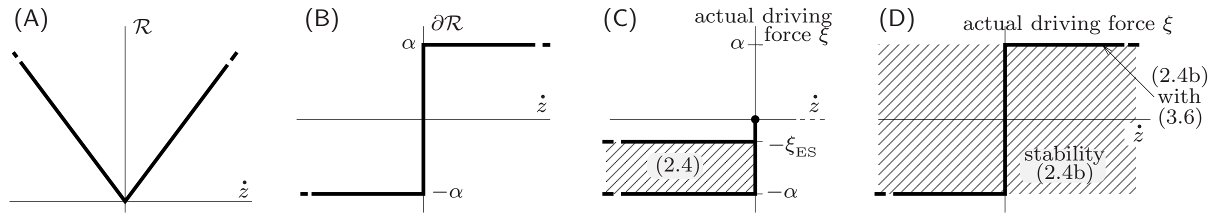

Fig. 2. Illustration of the selection role of the maximum-dissipation principle on the example (3.1). The graphs (C) and (D) are in terms of $\dot z$ and the driving force ξ initiating evolution of z, i.e. the left-limit of $-\partial_z\mathcal{E}$ at the time of rupture. The maximally-dissipative local solutions are the intersection (B)∩(C).
(A) The degree-1 homogeneous dissipation potential $\mathcal{R}$ from (3.1).
(B) The degree-0 homogeneous set-valued subdifferential $\partial\mathcal{R}$ of $\mathcal{R}$.
(C) The possible relation between the driving force ξ and the rate $\dot z$ complying with the local solution (2.4).
(D) The semistability (2.4b) used for local solutions according Definition 2.1 and its combination with the maximum-dissipation principle.

Actually, the argumentation (3.7) and (3.6) itself is unfortunately only very formal because $\dot z$ has values in X rather than in Z and also because it is only a general measure in time so that the validity of (3.6) only a.e. does not say much. Moreover, it is desirable to devise such condition amenable for various limit passages but simultaneously not to destroy its selectivity. Here a problem is that $\partial_z\mathcal{E}(t,u,z)$ may naturally jump in time and thus one cannot expect L^∞-strong convergence while an L^p-convergence obviously does not guarantee elimination of too early jumps and thus would destroy selectivity of such a condition.

In fact, in this example, (3.6) is satisfied even in a rather classical sense of measures: first, note that for any local solution that ruptures at $t_{\rm MD}$ (in particular also the left-continuous one) there is a continuous selection $\xi(t) \in -\partial_z\mathcal{E}(t,u(t),z(t))$ for all $t\in I$. (For example one can take $\xi(t) = -\partial_z\mathcal{E}(t,u(t),1)$ for $t \leq t_{\rm MD}$ and $\xi = -\alpha$ for $t > t_{\rm MD}$.) In this example, $\dot z$ is a Dirac measure with the mass -1 supported at $t = t_{\rm MD}$. The left-hand side of (3.6) is thus a Dirac measure of the magnitude α at $t = t_{\rm MD}$. The right-hand side of (3.6) is the same as $\mathcal{R}(\dot z)$, i.e. the variation of $\dot z$ with respect to $\mathcal{R}$. Here it means again the Dirac measure of the magnitude α at $t = t_{\rm MD}$. Thus the maximum-dissipation principle (3.6) holds in the sense of measures on I.

On the other hand, intuitively, any other local solution violates this principle in a certain sense. This is indeed obvious if z is absolutely continuous in time. For the right-continuous energetic solution (which ruptures at $t = t_{\rm ES}$ and allows for a driving force jumping to the magnitude α already immediately at this rupture time, cf. Fig. 1–upper row) it is not so clear, however. Thus, to reflect also causality and similarly as e.g. in [31, 48], we should rather consider only left-continuous local solutions. Note that any BV-function z and also any local solution (u,z) admits a left-continuous modification which is still a local solution just by taking left limit at all jump points.

Then, instead of the very formal pointwise principle (3.6), we try to formulate its integrated version which would also care about the fact that $\dot z$ is not in duality with z. To this goal, we employ the standard construction of the lower Riemann-Stieltjes integral defined by the supremum of lower Darboux sums as:

$$\text{(3.8)} \qquad \underline{\int_r^s} \xi(t)\,\mathrm{d}z(t) := \sup_{\substack{N\in\mathbb{N}\\ r=t_0<t_1<\ldots<t_{N-1}<t_N=s}} \sum_{j=1}^N \inf_{t\in[t_{j-1},t_j]} \big\langle \xi(t), z(t_j) - z(t_{j-1})\big\rangle;$$

actually, the standard definition in textbooks (as e.g. [47]) works with real-valued functions ξ and z but for our purposes it works equally as far as they range Banach spaces in duality so that the result is again real-valued and allows for the sup/inf-manipulation. In particular, it is important that this definition yields the expected additivity in ξ, in z, and in the integration domain, too. Also it is also convenient that the sum in (3.8) depends monotonically on the partition: any finer partition cannot make it lower. Of course, for $\dot z$ absolutely continuous valued in Z and ξ continuous, we have the expected equality $\underline{\int}_r^s \xi(t)\,\mathrm{d}z(t) = \int_r^s \langle \xi(t), \dot z(t)\rangle\,\mathrm{d}t$ with the later integral being the conventional Lebesgue integral.

With such definition of an integral and having in mind that formally $\max_{\widetilde\xi\in K}\langle\widetilde\xi,\dot z\rangle = \mathcal{R}(\dot z)$ for $K=\partial\mathcal{R}(0)$, the pointwise maximum dissipation principle (3.6) can be expressed integrated over time to yield:

Definition 3.1 (Maximally-dissipative local solutions). *The pair (u,z) with $u\in \mathrm{B}(I;U)$ and with $z\in\mathrm{BV}(I;Z)$ is called a maximally-dissipative local solution to (1.1) if, beside (1.1c) and (2.4), it holds that, for some selection $\xi(t)\in\partial\mathcal{R}(0)$:*

$$\forall\, t\in I:\ \xi(t)\in-\partial_z\mathcal{E}(t,u(t),z(t))\Big)\quad\text{and}\quad \forall\, 0\le t_1<t_2\le T:\quad \underline{\int}_{t_1}^{t_2}\xi(t)\,\mathrm{d}z(t)\ge \mathrm{Diss}_{\mathcal{R}}(z;[t_1,t_2]). \tag{3.9}$$

Note that we formulated (3.9) as an inequality rather than equality. This weaker variant has the same ability to select out solutions which evolve under not enough big driving force but may be better used also for viscous regularization of $\mathcal{R}$. Yet, some stability of this integral principle (to be used e.g. for passing to the limit with the mentioned viscous regularization) does not seem an easy task. E.g. it holds if u's converge pointwise weakly in Z (which is the typical situation) and ξ's converge strongly in Z^* uniformly in time (which is indeed rather strong requirement).

Also note that the temptation to substitute the troublesome term $\int_{t_1}^{t_2}\langle\xi,\dot z\rangle\,\mathrm{d}t$ by $\mathcal{E}(t_1,u(t_1),z(t_1)) + \int_{t_1}^{t_2}\langle\dot u,\partial_u\mathcal{E}(t,u,z)\rangle+\partial_t\mathcal{E}(t,u,z)\,\mathrm{d}t-\mathcal{E}(t_2,u(t_2),z(t_2))$ and use $\partial_u\mathcal{E}(t,u,z)=0$, like we did for the definition of the weak solutions (2.5) in (2.6), leads just to the standard upper energy estimate. The capacity of detecting possible too-early jumps (as may occur e.g. in energetic solutions) would then be lost. So we must really handle the integration problem of $\langle\xi,\dot z\rangle$ in an appropriate way.

In our 0-dimensional example, this integrated maximum-dissipation principle will indeed select out any left-continuous local solution which starts evolving damage z before the actual driving force ξ achieves the prescribed activation threshold. In particular, let us consider such a solution which makes a complete rupture at time $t_{\rm LS}$, i.e.

$$u(t)=\begin{cases}\frac{C}{C+K}v_{\rm D}t,\\ v_{\rm D}t,\end{cases}\qquad z(t)=\begin{cases}1,\\ 0\end{cases}\qquad \xi(t)\begin{cases}=-\frac12Ku(t)^2=-\frac{C^2K}{2(C+K)^2}v_{\rm D}^2t^2 & \text{for }\ t\le t_{\rm LS},\\ \in[-\alpha,\alpha]\ \text{arbitrary} & \text{for }\ t>t_{\rm LS}.\end{cases}$$

The value of the integral on the left-hand side of (3.9) depends on the definition of ξ on $(t_{\rm LS},T]$ but certainly is not bigger than $-\xi(t_{\rm LS})=\frac12C^2Kv_{\rm D}^2t_{\rm LS}^2/(C+K)^2$; indeed, due to the definition (3.8), it is easy to check that

$$\begin{aligned}\underline{\int}_0^T\xi(t)\,\mathrm{d}z(t) &= \underline{\int}_0^{t_{\rm LS}}\xi(t)\,\mathrm{d}z(t)+\underline{\int}_{t_{\rm LS}}^T\xi(t)\,\mathrm{d}z(t)\\ &=0\ +\sup_{0<\varepsilon\le T-t_{\rm LS}}\ \inf_{t\in[t_{\rm LS},t_{\rm LS}+\varepsilon]}\xi(t)\big(z(t_{\rm LS}+\varepsilon)-z(t_{\rm LS})\big)\\ &=0\ +\sup_{0<\varepsilon\le T-t_{\rm LS}}\min\Big(-\xi(t_{\rm LS}),\inf_{t\in(t_{\rm LS},t_{\rm LS}+\varepsilon]}-\xi(t)\Big)\ \le\ -\xi(t_{\rm LS}).\end{aligned}$$

For $t_{\rm LS}<t_{\rm MD}$, we have $-\xi(t_{\rm LS})<\alpha=\mathrm{Diss}_{\mathcal{R}}(z;[0,T])$ and thus (3.9) is not satisfied. In particular, left-continuous energetic solutions are not maximally dissipative. Similar argumentation holds for left-continuous local solutions with more jumps before $t_{\rm MD}$ or with absolutely-continuous parts valued in $(0,1)$ as in Fig. 1(middle row). On the other hand, the solution which ruptures at $t_{\rm MD}$ as in Fig. 1(lower row) is maximally dissipative; the selection $\xi(\cdot)\in\partial\mathcal{R}(0)$ used in Definition 3.1 might be taken on $(t_{\rm MD},T]$ as any continuous extension of ξ uniquely defined on $[0,t_{\rm MD}]$.

4. Semi-implicit approximation scheme of a fractional-step type. As we are confining on the case when $\mathcal{E}(t,\cdot,\cdot)$ is separately convex, both $\partial_u\mathcal{E}(t,\cdot,z)$ and $\partial_z\mathcal{E}(t,u,\cdot)$ are monotone. Sometimes $\partial_u\mathcal{E}(t,\cdot,z)$ has even a certain strong-monotonicity-like property, which can improve convergence of approximate solutions. More specifically, sometimes, it is realistic and advantageous to qualify $\partial_u\mathcal{E}(t,\cdot,\cdot)$

by requiring a so-called *(S$^+$)-property* of the family $\{\partial_u\mathcal{E}(t,\cdot,z):U\to U^*\}_{z\in(\mathcal{Z},\mathscr{T}_Z)}$ with the topology $\mathscr{T}_Z$ of $\mathcal{Z}\subset Z$ to be specified (later, here always as the weak one), namely

$$\left.\begin{array}{l}u_k\rightharpoonup u \text{ in } U,\qquad z_k\stackrel{\mathscr{T}_Z}{\to} z \text{ in } Z,\qquad z_k\in\mathcal{Z},\\ \limsup_{k\to\infty}\big\langle\partial_u\mathcal{E}(t,u_k,z_k)-\partial_u\mathcal{E}(t,u,z_k),u_k{-}u\big\rangle\le 0\end{array}\right\}\quad\Rightarrow\quad u_k\to u \text{ in } U. \tag{4.1}$$

For z fixed, the property of $\partial_u\mathcal{E}(t,\cdot,z):U\to U^*$ being a mapping of the type (S$^+$) has been invented by Browder [3, p.279].

Analogously, we define also the (S$^+$)-property of the family $\{\partial_z\mathcal{E}(t,u,\cdot):Z\to Z^*\}_{u\in(\mathcal{U},\mathscr{T}_U)}$ that can turn the weak convergence on Z into the strong one:

$$\left.\begin{array}{l}z_k\rightharpoonup z \text{ in } Z,\qquad u_k\stackrel{\mathscr{T}_U}{\to} u \text{ in } U,\qquad u_k\in\mathcal{U},\\ \limsup_{k\to\infty}\big\langle\partial_z\mathcal{E}(t,u_k,z_k)-\partial_z\mathcal{E}(t,u_k,z),z_k{-}z\big\rangle\le 0\end{array}\right\}\quad\Rightarrow\quad z_k\to z \text{ in } Z. \tag{4.2}$$

To analyze the problem, we use the time discretisation. This may (and here will) also suggest an *efficient numerical strategy*, cf. e.g. [29,43,44,46,51]. We use an equidistant partition of the time interval $I=[0,T]$ with a time step $\tau>0$, assuming $T/\tau\in\mathbb{N}$, and denote by $\{u_\tau^k\}_{k=0}^{T/\tau}$ an approximation of the desired values $u(k\tau)$, and similarly z_τ^k is to approximate $z(k\tau)$.

An intuitive approach, often really used in engineering calculations, exploits the separate convexity of $\mathcal{E}(t,\cdot,\cdot)$ and makes the corresponding splitting leading, instead of a fully implicit formula $\partial_u\mathcal{E}(k\tau,u_\tau^k,z_\tau^k)\ni 0$ and $\partial\mathcal{R}(z_\tau^k{-}z_\tau^{k-1})+\partial_z\mathcal{E}(k\tau,u_\tau^k,z_\tau^k)\ni 0$, to a *semi-implicit formula*

$$\partial_u\mathcal{E}(k\tau,u_\tau^k,z_\tau^{k-1})\ni 0\qquad\text{and}\qquad\partial\mathcal{R}(z_\tau^k{-}z_\tau^{k-1})+\partial_z\mathcal{E}(k\tau,u_\tau^k,z_\tau^k)\ni 0. \tag{4.3}$$

Note that the two inclusions in (4.3) are decoupled. The mentioned separate convexity makes (4.3) equivalent to solving two alternating recursive incremental problems: given z_τ^{k-1}, we seek

$$u_\tau^k \text{ minimizes } u\mapsto\mathcal{E}(k\tau,u,z_\tau^{k-1})\qquad\text{subject to } u\in U,\ \text{ and} \tag{4.4a}$$

$$z_\tau^k \text{ minimizes } z\mapsto\mathcal{E}(k\tau,u_\tau^k,z)+\mathcal{R}(z{-}z_\tau^{k-1})\qquad\text{subject to } z\in\mathcal{Z} \tag{4.4b}$$

for $k=1,\dots,T/\tau$, starting from the initial condition $z_\tau^0=z_0$. Solutions (u_τ^k,z_τ^k) of both problems in (4.4) standardly do exist due to compactness/coercivity arguments. A definite algorithmic advantage is that both problems in (4.4) are decoupled and a possible difficulty with global minimization, which would arise in a fully implicit discretisation if $\mathcal{E}(t,\cdot,\cdot)$ is nonconvex, is thus eliminated. Such a decoupled scheme can be understood as a popular *fractional-step method*: first solve in u, and after this solve for z, and go to next time level, i.e. the nonlinear operator $(\partial_u\mathcal{E},\partial_z\mathcal{E})$ acting in (1.1) is split as $(\partial_u\mathcal{E},0)+(0,\partial_z\mathcal{E})$ and then the formula (4.3) arises by applying these two operators successively; cf. [41, Remark 8.25] for a general discussion.

We will use the notation for the piecewise-constant interpolants

$$\left.\begin{array}{l}\bar u_\tau(t):=u_\tau^k\quad\&\quad\underline{u}_\tau(t):=u_\tau^{k-1},\\ \bar z_\tau(t):=z_\tau^k\quad\&\quad\underline{z}_\tau(t):=z_\tau^{k-1}\\ \bar{\mathcal{E}}_\tau(t,u,z):=\mathcal{E}(k\tau,u,z),\end{array}\right\}\text{ for } (k{-}1)\tau<t\le k\tau. \tag{4.5}$$

LEMMA 4.1. *The discrete solution obtained by (4.4) satisfies*

$$\partial_u\bar{\mathcal{E}}_\tau(t,\bar u_\tau(t),\underline{z}_\tau(t))\ni 0, \tag{4.6a}$$

$$\bar{\mathcal{E}}_\tau(t,\bar u_\tau(t),\bar z_\tau(t))\le\bar{\mathcal{E}}_\tau(t,\bar u_\tau(t),\tilde z)+\mathcal{R}\big(\tilde z{-}\bar z_\tau(t)\big), \tag{4.6b}$$

$$\mathcal{E}(t_2,\bar u(t_2),\bar z(t_2))+\mathrm{Diss}_{\mathcal{R}}(\bar z_\tau;[t_1,t_2])\le\mathcal{E}(t_1,\bar u_\tau(t_1),\bar z_\tau(t_1))+\int_{t_1}^{t_2}\mathcal{E}_t'(t,\underline{u}_\tau(t),\underline{z}_\tau(t))\,\mathrm{d}t \tag{4.6c}$$

for all $t\in I$ and all $0{\le}t_1{<}t_2{\le}T$ of the form $t_1=k_1\tau$ and $t_2=k_2\tau$ with $k_1,k_2\in\mathbb{N}$.

Proof. Obviously, (4.6a) means just $\partial_u\mathcal{E}(k\tau,u_\tau^k,z_\tau^{k-1})\ni 0$ for all $k\ge 1$, which just represents the 1st-order necessary optimality condition for (4.4a). Testing (4.4b) by a general $\tilde z$, we obtain the semistability

$$\mathcal{E}\big(k\tau,u_\tau^k,z_\tau^k\big)\le\mathcal{E}\big(k\tau,u_\tau^k,\tilde z\big)-\mathcal{R}\big(z_\tau^k{-}z_\tau^{k-1}\big)+\mathcal{R}\big(\tilde z{-}z_\tau^{k-1}\big)\le\mathcal{E}\big(k\tau,u_\tau^k,\tilde z\big)+\mathcal{R}\big(\tilde z{-}z_\tau^k\big) \tag{4.7}$$

for any k, which means exactly (4.6b). Eventually, to obtain the discrete energy balance, we compare the value of (4.4a) for u_τ^k and u_τ^{k-1}, obtaining $\mathcal{E}(k\tau,u_\tau^k,z_\tau^{k-1})\le\mathcal{E}(k\tau,u_\tau^{k-1},z_\tau^{k-1})$. Further, comparing the value of (4.4b) for z_τ^k and z_τ^{k-1}, we obtain $\mathcal{E}(k\tau,u_\tau^k,z_\tau^k)+\mathcal{R}(z_\tau^k{-}z_\tau^{k-1})\le\mathcal{E}(k\tau,u_\tau^k,z_\tau^{k-1})$. Summing these two estimates, the terms $\pm\mathcal{E}(k\tau, u_\tau^k, z_\tau^{k-1})$ mutually cancel, and one gets the discrete upper energy estimate

$$\begin{aligned}(4.8)\qquad \mathcal{E}\big(k\tau, u_\tau^k, z_\tau^k\big) + \mathcal{R}\big(z_\tau^k - z_\tau^{k-1}\big) &\le \mathcal{E}\big(k\tau, u_\tau^{k-1}, z_\tau^{k-1}\big)\\ &= \mathcal{E}\big((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}\big) + \int_{(k-1)\tau}^{k\tau} \mathcal{E}_t'\big((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}\big)\,\mathrm{d}t.\end{aligned}$$

After summing (4.8) for $k = k_1+1, ..., k_2$, one gets (4.6c). □

To pass to the limit in the discrete semistability (4.6b), like in [30] we will rely on constructions of *mutual recovery sequences* but here modified by considering the family of functionals parameterized by u, namely

$$\begin{aligned}(4.9)\qquad &\forall \text{ semistable sequence } (t_k, u_k, z_k) \xrightarrow{I\times U\times Z} (t, u, z) \;\; \forall \tilde z \in \mathcal{Z} \;\; \exists (\tilde z_k)_{k\in\mathbb{N}} \subset \mathcal{Z}:\\ &\limsup_{k\to\infty} \big(\mathcal{E}(t_k, u_k, \tilde z_k) + \mathcal{R}(\tilde z_k - z_k) - \mathcal{E}(t_k, u_k, z_k)\big) \le \mathcal{E}(t, u, \tilde z) + \mathcal{R}(\tilde z - z) - \mathcal{E}(t, u, z),\end{aligned}$$

where we say that $(t_k, u_k, z_k)_{k\in\mathbb{N}}$ is a *semistable sequence* if

$$(4.10)\qquad \sup_{k\in\mathbb{N}} \mathcal{E}(t_k, u_k, z_k) < \infty \quad\text{and}\quad \forall\, k\in\mathbb{N}\;\forall\,\tilde z\in\mathcal{Z}:\;\; \mathcal{E}(t_k, u_k, z_k) \le \mathcal{E}(t_k, u_k, \tilde z) + \mathcal{R}(\tilde z - z_k).$$

Moreover, to cover some constrained problems without generalizing the (S$^+$)-property (4.1) for set-valued mappings, we confine ourselves to the bit simplified ansatz for the Gibbs-type energy $\mathcal{E}$:

$$(4.11a)\quad \mathcal{E}(t, u, z) = \Phi(u, z) + \delta_{\mathcal{U}}(u) + \delta_{\mathcal{Z}}(z) - \langle \mathcal{F}_u(t), u\rangle - \langle \mathcal{F}_z(t), z\rangle$$

for some closed sets $\mathcal{U} \subset U$ and $\mathcal{Z} \subset Z$ with "δ" denoting the indicator function valued in $\{0, \infty\}$, and for some Helmholtz-type energy Φ Gâteaux differentiable, and the loading $\mathcal{F}(t) = (\mathcal{F}_u(t), \mathcal{F}_z(t)) \in U^*\times Z^*$. Furthermore, we assume

(4.11b) $\exists c_\Phi > 0\ \forall (u, z)\in U\times Z:\quad \Phi(u, z) \ge c_\Phi(\|u\|_U + \|z\|_Z)$,

(4.11c) Φ (strong×weak)-continuous on $\mathcal{U}\times\mathcal{Z}$,

(4.11d) $\forall \tilde u \in \mathcal{U}:\qquad (u, z) \mapsto \Phi(u, z) - \Phi(\tilde u, z)$ weakly l.s.c. on $\mathcal{U}\times\mathcal{Z}$,

(4.11e) $\Phi(\cdot, z) : U \to \mathbb{R}\cup\{\infty\}$ strictly convex, $\mathcal{U}$ and $\mathcal{Z}$ convex,

(4.11f) $\forall u \in \mathcal{U}:\qquad \partial_u\Phi(u, \cdot) : \mathcal{Z} \to U^*$ (weak,strong)-continuous,

(4.11g) the family $\{\partial_u\Phi(\cdot, z)\}_{z\in(Z,\text{weak})}$ satisfy the (S$^+$)-property, i.e. (4.1) with $\mathscr{T}_Z =$ weak topology.

Note that (4.11d) is formulated carefully to avoid a requirement for $\Phi(u, \cdot)$ to be weakly continuous, which would otherwise exclude some interesting applications, in particular the gradient damage in Section 6 below.

We will prove convergence of the approximate solutions obtained by means of (4.4) towards local solutions in the sense of Definition 2.1; in fact, we will arrive even to a slightly strengthened property as the set of exceptional points, beside having zero measure, will be proved to be at most countable and, on top of it, it acts only in (2.4a) but not in (2.4b).

Proposition 4.2 (Weak convergence towards local solutions). *Let (1.2) hold with $\mathcal{E}$ satisfying (4.11) with $\mathcal{F} \in W^{1,1}(I; U^*\times Z^*)$, and $\mathcal{R}$ being coercive, i.e. $\inf_{v\ne 0} \mathcal{R}(v)/\|v\|_X > 0$. Furthermore, let (4.9) with the weak topology on $\mathcal{Z}$ hold. Then there exists a subsequence and $(u, z) \in \mathrm{B}(I; U\times Z)$ valued in $\mathcal{U}\times\mathcal{Z}$ such that*

$$(4.12a)\qquad \bar u_\tau(t) \to u(t) \qquad \textit{in } U \textit{ for all } t \in I,$$

$$(4.12b)\qquad \bar z_\tau(t) \rightharpoonup z(t) \qquad \textit{in } Z \textit{ for all } t \in I,$$

and every (u, z) obtained by this way is a local solution; more specifically, $z \in \mathrm{BV}(I; X)$ and (2.4) can be slightly strengthened so that, for some at most countable set $J \subset I$, it holds that

$$(4.13a)\qquad \forall t\in I\backslash J\ \forall \tilde u\in\mathcal{U}:\;\; \big\langle \partial_u\Phi(u(t), z(t)), \tilde u - u(t)\big\rangle \ge \big\langle \mathcal{F}_u(t), \tilde u - u(t)\big\rangle,$$

$$(4.13b)\qquad \forall t\in I\quad \forall \tilde z\in\mathcal{Z}:\;\; \Phi(u(t), z(t)) \le \Phi(u(t), \tilde z) + \big\langle \mathcal{F}_z(t), z(t) - \tilde z\big\rangle + \mathcal{R}(\tilde z - z(t)),$$

$$\begin{aligned}(4.13c)\qquad \forall\, 0\le t_1 < t_2 \le T:\;\; &\big[\Phi - \mathcal{F}(t_2)\big](u(t_2), z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(z; [t_1, t_2])\\ &\le \big[\Phi - \mathcal{F}(t_1)\big](u(t_1), z(t_1)) - \int_{t_1}^{t_2} \big\langle \dot{\mathcal{F}}, (u, z)\big\rangle\,\mathrm{d}t.\end{aligned}$$

Proof. From (4.8), by using the coercivity (4.11b) of Φ and the (discrete) Gronwall inequality, one gets standardly the a-priori estimates:

$$\big\|\bar u_\tau\big\|_{\mathrm{B}(I;U)} \le C, \tag{4.14a}$$

$$\big\|\bar z_\tau\big\|_{\mathrm{B}(I;Z)\cap\mathrm{BV}(I;X)} \le C. \tag{4.14b}$$

By Helly's principle, we choose a subsequence and $z, \underline{z} \in \mathrm{BV}(I;X)$ so that

$$\bar z_\tau(t) \rightharpoonup z(t) \qquad \& \qquad \underline{z}_\tau(t) \rightharpoonup \underline{z}(t) \quad \text{ in } \ Z \ \text{ for all } \ t\in I; \tag{4.15}$$

here we used that Z is reflexive and separable. Now, for a fixed t, we select (for a moment) further subsequence so that $\bar u_\tau(t) \rightharpoonup u(t)$ in U. Let again $t_\tau := \min\{k\tau \ge t;\ k \in \mathbb{N}\}$. Obviously, $t_\tau \to t$ for $\tau \to 0$. By using that $\bar u_\tau(t)$ minimizes $\mathcal{E}(t_\tau, \cdot, \underline{z}_\tau(t))$ and by (4.11d), for all $\tilde u \in \mathcal{U}$, we pass to the limit in

$$0 \le \limsup_{\tau\to 0} \Big(\mathcal{E}(t_\tau, \tilde u, \underline{z}_\tau(t)) - \mathcal{E}(t_\tau, \bar u_\tau(t), \underline{z}_\tau(t))\Big) \le \mathcal{E}(t, \tilde u, \underline{z}(t)) - \mathcal{E}(t, u(t), \underline{z}(t)) \tag{4.16}$$

and we can thus see that $u(t)$ minimizes the strictly convex functional $\mathcal{E}(t, \cdot, \underline{z}(t))$. Thus $u(t)$ is determined uniquely so that, in fact, we did not need to make further selection of a subsequence, and this procedure can be performed for any t. Also, $u : I \to U$ is measurable because $\underline{z}$ and $\mathcal{F}$ are measurable, and $\partial_u\mathcal{E}(t, u(t), \underline{z}(t)) \ni 0$ for all t.

By continuity of both BV-functions $z(\cdot)$ and $\underline{z}(\cdot)$ on $I\backslash J$ for some at most countable set J, we have $z(t) = \underline{z}(t)$ for any $t \in I\backslash J$. This can be seen by realizing that $\bar z_\tau - \underline{z}_\tau \overset{*}{\rightharpoonup} 0$ in $L^\infty(I;Z)$ for $\tau \to 0$, cf. [41, Sect. 8.2], so that $z - \underline{z} = 0$ a.e. on I and in particular at every joint continuity points. Alternatively, one can use the $W^{1,1}(I;X)$-boundedness of the piecewise affine interpolants to see that

$$\begin{aligned}\big\|\bar z_\tau - \underline{z}_\tau\big\|_{L^1(I;X)} &= \int_0^T \big\|\bar z_\tau(t) - \underline{z}_\tau(t)\big\|_X \,\mathrm{d}t = \sum_{k=1}^{T/\tau} \int_{(k-1)\tau}^{k\tau} \big\|z_\tau^k - z_\tau^{k-1}\big\|_X \,\mathrm{d}t \\ &= \tau \sum_{k=1}^{T/\tau} \int_{(k-1)\tau}^{k\tau} \Big\|\frac{z_\tau^k - z_\tau^{k-1}}{\tau}\Big\|_X \,\mathrm{d}t = \tau \sum_{k=1}^{T/\tau} \int_{(k-1)\tau}^{k\tau} \big\|\dot z_\tau(t)\big\| \,\mathrm{d}t = \tau\big\|\dot z_\tau\big\|_{L^1(I;X)} = \tau\big\|\bar z_\tau\big\|_{\mathrm{BV}(I;X)} \le \tau C\end{aligned} \tag{4.17}$$

with C referring to the BV-estimate in (4.14b), cf. also [41, Remark 8.10]. In particular, $\partial_u\mathcal{E}(t, u(t), z(t)) = \partial_u\mathcal{E}(t, u(t), \underline{z}(t)) \ni 0$ for such t, which proves (4.13a).

Let us recall the notation $\mathcal{F} = (\mathcal{F}_u, \mathcal{F}_z)$ with $\mathcal{F}_u \in U^*$ and $\mathcal{F}_z \in Z^*$. We realize that (4.6a) means that $\bar u_\tau(t) \in \mathcal{U}$ satisfies the variational inequality

$$\forall \tilde u \in \mathcal{U}: \qquad \langle \partial_u\Phi(\bar u_\tau(t), \underline{z}_\tau(t)) \,,\, \tilde u - \bar u_\tau(t)\rangle \ge \langle \mathcal{F}_u(t_\tau) \,,\, \tilde u - \bar u_\tau(t)\rangle. \tag{4.18}$$

We can rely on having $u(t) \in \mathcal{U}$ and we can thus use the test $\tilde u = u(t)$. By this way, we obtain

$$\begin{aligned}&\big\langle \partial_u\Phi(\bar u_\tau(t), \underline{z}_\tau(t)) - \partial_u\Phi(u(t), \underline{z}_\tau(t)) \,,\, \bar u_\tau(t) - u(t)\big\rangle \\ &\qquad\qquad \le \big\langle \mathcal{F}_u(t_\tau) - \partial_u\Phi(u(t), \underline{z}_\tau(t)) \,,\, \bar u_\tau(t) - u(t)\big\rangle \to 0\end{aligned} \tag{4.19}$$

thanks to (4.11f) and $\bar u_\tau(t) \rightharpoonup u(t)$. By (4.11g), one then gets $\bar u_\tau(t) \to u(t)$ in U. As it holds for any t, the improved convergence (4.12a) is shown.

One can rewrite (4.6b) in terms of the original energy $\mathcal{E}$ as

$$\mathcal{E}(t_\tau, \bar u_\tau(t), \bar z_\tau(t)) \le \mathcal{E}(t_\tau, \bar u_\tau(t), \tilde z) + \mathcal{R}\big(\tilde z - \bar z_\tau(t)\big) \tag{4.20}$$

and then pass it to the limit by the assumption about the mutually recovery sequence for the semi-stability condition (4.9).

It remains to pass to the limit in the discrete energy inequality (4.6c). One can rewrite (4.6c) for $t_1 < t_2$ arbitrary as

$$\begin{aligned}&\mathcal{E}(t_2, \bar u(t_2), \bar z(t_2)) + \mathrm{Diss}_{\mathcal{R}}(\bar z_\tau; [t_1, t_2]) \\ &\qquad\qquad \le \mathcal{E}(t_1, \bar u_\tau(t_1), \bar z_\tau(t_1)) + \int_{t_1}^{t_2} \mathcal{E}_t'(t, \underline{u}_\tau(t), \underline{z}_\tau(t)) \,\mathrm{d}t + 4C\omega(\tau);\end{aligned} \tag{4.21}$$

here C is from (4.14) and ω is the modulus of continuity of the (uniformly) continuous mapping $\mathcal{F}$ and of the mapping (uniformly) continuous $t \mapsto \int_0^t \|\dot{\mathcal{F}}\|_{U^*\times Z^*} \,\mathrm{d}t : I \to U^*\times Z^*$. Therefore, coming from (4.6c) to (4.21), we could use that $|\mathcal{E}(t_{i,\tau},u,z)-\mathcal{E}(t_i,u,z)|\le\omega(\tau)\|(u,z)\|_{L^\infty(I;U\times Z)}$ with $t_{i,\tau}:=\min\{k\tau\ge t_i;\ k\in\mathbb{N}\}$ for $i=1,2$ and that the difference of the integrals in (4.6c) and (4.21) can be estimated by $\sum_{i=1,2}\int_{t_i}^{t_{i,\tau}}\|\dot{\mathcal{F}}\|_{U^*\times Z^*}\mathrm{d}t\|(u,z)\|_{L^\infty(I;U\times Z)}\le 2\omega(\tau)\|(u,z)\|_{L^\infty(I;U\times Z)}$. Now we can pass to the limit in (4.21). The important fact is that we have proved the strong convergence (4.12a), so that we can pass to the limit in $\mathcal{E}(t_1,\bar u_\tau(t_1),\bar z_\tau(t_1))$ by (4.11c), not merely use lower semicontinuity which suffices for limiting $\mathcal{E}(t_2,\bar u(t_2),\bar z(t_2))$ only. Eventually, the integral in (4.6c) is to be treated by the Lebesgue theorem. □

Proposition 4.3 (Strong convergence towards local solutions). *Let all assumptions of Proposition 4.2 hold with (4.9) and (4.11c) even weakened by taking the strong topology on $\mathcal{Z}$. Let, in addition, $\mathcal{Z}=Z$, $\mathcal{R}=\delta_S^*$ for some compact $S\subset Z^*$, $\partial_z\Phi(\cdot,z):\mathcal{U}\to Z^*$ is continuous, and the family $\{\partial_z\Phi(u,\cdot)\}_{u\in(\mathcal{U},strong)}$ satisfy the (S^+)-property, i.e. (4.2) holds with $\mathscr{T}_U=$ strong topology. Then there exists a subsequence and $(u,z)\in\mathrm{B}(I;U\times Z)$ valued in $\mathcal{U}\times\mathcal{Z}$ such that*

(4.22a) $$\bar u_\tau(t)\to u(t)\qquad \text{in } U \text{ for all } t\in I,$$

(4.22b) $$\bar z_\tau(t)\to z(t)\qquad \text{in } Z \text{ for all } t\in I,$$

and every (u,z) obtained by this way is a local solution again in the sense that $z\in\mathrm{BV}(I;X)$ and (4.13) holds for some at most countable set $J\subset I$.

Proof. Let us realize that, due to $\mathcal{Z}=Z$, the flow rule (1.1b), which generally involves two set-valued mappings and thus can be written as one equality and two inclusions, reads as an equality $\bar\xi_\tau+\partial_z\Phi(\bar u_\tau,\bar z_\tau)=\bar{\mathcal{F}}_{z,\tau}$ combined with the only one inclusion $\bar\xi_\tau\in\partial\mathcal{R}(\dot z_\tau)$. This will allow for reading some estimates for $\bar\xi_\tau$ from the corresponding estimate for the driving force $\bar{\mathcal{F}}_{z,\tau}-\partial_z\Phi(\bar u_\tau,\bar z_\tau)$, which would not be possible otherwise. As $\mathcal{R}=\delta_S^*$ for some compact $S\subset Z^*$, we have granted that $\bar\xi_\tau$ ranges over a compact set in Z^*, namely just S. Using $\bar z_\tau(t)\rightharpoonup z(t)$, we can pass to the limit in

$$\Big\langle\partial_z\Phi(\bar u_\tau(t),\bar z_\tau(t))-\partial_z\Phi(\bar u_\tau(t),z(t))\,,\,\bar z_\tau(t)-z(t)\Big\rangle\le\Big\langle\mathcal{F}_z(t_\tau)-\bar\xi_\tau(t)-\partial_z\Phi(\bar u_\tau(t),z(t))\,,\,\bar z_\tau(t)-z(t)\Big\rangle$$
$$\to\Big\langle\mathcal{F}_z(t)-\xi(t)-\partial_z\Phi(u(t),z(t))\,,\,z(t)-z(t)\Big\rangle=0;$$

note that we used that $\partial_z\Phi(\cdot,z):\mathcal{U}\to Z^*$ is assumed continuous and that $\bar u_\tau(t)\to u(t)$ so that $\partial_z\Phi(\bar u_\tau(t),z(t))$ converges strongly in Z^* and also that $\bar\xi_\tau(t)\to\xi(t)$ strongly in Z^* by the mentioned compactness of S, although these limits are not important here. By the (S^+)-property for the family $\{\partial_z\Phi(u,\cdot)\}_{u\in(\mathcal{U},\mathrm{strong})}$, we then get $\bar z_\tau(t)\to z(t)$. The rest is as in the proof of Proposition 4.2. □

Remark 4.4 (*Application to example (3.1)*). An interesting feature of this algorithm that, when applied to the example from Section 3, it gives "generically" the left-continuous maximally-dissipative local solution. The adjective "generically" means that, for almost all data, e.g. for almost all choices of the loading velocity $v_{\rm D}>0$, the rupture time $t_{\rm MD}$ from (3.2) does not belong to any considered partition of I. The algorithm (4.4) yields always a solution (u_τ,z_τ) with z_τ staying constant (equal 1) until the time $t_{{\rm MD},\tau}=\max\{k\tau;k\in\mathbb{N};\ k\tau<t_{\rm MD}\}$ when it breaks to 0. In the limit for $\tau\to0$, one gets the mentioned maximally-dissipative local solution.

Remark 4.5 (*An enhanced example: two damageable springs*). An interesting illustration is provided by modifying the example (3.1) by considering both springs undergoing damage, i.e.

(4.23)
$$\mathcal{E}(t,u,z_1,z_2)=\begin{cases}\frac12 z_1Ku^2+\frac12 z_2C|u-v_{\rm D}t|^2 & \text{if } 0\le z_1,\ 0\le z_2,,\\ +\infty & \text{otherwise,}\end{cases}\quad\text{and}\quad\mathcal{R}(\dot z_1,\dot z_2)=\alpha_1|\dot z_1|+\alpha_2|\dot z_2|.$$

Again, we start stretching this two-spring structure from the undamaged state $z_1(0)=1=z_2(0)$. We focus on the fully symmetric situation, i.e. for $C=K$ and $\alpha_1=\alpha_2$. There are two left-continuous energetic solutions rupturing again at $t_{\rm ES}$, namely that either z_1 or z_2 jumps to 0 at $t_{\rm ES}$. In particular, neither of these two solutions inherit symmetry of the problem. Interestingly, there is a continuum of left-continuous maximally-dissipative local solutions, namely that z_1 or z_2 (meaning that possibly both) jump to 0 at $t_{\rm MD}$ but either z_1 or z_2 may possibly not jump completely up to 0. One of these solutions is symmetric, namely this one which make complete damage of both springs. Although all these solutions rupture at the same time and dissipate maximal work of external load, the contribution to $\mathrm{Diss}(z;0,t)$ for $t>t_{\rm MD}$ is different, ranging from α to 2α for the symmetric maximal-dissipative local solutions. This symmetric solution is also maximally dissipative in the sense of [48], in contrast to the others. It is also the vanishing-viscosity solution attainable by limiting to zero a viscosity added to the flow-rule for z_1 and z_2 in a symmetric way. On the other hand, non-symmetric viscosity may attain another, non-symmetric maximally dissipative solution. The viscosity added to the spring (analogous to visco-elastic materials in Kelvin-Voigt rheology like in [43]) may yield all those solutions.

Remark 4.6 (*Approximate maximum-dissipation principle*). One can devise the discrete analog of the integrated maximum-dissipation principle (3.9) straightforwardly for the left-continuous interpolants, required however to hold only asymptotically. More specifically, one can expect

$$\underline{\int_0^T} \bar{\xi}_\tau(t)\,\mathrm{d}\bar{z}_\tau(t) \overset{?}{\to} \mathrm{Diss}_{\mathcal{R}}(\bar{z}_\tau;[0,T]) \qquad \text{with} \quad \bar{\xi}_\tau(t) \in -\partial_z\bar{\mathcal{E}}_\tau(t,\bar{u}_\tau(t),\bar{z}_\tau(t)). \tag{4.24}$$

We can explicitly evaluate the left-hand side as

$$\underline{\int_0^T} \bar{\xi}_\tau(t)\,\mathrm{d}\bar{z}_\tau(t) = \sum_{k=1}^{T/\tau}\langle \xi_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}\rangle \qquad \text{with} \quad \xi_\tau^{k-1} \in -\partial_z\mathcal{E}((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}); \tag{4.25}$$

indeed, in view of the definition of the integral $\underline{\int_0^T}$, the supremum in (3.8) is attained already just on the partition $\{k\tau;\ k=0,...,T/\tau\}$. For the fractional-step-type semi-implicit algorithm (4.4), we unfortunately cannot expect equality in (4.24) and we unfortunately even cannot prove the convergence (4.24) in a general case. However, we can at least test it on our 0-dimensional example, where generically the left-hand side of (4.24) equals $\frac{1}{2}K(v_{\mathrm{D}}Ct_{\mathrm{MD},\tau}/(K{+}C))^2 = \alpha$ with $t_{\mathrm{MD},\tau}$ from Remark 4.4, while the right-hand side of (4.24) equals α. In particular, it is always below the right-hand side $\mathrm{Diss}_{\mathcal{R}}(\bar{z}_\tau;[0,T])$ and, as $t_{\mathrm{MD},\tau} \nearrow t_{\mathrm{MD}}$, we indeed have the convergence (4.24) for $\tau \to 0$. Simultaneously, for the fully-implicit global-minimization algorithm leading to the energetic solution, (4.24) would not hold. More specifically, the left-hand side of (4.24) would converge to ξ_{ES} from (3.3) which is less than α, i.e. the right-hand side of (4.24). On the other hand, for problems or loadings leading to rate-independent slides with $\{z_\tau\}_{\tau>0}$ bounded in $W^{1,p}(I;Z)$ and $\{\xi_\tau\}_{\tau>0}$ bounded in $W^{1,p'}(I;Z^*)$ both discretisation schemes satisfy (4.24). Indeed, we know that $\sum_{k=1}^{T/\tau}\langle \xi_\tau^k, z_\tau^k - z_\tau^{k-1}\rangle = \mathrm{Var}_{\mathcal{R}}(\bar{z}_\tau;[0,T])$ because $\xi_\tau^k \in -\partial_z\mathcal{E}(k\tau,u_\tau^k,z_\tau^k)$ and $\xi_\tau^k \in \partial\mathcal{R}(z_\tau^k - z_\tau^{k-1})$, and also we know that

$$\Big|\underline{\int_0^T} \bar{\xi}_\tau(t)\,\mathrm{d}\bar{z}_\tau(t) - \sum_{k=1}^{T/\tau}\langle \xi_\tau^k, z_\tau^k - z_\tau^{k-1}\rangle\Big| \le \tau\big\|\dot{\xi}_\tau\big\|_{L^{p'}(I;Z^*)}\big\|\dot{z}_\tau\big\|_{L^p(I;Z)} \to 0. \tag{4.26}$$

Interestingly, in the problem from Remark 4.5, our semi-implicit algorithm generically (if always $k\tau \ne t_{\mathrm{MD}}$) approximates only the fully symmetric maximally-dissipative local solutions. In general, the fractional-step-type semi-implicit algorithm hardly can be expected to yield a maximally-dissipative local solution (even after a left-continuous modification) and to comply with (4.24). Nevertheless, always the residuum in (4.24) can easily be checked with the goal to justify, at least in specific computational experiments, usage of the (physically not fully justified but) simple and computationally efficient semi-implicit algorithm (4.4); cf. [51] for computational experiments in this direction. The philosophy is to check or to achieve (e.g. by adaptive refinement of the time step τ or of a spacial discretisation, not considered in this paper however) that, in particular situations, even jump regimes in multi-dimensional problems are "locally close" to the 0-dimensional example where the maximum-dissipation principle makes a good job as far as to select force-driven local solutions, as we saw above. Even one can think about an adaptive finer splitting of (u,z) to more than two components u and z so that more than two fractional steps at each time level are performed, again not considered in this paper, however. This is likely also the explanation behind a surprising very good match with physically relevant (but very hardly computable) vanishing-viscosity solutions observed on specific multidimensional experiments in [43] where successively propagating delamination on a 1-dimensional surface well imitates locally the 0-dimensional situation.

Remark 4.7 (*Generalization for dissipation also on u*). One can straightforwardly generalize the definitions and the results for the case when also (1.1a) would involve some 1-homogeneous dissipation energy acting possibly on some components of the abstract variable u. E.g. Definition 2.1 then involves two semi-stability conditions. This generalization was applied in [44] to a mixed-mode delamination, showing mechanically relevant response on particular loading experiment(s) in comparison with (approximate) weak solutions obtained by another, engineering-type model allowing for conventional mathematical analysis in [20].

Remark 4.8 (*Nonconvexity of $\mathcal{E}(t,u,\cdot)$*). Actually, most of the results above hold for $\mathcal{E}(t,u,\cdot)$ nonconvex. Obviously, the minimization strategy (4.4) then does not fully dismantle the difficulty of nonconvex minimization but only reduces it somehow. Of course, like in energetic solutions, one cannot expect the maximum-dissipation principle to be satisfied even approximately in the sense of Remark 4.6 above.

5. Example for weak convergence in z-variable: a delamination problem. Let us illustrate application of Proposition 4.2 to a specific problem from continuum mechanics, namely the rate-independent, *unidirectional* (i.e. no healing) *delamination problem at small strains*, cf. [29, 43–45] and references therein.

For notational simplicity, we consider a single elastic body occupying a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$, $d=2,3$, with $\vec{n}$ the unit outward normal on its boundary $\partial\Omega$, and the *adhesive unilateral contact* on a part $\Gamma_{\rm C}$ of the boundary $\partial\Omega$, so that we consider $\partial\Omega=\Gamma_{\rm C}\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}\cup N$ with disjoint relatively open subsets $\Gamma_{\rm C}$, $\Gamma_{\rm D}$, and $\Gamma_{\rm N}$ of $\partial\Omega$ and with N having a zero $(d{-}1)$-dimensional measure.

The meaning of the above used variables (u,z) is now the $\mathbb{R}^d$-valued displacement u (defined on Ω) and a scalar-valued delamination parameter z (defined on $\Gamma_{\rm C}$). In the most simplest scenario, the functionals $\mathcal{E}$ and $\mathcal{R}$ are considered as

$$
\text{(5.1a)}\quad \mathcal{E}(t,u,z):=\begin{cases}\displaystyle\int_\Omega\frac12\mathbb{C}e(u){:}e(u)\mathrm{d}x-\big\langle\mathcal{F}_u(t),u\big\rangle+\int_{\Gamma_{\rm C}}\frac12 z\mathbb{K}u{\cdot}u\mathrm{d}S & \text{if }\ u{\cdot}\vec{n}\ge0,\ \ 0\le z\le1\ \text{ on }\Gamma_{\rm C},\\ & \text{and if }\ u=0\ \text{ on }\Gamma_{\rm D},\\ +\infty & \text{else,}\end{cases}
$$

$$
\text{with }\big\langle\mathcal{F}_u(t),v\big\rangle:=\int_{\Gamma_{\rm N}}g(t){\cdot}v\,\mathrm{d}S-\int_\Omega\mathbb{C}e\big(u_{\rm D}(t)\big){:}e(v)\,\mathrm{d}x,
$$

$$
\text{(5.1b)}\quad \mathcal{R}(\dot z)=\begin{cases}\displaystyle\int_{\Gamma_{\rm C}}\alpha|\dot z|\,\mathrm{d}S & \text{if }\dot z\le0\text{ a.e. on }\Gamma_{\rm C},\\ +\infty & \text{otherwise,}\end{cases}
$$

where $\mathbb{C}$ is a positive-definite tensor of elastic moduli of the $4^{\rm th}$-order, $\mathbb{K}$ is a positive-definite matrix of elastic moduli of the adhesive, $\alpha>0$ is a so-called fracture toughness, $u_{\rm D}=u_{\rm D}(t)$ given, and g a given surface load (acting on $\Gamma_{\rm N}$). Of course, the very original problem uses the shifted displacement $u{+}u_{\rm D}$ rather than u, and the non-homogeneous Dirichlet boundary condition $u_{\rm D}|_{\Sigma_{\rm D}}=w_{\rm D}$ on $\Gamma_{\rm D}$ with $w_{\rm D}$ a given surface displacement loading (acting on $\Gamma_{\rm D}$); then $u_{\rm D}$ is a suitable prolongation of $w_{\rm D}$ defined on Ω, i.e. $u_{\rm D}|_{\Sigma_{\rm D}}=w_{\rm D}$. We also used the notation of " $\cdot$ " and " $:$ " for a scalar product of vectors and 2nd-order tensors, respectively.

We will use the standard notation $W^{k,p}(\Omega)$ for the Sobolev space of functions having all $k^{\rm th}$-order derivatives in $L^p(\Omega)$. If valued in $\mathbb{R}^n$ with $n\ge2$, we will write $W^{k,p}(\Omega;\mathbb{R}^n)$, and furthermore, if $p=2$, we use the shorthand notation $H^k(\Omega;\mathbb{R}^n)=W^{k,2}(\Omega;\mathbb{R}^n)$.

We qualify the data of this problem by requiring

$$
\text{(5.2a)}\qquad \mathbb{C},\ \mathbb{K}\ \text{symmetric positive definite,}
$$

$$
\text{(5.2b)}\qquad w_{\rm D}\in W^{1,1}(I;H^{1/2}(\Gamma_{\rm D};\mathbb{R}^d)),\quad g\in W^{1,1}(I;L^p(\Gamma_{\rm N};\mathbb{R}^d))\quad\text{with}\quad p\begin{cases}>1 & \text{for }d=2,\\ =2{-}2/d & \text{for }d\ge3\end{cases}
$$

$$
\text{(5.2c)}\qquad z_0\in L^\infty(\Gamma_{\rm C}),\qquad 0\le z_0\le1\quad\text{a.e. on }\Gamma_{\rm C}.
$$

Let us note that the closed convex sets $\mathcal{U}$ and $\mathcal{Z}$ used in the ansatz (4.11a) are now $\mathcal{U}=\{u\in H^1(\Omega;\mathbb{R}^d);\ u{=}0\text{ on }\Gamma_{\rm D},\ \ u{\cdot}\vec{n}\ge0\text{ on }\Gamma_{\rm C}\}$ and $\mathcal{Z}=\{z{\in}L^\infty(\Gamma_{\rm C});\ 0\le z\le1\}$, and that $X=L^1(\bar\Gamma_{\rm C})$.

Proposition 5.1. *Let (5.2) hold and let $(\bar u_\tau,\bar z_\tau)$ be an approximate solution obtained by the semi-implicit formula (4.4). Then there exists a subsequence and $u\in\mathrm{B}(I;H^1(\Omega;\mathbb{R}^d))$ with $u{\cdot}\vec{n}\ge0$ on $I{\times}\Gamma_{\rm C}$ and $z\in\mathrm{B}(I;L^\infty(\Gamma_{\rm C}))\cap\mathrm{BV}(I;L^1(\Gamma_{\rm C}))$ such that*

$$
\text{(5.3a)}\qquad \bar u_\tau(t)\to u(t)\qquad \textit{in } H^1(\Omega;\mathbb{R}^d)\qquad \textit{for all } t\in I,
$$

$$
\text{(5.3b)}\qquad \bar z_\tau(t)\stackrel{*}{\rightharpoonup} z(t)\qquad \textit{in } L^\infty(\Gamma_{\rm C})\qquad \textit{for all } t\in I.
$$

Moreover, any (u,z) obtained by this way is a local solution to the delamination problem in that sense that $u{\cdot}\vec{n}\ge0$ on $I\times\Gamma_{\rm C}$ and, for some $J\subset I$ at most countable, it holds that:

$$
\text{(5.4a)}\quad \forall t{\in}I\backslash J\ \forall v{\in}H^1(\Omega;\mathbb{R}^d),\ v{\cdot}\vec{n}\ge0:
$$

$$
\int_\Omega\mathbb{C}e(u(t)){:}e(v{-}u(t))\,\mathrm{d}x+\int_{\Gamma_{\rm C}}z(t)\mathbb{K}u(t){\cdot}(v{-}u(t))\mathrm{d}S\mathrm{d}t\ge\langle\mathcal{F}_u(t),v{-}u(t)\rangle,
$$

$$
\text{(5.4b)}\quad \forall t{\in}I\ \forall\tilde z{\in}L^\infty(\Gamma_{\rm C}),\ 0{\le}\tilde z{\le}z(t):\qquad \int_{\Gamma_{\rm C}}(z(t){-}\tilde z)\Big(\mathbb{K}u(t){\cdot}u(t)-2\alpha\Big)\,\mathrm{d}S\le0,
$$

$$
\text{(5.4c)}\quad \forall0{\le}t_1{\le}t_2{\le}T:\quad \mathcal{E}\big(t_2,u(t_2),z(t_2)\big)+\mathcal{R}\big(z(t_2){-}z(t_1)\big)\le\mathcal{E}\big(t_1,u(t_1),z(t_1)\big)-\int_{t_1}^{t_2}\langle\dot{\mathcal{F}}_u,u\rangle\,\mathrm{d}t.
$$

Proof. We just use Proposition 4.2. Let us verify the assumptions:

The form (5.1a) of $\mathcal{E}$ obviously complies with the ansatz (4.11a) with $\mathcal{F}_z = 0$. The strict convexity of $\mathcal{E}(t,\cdot,z)$ required in (4.11e) is due to positive-definiteness of $\mathbb{C}$ and $\mathbb{K}$ via Korn's inequality.

Further, (4.11d) requires that, for a fixed $\widetilde{u}$, the functional $\mathcal{E}(t,u,z)-\mathcal{E}(t,\widetilde{u},z) = \frac{1}{2}\int_\Omega \mathbb{C}e(u){:}e(u)\,\mathrm{d}x+ \frac{1}{2}\int_{\Gamma_\mathrm{C}} z\mathbb{K}(u{+}\widetilde{u}){\cdot}(u{-}\widetilde{u})\mathrm{d}S - \langle\mathcal{F}_u(t),u\rangle + C(t)$ is weakly lower semicontinuous on $\mathcal{U}{\times}\mathcal{Z}$, which actually easily follows by compactness of the mappings $u\mapsto \mathbb{K}u{\cdot}u : H^1(\Omega;\mathbb{R}^d)\to L^2(\Gamma_\mathrm{C})$ and $u\mapsto\mathbb{K}u : H^1(\Omega;\mathbb{R}^d)\to L^2(\Gamma_\mathrm{C};\mathbb{R}^d)$. The unimportant constant $C(t)$ is here $\langle\mathcal{F}_u(t),\widetilde{u}\rangle - \frac{1}{2}\int_\Omega \mathbb{C}e(\widetilde{u}){:}e(\widetilde{u})\,\mathrm{d}x$.

The assumption (4.11f) asking for $\partial_u\mathcal{E}(t,u,\cdot):\mathcal{Z}\to U^*$ to be (weak,strong)-continuous means that $\sup_{\|v\|_{H^1}\le 1}\int_{\Gamma_\mathrm{C}}(z_k{-}z)\mathbb{K}u{\cdot}v\mathrm{d}S \to 0$ for $z_k \overset{*}{\rightharpoonup} z$ in $L^\infty(\Gamma_\mathrm{C})$, which follows from the fact that $z_k u \rightharpoonup zu$ in $L^{p/(p-1)}(\Gamma_\mathrm{C};\mathbb{R}^d)$ with p from (5.2b), which is compactly embedded into $H^{-1/2}(\Gamma_\mathrm{C};\mathbb{R}^d)$ hence $\|(z_k{-}z)\mathbb{K}u\|_{H^{-1/2}((\Gamma_\mathrm{C};\mathbb{R}^d)} \to 0$.

The (strong$\times$weak)-continuity of $\mathcal{E}(t,\cdot,\cdot)$ required in (4.11c) is obvious.

Further ingredient is the (S$^+$)-property of the family $\{\partial_u\mathcal{E}(t,\cdot,z)\}_{z\in(\mathcal{Z},\mathrm{weak})}$, as required in (4.11g). This means here that $u_k\rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^d)$ and $z_k\overset{*}{\rightharpoonup} z$ in $L^\infty(\Gamma_\mathrm{C})$, $0\le z_k\le 1$ on Γ_C, together with

$$
\begin{aligned}
0 &\ge \limsup_{k\to\infty}\big\langle \partial_u\mathcal{E}(t,u_k,z_k) - \partial_u\mathcal{E}(t,u,z_k), u_k{-}u\big\rangle \\
&= \limsup_{k\to\infty}\int_\Omega \mathbb{C}e(u_k{-}u){:}e(u_k{-}u)\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} z_k\mathbb{K}(u_k{-}u){\cdot}(u_k{-}u)\mathrm{d}S - \big\langle\mathcal{F}_u(t),u_k{-}u\big\rangle \\
&\ge \limsup_{k\to\infty}\int_\Omega \mathbb{C}e(u_k{-}u){:}e(u_k{-}u)\,\mathrm{d}x
\end{aligned}
$$

implies the strong convergence of displacements $u_k\to u$ in $H^1(\Omega;\mathbb{R}^d)$, which is indeed obvious.

The condition (4.9) is realized by a mutual-recovery sequence

$$
\widetilde{z}_\tau(x) := \begin{cases} \bar{z}_\tau(t,x)\widetilde{z}(x)/z(t,x) & \text{if } z(t,x)>0,\\ 0 & \text{if } z(t,x)=0 \end{cases} \tag{5.5}
$$

for all t; cf. also [25, Lemma 6.1] or [45, Formula (3.71)]. Eventually, $\mathcal{F}_u\in W^{1,\infty}(I;H^1(\Omega;\mathbb{R}^d)^*)$ follows from the assumptions on w_D and f. □

This maximally-dissipative local-solution approach based on the semi-implicit discretisation (4.4) has already been tested computationally on a two-dimensional elastic specimen and, in all investigated configurations, there has been a surprisingly good match with solutions obtained by considering a very small viscosity in the bulk (involving therefore u-variable) in the Kelvin-Voigt rheology, see [43, Figures 10 and 13]. Let us emphasize that such a "vanishing-viscosity" concept in general has received a considerable attention in literature in particular when the viscosity concerns z-variable rather than u-variable, cf. e.g. [5, 7, 10, 18, 26–28, 42, 50], and is considered as a physically relevant approach. As already outlined in Remark 4.5, these two approaches produces not the same effects. The question on what conditions the noteworthy phenomenon observed in [43], i.e. that local solutions arising by some vanishing viscosity are maximally dissipative, has not been studied yet, however. A conjecture is that it will be observed whenever, roughly speaking, the delamination localizes at each (t,x) essentially to a 0-dimensional situation imitating the example from Section 3, as already suggested in Remark 4.6 above.

6. Example for strong convergence in z-variable: a damage problem. Another continuum-mechanical problem at small strains nicely illustrates application of Proposition 4.3, namely the rate-independent, *incomplete, gradient damage with a possible healing* at small strains. The (rather formal) healing is needed to make $\mathcal{R}$ finite, as required in Proposition 4.3.

In contrast to Section 5, z will be distributed over Ω and the functionals $\mathcal{E}$ and $\mathcal{R}$ are smooth, considered as

$$
\mathcal{E}(t,u,z) := \int_\Omega \frac{1}{2}\gamma(z)\mathbb{C}e(u){:}e(u) + \frac{\kappa}{r}|\nabla z|^r\,\mathrm{d}x - \big\langle\mathcal{F}_u(t),u\big\rangle, \tag{6.1a}
$$

$$
\mathcal{R}(\dot{z}) = \int_\Omega \alpha\big(\dot{z}\big)^- + \beta\big(\dot{z}\big)^+\,\mathrm{d}x, \tag{6.1b}
$$

where $\big(\dot{z}\big)^\pm = \max(0,\pm\dot{z})$ and $\mathcal{F}_u(t)$ is again from (5.1a). Again, $\alpha>0$ is a phenomenological energy needed (and thus dissipated) to disintegrate the material (similarly as it was used for debonding in Section 5 but, if $d=3$, the physical dimension is now J/m^3 instead of J/m^2 there). Now, we also consider $\beta>0$ as a phenomenological energy to integrate the material back (=healing). In fact, due to an essentially missing driving force for healing (up to the ∇z-term in the stored energy), the model will mostly work as if the damage were unidirectional (i.e. $\beta=\infty$), as usually considered in engineering models (in contrast e.g. to geophysical models where healing is considered as a vital part of the model although it is naturally rather rate dependent). To be more precise, the healing can occur if the driving force $\kappa\,\mathrm{div}(|\nabla z|^{r-2}\nabla z)-\frac12\gamma'(z)\mathbb{C}e(u){:}e(u)$ exceeds β. For $\kappa>0$ small and β very big, this may practically be assumed only very locally at regions where the material is considerably damaged but surrounded by well undamaged material. If the damage profile does not have too big (depending on β) gradient ∇z in the $W^{1,\infty}(\Omega;\mathbb{R}^d)$-norm, no healing will occur. So this model will well imitate the situation when healing is truly forbidden by setting $\beta=\infty$ but for which our convergence arguments break.

We qualify the data of this problem by requiring, in addition to (5.2b), that

$$\text{(6.2a)}\qquad \mathbb{C}\ \text{symmetric positive definite},\qquad \kappa>0,\qquad r>d,$$

$$\text{(6.2b)}\qquad \gamma\in C^1(\mathbb{R})\ \ \text{positive, convex, and constant on } (-\infty,0],$$

$$\text{(6.2c)}\qquad z_0\!\in W^{1,r}(\Omega),\quad 0\le z_0\le 1\quad \text{a.e. on } \Omega.$$

The coefficient $\kappa>0$ determines a certain length-scale, as usual in gradient models for internal parameters. The assumption (6.2b) together with (6.2c) guarantees that z will be valued in the interval $[0,1]$. Here, an important feature of this model is that, due to the mentioned absence of healing driving force related to the elastic bulk energy and due to the character of the only healing contribution related to the gradient term $\kappa\,\mathrm{div}(|\nabla z|^{r-2}\nabla z)$ complying with the maximum principle, the damage z will never exceed the value ess sup $z_0\le 1$. Very heuristically, one can see it by classical contradiction argument like this one used for the parabolic equation $\dot z-\kappa\,\mathrm{div}(|\nabla z|^{r-2}\nabla z)=-\frac12\gamma'(z)\mathbb{C}e(u){:}e(u)$, assuming that $z(t,\cdot)$ has a maximum at x such that $z(t,x)\ge$ ess sup z_0 and using that both $\kappa\,\mathrm{div}(|\nabla z(t,x)|^{r-2}\nabla z(t,x))\le 0$ and $-\frac12\gamma'(z)\mathbb{C}e(u){:}e(u)\le 0$ so that $\dot z$ cannot be positive and such situation cannot arise during the evolution. For similar argumentation used for the time discrete solution and then the limit cf. also [18, Prop. 4.2]. In particular, $\gamma(\cdot)$ need not be qualified for $z>1$ at all.

Also, we consider only an uncomplete damage; this is ensured by (6.2b) because the minimal value of γ, i.e. $\gamma(0)$, is assumed still positive there and thus $\mathcal{E}(t,\cdot,z)$ is always strictly convex and even uniformly strongly convex, as exploited for (4.1).

As for the sets $\mathcal{U}$ and $\mathcal{Z}$ used in the ansatz (4.11a) concerns, we can now simply consider $U=\mathcal{U}=\{u\!\in H^1(\Omega;\mathbb{R}^d);\ u{=}0 \text{ on } \Gamma_{\rm D}\}$ and $Z=\mathcal{Z}=W^{1,r}(\Omega)$, while $X=L^1(\Omega)$.

Proposition 6.1. *Let (5.2b) and (6.2) hold and let $(\bar u_\tau,\bar z_\tau)$ be an approximate solution obtained by the semi-implicit formula (4.4). Then there exists a subsequence and $u\in \mathrm{B}(I;H^1(\Omega;\mathbb{R}^d))$ and $z\in \mathrm{B}(I;W^{1,r}(\Omega))\cap\mathrm{BV}(I;L^1(\Omega))$ such that*

$$\text{(6.3a)}\qquad \bar u_\tau(t)\to u(t)\qquad \textit{in } H^1(\Omega;\mathbb{R}^d)\qquad \textit{for all } t\in I,$$

$$\text{(6.3b)}\qquad \bar z_\tau(t)\to z(t)\qquad \textit{in } W^{1,r}(\Omega)\qquad \textit{for all } t\in I.$$

Moreover, any (u,z) obtained by this way is a local solution to the damage problem in that sense that, for some $J\subset I$ at most countable, it holds that:

$$\text{(6.4a)}\qquad \forall t\!\in I\backslash J\ \forall v\!\in H^1(\Omega;\mathbb{R}^d):\quad \int_\Omega\gamma(z)\mathbb{C}e(u(t)){:}e(v)\,\mathrm{d}x=\langle\mathcal{F}_u(t),v\rangle,$$

$$\text{(6.4b)}\qquad \forall t\!\in I\ \forall\tilde z\!\in W^{1,r}(\Omega):\quad \int_\Omega\alpha\big(\tilde z{-}z(t)\big)^-+\beta\big(\tilde z{-}z(t)\big)^++\frac12\gamma(\tilde z)\mathbb{C}e(u(t)){:}e(u(t))$$
$$+\frac{\kappa}{r}\big|\nabla\tilde z\big|^r\,\mathrm{d}x\ge\int_\Omega\frac12\gamma(z(t)\mathbb{C}e(u(t)){:}e(u(t))+\frac{\kappa}{r}\big|\nabla z(t)\big|^r\,\mathrm{d}x,$$

$$\text{(6.4c)}\qquad \forall 0\!\le t_1\!\le t_2\!\le T:\quad \mathcal{E}\big(t_2,u(t_2),z(t_2)\big)+\mathcal{R}\big(z(t_2){-}z(t_1)\big)\le\mathcal{E}\big(t_1,u(t_1),z(t_1)\big)-\int_{t_1}^{t_2}\langle\dot{\mathcal{F}}_u,u\rangle\,\mathrm{d}t.$$

Proof. We just use Proposition 4.3. To this goal, we verify the assumptions (4.9) and (4.11) together with that $\mathcal{R}=\delta_S^*$ for some compact $S\subset Z^*$, $\partial_z\mathcal{E}(t,\cdot,z):U\to Z^*$ is continuous, and the family $\{\partial_z\mathcal{E}(t,u,\cdot)\}_{u\in(U,\text{strong})}$ satisfies the (S^+)-property.

The form (6.1a) of $\mathcal{E}$ obviously complies with the ansatz (4.11a) again with $\mathcal{F}_z=0$. The strict convexity of $\mathcal{E}(t,\cdot,z)$ required in (4.11e) is due to positive-definiteness of $\mathbb{C}$ via Korn's inequality; here it is important that only uncomplete damage is considered.

Further, (4.11d) requires that, for a fixed $\tilde u$, the functional

$$\mathcal{E}(t,u,z)-\mathcal{E}(t,\tilde u,z)=\int_\Omega\gamma(z)(\mathbb{C}e(u){:}e(u){-}\mathbb{C}e(\tilde u){:}e(\tilde u))\,\mathrm{d}x-\langle\mathcal{F}_u(t),u\rangle+C(t)$$

with $C(t) = \langle \mathcal{F}_u(t), \tilde{u}\rangle$ is weakly lower semicontinuous on $U\times Z$, which actually follows by compactness of the embedding $W^{1,r}(\Omega) \Subset L^\infty(\Omega)$. Also the strong continuity of $\mathcal{E}(t,\cdot,\cdot)$ on its domain, as required in Proposition 4.3, is obvious.

The assumption (4.11f) asking for $\partial_u\mathcal{E}(t,u,\cdot) : Z \to U^*$ (weak,strong)-continuous means that $\sup_{\|v\|_{H^1}\le 1}\int_\Omega(\gamma(z_k)-\gamma(z))\mathbb{C}e(u){:}e(v)\,\mathrm{d}x \to 0$ for $z_k \rightharpoonup z$ in $W^{1,r}(\Omega)$, which can be seen when realizing that $(\gamma(z_k)-\gamma(z))\mathbb{C}e(u) \to 0$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ again because of the compact embedding $W^{1,r}(\Omega) \Subset L^\infty(\Omega)$. Note that we used the cancellation of the ∇z-terms, which was the motivation why (4.11d) have been designed in such a way.

Further ingredient is the (S$^+$)-property of the family $\{\partial_u\mathcal{E}(t,\cdot,z)\}_{z\in(Z,\text{weak})}$ as required in (4.11g). This means here that $u_k \rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^d)$ and $z_k \rightharpoonup z$ in $W^{1,r}(\Omega)$ together with

$$0 \ge \limsup_{k\to\infty}\big\langle \partial_u\mathcal{E}(t,u_k,z_k) - \partial_u\mathcal{E}(t,u,z_k), u_k{-}u\big\rangle = \limsup_{k\to\infty}\int_\Omega\gamma(z_k)\mathbb{C}e(u_k{-}u){:}e(u_k{-}u)\,\mathrm{d}x - \big\langle\mathcal{F}_u(t), u_k{-}u\big\rangle$$

which further implies $0 \ge \limsup_{k\to\infty}\int_\Omega\gamma(0)\mathbb{C}e(u_k{-}u){:}e(u_k{-}u)\,\mathrm{d}x$, yields the strong convergence of displacements $u_k \to u$ in $H^1(\Omega;\mathbb{R}^d)$, which is indeed obvious; here the assumptions (6.2a,b) together with the Korn inequality and the concept of uncomplete damage have been used.

We also need to comply with the assumption that $\Psi = \delta^*_S$ for some compact set $S \subset Z^*$, which here follows from that $S = \{z^* \in L^\infty(\Omega);\ -\alpha \le z^* \le \beta \text{ a.e.}\}$ is compact in $W^{1,r}(\Omega)^*$; here the concept of healing (which is, however, rather formal from the modelling viewpoint here) has vitally been used for this analytical reason.

The required continuity of $\partial_z\mathcal{E}(t,\cdot,z) : U \to Z^*$ means that $\|\partial_z\mathcal{E}(t,u_k,z) - \partial_z\mathcal{E}(t,u,z)\|_{W^{1,r}(\Omega)^*} = \sup_{\|v\|_{W^{1,r}(\Omega)}\le 1}\int_\Omega\gamma'(z)v\mathbb{C}e(u_k{-}u){:}e(u_k{+}u)\,\mathrm{d}x \le N\max_{[0,1]}\gamma'(\cdot)\,|\mathbb{C}|\,\|e(u_k{-}u){:}e(u_k{+}u)\|_{L^1(\Omega)} \to 0$ with N denoting the norm of the embedding $W^{1,r}(\Omega) \subset L^\infty(\Omega)$, which is obvious if $u_k \to u$ in $H^1(\Omega;\mathbb{R}^d)$.

Eventually, the family $\{\partial_z\mathcal{E}(t,u,\cdot)\}_{u\in(U,\text{strong})}$ satisfy the (S$^+$)-property. Indeed, this means that $u_k \to u$ in $H^1(\Omega;\mathbb{R}^d)$ and $z_k \rightharpoonup z$ in $W^{1,r}(\Omega)$ together with

$$\begin{aligned}(6.5)\quad 0 &\ge \limsup_{k\to\infty}\big\langle\partial_z\mathcal{E}(t,u_k,z_k) - \partial_z\mathcal{E}(t,u_k,z), z_k{-}z\big\rangle\\
&= \limsup_{k\to\infty}\int_\Omega\big(\gamma(z_k){-}\gamma(z)\big)\mathbb{C}e(u_k){:}e(u_k) + \kappa\big(|\nabla z_k|^{r-2}\nabla z_k - |\nabla z|^{r-2}\nabla z\big){\cdot}\nabla(z_k{-}z)\,\mathrm{d}x\\
&= \lim_{k\to\infty}\int_\Omega\big(\gamma(z_k){-}\gamma(z)\big)\mathbb{C}e(u_k){:}e(u_k)\,\mathrm{d}x + \kappa\limsup_{k\to\infty}\int_\Omega\big(|\nabla z_k|^{r-2}\nabla z_k - |\nabla z|^{r-2}\nabla z\big){\cdot}\nabla(z_k{-}z)\,\mathrm{d}x\end{aligned}$$

should yield $z_k \to z$ in $W^{1,r}(\Omega)$. Due to the estimate $|\int_\Omega(\gamma(z_k){-}\gamma(z))\mathbb{C}e(u_k){:}e(u_k)\,\mathrm{d}x| \le |\mathbb{C}|\,m_\gamma(\|z_k{-}z\|_{L^\infty(\Omega)})\|e(u_k)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}$ with m_γ the modulus of continuity of the uniformly continuous function γ, and due to the compact embedding $W^{1,r}(\Omega) \Subset L^\infty(\Omega)$, the limit in (6.5) is zero, and we have

$$\begin{aligned}(6.6)\qquad 0 &\ge \kappa\limsup_{k\to\infty}\int_\Omega\big(|\nabla z_k|^{r-2}\nabla z_k - |\nabla z|^{r-2}\nabla z\big){\cdot}\nabla(z_k{-}z)\,\mathrm{d}x\\
&= \kappa\limsup_{k\to\infty}\bigg(\|\nabla z_k\|^r_{L^r(\Omega;\mathbb{R}^d)} - \int_\Omega\Big(|\nabla z_k|^{r-2}\nabla z_k{\cdot}\nabla z + |\nabla z|^{r-2}\nabla z{\cdot}\nabla z_k\Big)\,\mathrm{d}x\bigg) + \|\nabla z\|^r_{L^r(\Omega;\mathbb{R}^d)}\\
&\ge \kappa\limsup_{k\to\infty}\Big(\big\|\nabla z_k\big\|^r_{L^r(\Omega;\mathbb{R}^d)} - \big\||\nabla z_k|^{r-2}\nabla z_k\big\|^{r'}_{L^{r'}(\Omega;\mathbb{R}^d)}\big\|\nabla z\big\|_{L^r(\Omega;\mathbb{R}^d)}\\
&\qquad - \big\||\nabla z|^{r-2}\nabla z\big\|^{r'}_{L^{r'}(\Omega;\mathbb{R}^d)}\big\|\nabla z_k\big\|_{L^r(\Omega;\mathbb{R}^d)}\Big) + \big\|\nabla z\big\|^r_{L^r(\Omega;\mathbb{R}^d)}\\
&= \kappa\limsup_{k\to\infty}\Big(\big\|\nabla z_k\big\|^r_{L^r(\Omega;\mathbb{R}^d)} - \big\|\nabla z_k\big\|^{r-1}_{L^r(\Omega;\mathbb{R}^d)}\big\|\nabla z\big\|_{L^r(\Omega;\mathbb{R}^d)}\\
&\qquad - \big\|\nabla z\big\|^{r-1}_{L^r(\Omega;\mathbb{R}^d)}\big\|\nabla z_k\big\|_{L^r(\Omega;\mathbb{R}^d)}\Big) + \big\|\nabla z\big\|^r_{L^r(\Omega;\mathbb{R}^d)}\\
&= \kappa\limsup_{k\to\infty}\big(\|\nabla z_k\|^{r-1}_{L^r(\Omega;\mathbb{R}^d)} - \|\nabla z\|^{r-1}_{L^r(\Omega;\mathbb{R}^d)}\big)\big(\|\nabla z_k\|_{L^r(\Omega;\mathbb{R}^d)} - \|\nabla z\|_{L^r(\Omega;\mathbb{R}^d)}\big).\end{aligned}$$

The estimate on the 3rd line in (6.6) is by Hölder inequality. This yields the convergence $\|\nabla z_k\|_{L^r(\Omega;\mathbb{R}^d)} \to \|\nabla z\|_{L^r(\Omega;\mathbb{R}^d)}$. By the already obtained weak convergence $\nabla z_k \to \nabla z$ in $L^r(\Omega;\mathbb{R}^d)$ and the well-known attribute of $L^r(\Omega;\mathbb{R}^d)$ that its norm used here makes it a uniformly convex Banach space, we obtain the strong convergence of $\nabla z_k \to \nabla z$, hence also the strong convergence $z_k \to z$ in $W^{1,r}(\Omega)$. □

Acknowledgments. The author is very thankful to two anonymous referees for a lot of very thorough observations and very useful comments which led to very essential improvements of the original version. This research was performed within the grant 201/10/0357 (GA ČR), together with the institutional support RVO: 61388998 (ČR). The author is deeply thankful to Profs. Vladislav Mantič, Alexander Mielke, and Rodica Toader for inspiring conceptual discussions.

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