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Abstract

We consider stationary heat transfer near contact line of an evaporating liquid wedge surrounded by the atmosphere of its pure vapor. In a simplified setting, the problem reduces to the Laplace equation in a half circle, subject to a non-homogeneous and singular boundary condition.

By the means of classical tools (conformal mapping, Green's function), we reformulate the problem as an integral equation for the unknown Neumann boundary condition in the setting of appropriate fractional Sobolev and weighted space. The unique solvability is then obtained by means of the Fredholm theorem.

1 Physical background of the problem

Evaporation of liquid in a contact with solid substrate is a complex phenomenon with crucial importance in industrial applications, e.g. boiling heat exchangers or heat pipes. Special case of such problem is a stationary evaporation of liquid into the atmosphere of its pure vapor. The phase transformation rate is in this case controlled by the heat amount supplied from the liquid side of the free interface and spent mainly to compensate the latent heat of vaporisation. Under partial wetting conditions, a wedge shaped liquid region (frequently called microregion) bordered from one side by liquid-vapor-solid contact line and from the other side by bulk liquid region is formed in the vicinity of the solid wall. The fluid is considered out of equilibrium due to heating of the solid substrate.

Such situation has been extensively studied by a number of authors, see e.g. [13, 11, 3, 6]. Majority of research publications rely on isothermal heater consideration, i.e. imposing constant overheating (with respect to saturation temperature given by ambient pressure) of the solid heater. Such assumption is justified for vanishing liquid-solid thermal conductivity ratio $\beta = k_L/k_S$ (e.g. for water on metallic heater $\beta \sim 10^{-3}$) for which the perturbation of temperature field in solid substrate is strongly localized near the contact line [8]. To our knowledge, there are only few research publications considering solid substrate heat conduction problem in the contact line vicinity, see e.g. [9, 12, 15, 2]. Consideration of such physical complexity of the thermal problem poses significant complication of the model with weak influence on the solution of the microregion problem (slope of the free interface far from contact line and total evaporated mass). Attention of researchers in this domain was thus focused rather on other phenomena such as slip length, interface thermal resistance or Kelvin effect related principally to liquid domain and its interfaces with surrounding phases.

In this paper, we focus on situation with high thermal conductivity liquids ($\beta \gtrsim 1$) for which solid heat conduction problem might be of considerable importance as the transmission of thermal energy between solid substrate and liquid vapor-interface is not significantly obstacled by the liquid thermal resistance. Free interface temperature is thus practically imprinted on the solid-liquid interface and heat conduction problems in solid and liquid domains are strongly coupled, e.g. the size of microregion itself might depend on solid substrate part of the problem. Practical example of industrial application where such question is of crucial importance can be find in advanced nuclear power reactors. A liquid metal, usually sodium, is used as heat exchange fluid. A common interest is avoiding boiling of this cooling medium, i.e. nucleation of vapor bubbles on the heat transfer solid surfaces.

For the purpose of this paper we consider a simple fixed 2D geometry similarly as in [9], see Figure 1. In particular, the contact line (here passing through the origin) does not move.

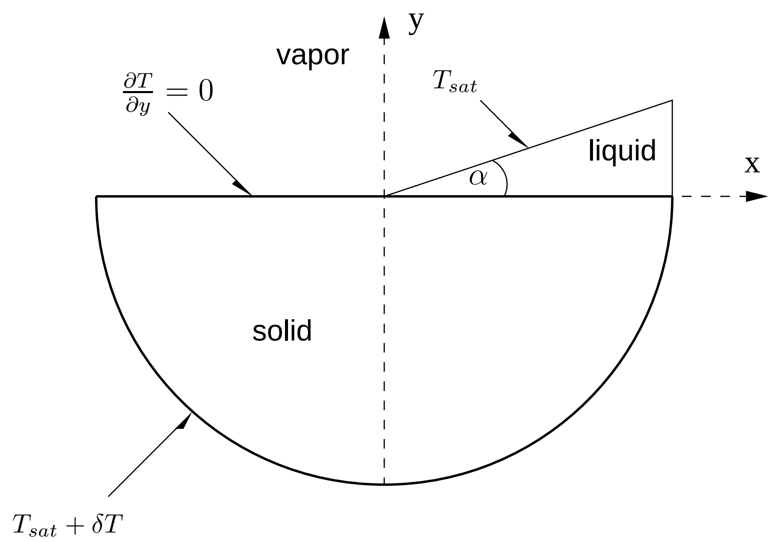

Figure 1: Geometry of the problem

As explained above, our focus here is the problem of the temperature distribution

$T^S = T^S(x, y)$ in the solid domain, described by the following system of equations

$$\Delta T^S = 0, \qquad x^2 + y^2 < R^2,\ y < 0 \tag{1}$$

$$T^S = T_{\text{sat}} + \delta T, \qquad x^2 + y^2 = R^2,\ y < 0 \tag{2}$$

$$\frac{\partial T^S}{\partial y} = 0, \qquad -R < x < 0,\ y = 0 \tag{3}$$

$$k_S \frac{\partial T^S}{\partial y} = k_L \frac{T_{\text{sat}} - T^S}{\alpha x}, \qquad 0 < x < R,\ y = 0 \tag{4}$$

Here T_{sat} is the saturation temperature of the liquid-gas interface, and $\delta T > 0$ is the external heating of the lower circular boundary of the solid. The heat transfer through the liquid-solid interface is captured by the boundary condition (4), which is a result of the following simplifying assumptions:

1. The temperature profile in the y-direction in the (thin) liquid layer is linear, meaning that $T_{\text{sat}} - T^S = -hq_L/k_L$, where h is the thickness of the fluid layer, $q_L = k_S \frac{\partial T^S}{\partial y}$ is the heat flux at the solid-liquid interface (positive for evaporation), and k_L, k_S stand for the thermal conductivity of the liquid and the solid, respectively.

2. The liquid layer has a fixed geometry of straight wedge of a fixed (small) angle α, hence $h = \alpha x$. We note that in a regular perturbation (w.r. to small δT), the linear wedge is a first order approximation of the isothermal outer contact line problem, cf. Morris [10].

There has been several studies, devoted to this or a similar problem from the applied mathematics point of view, see e.g. Anderson and Davis [2], Morris [10]. However, none of these papers seem to address the mathematical question of the existence and uniqueness of solution. In view of the singularity of the boundary condition (4) near the origin, this issue is certainly not trivial; in fact, it poses an interesting mathematical problem.

In the present paper, we establish the mathematical consistency of the above model. We show that the system (1)–(4) can be reduced to finding the unknown heat flux $\frac{\partial T^S}{\partial y}$ on the solid-liquid interface in a class of negative exponent Sobolev functions with singular weight. In particular, the condition (4) is reformulated as an integral equation, which is shown to be uniquely solvable by a Fredholm-type argument.

2 Formulation of the main theorem

After suitable rescaling and non-dimensionalization, the equation to be solved can be written as

$$\Delta T = 0 \qquad \text{in } M \tag{5}$$

$$T = \theta \qquad \text{on } \Gamma_1 \tag{6}$$

$$\frac{\partial T}{\partial Y} = 0 \qquad \text{on } \Gamma_2 \tag{7}$$

$$K\frac{\partial T}{\partial X} = -\frac{T+\sigma}{X} \qquad \text{on } \Gamma_3 \tag{8}$$

where $T = T(X, Y)$ is the unknown temperature distribution in $M = \{X^2 + Y^2 < 1,\ Y < 0\}$. The boundary of M is split into three parts $\Gamma_1 = \{X^2 + Y^2 = 1,\ Y < 0\}$, $\Gamma_2 = \{-1 < X < 0,\ Y = 0\}$ and $\Gamma_3 = \{0 < X < 1,\ Y = 0\}$; $\theta = \theta(X, Y)$, $\sigma = \sigma(X)$ are given functions on the respective parts of the boundary. We note that $K = \alpha/\beta$ is typically small in our setting.

The main result of our paper is the following. We remark that γ_{iN} or γ_{iD} denotes the Neumann or the Dirichlet boundary operator for $\Gamma_i \subset \partial M$. We refer the reader to Appendix for detailed treatment of these issues and definitions of function spaces.

Theorem 2.1. *For any $\theta \in H^{1/2}(\Gamma_1)$, $\sigma \in H^{1/2}(\Gamma_3)$, there exists unique $T \in H^1(M)$ such that $\gamma_{3N}(T) \in H^{-1/2}(\Gamma_3) \cap L^2_{xdx}(\Gamma_3)$, $\gamma_{3D}(T) + \sigma \in L^2_{dx/x}$, satisfying (5)–(8).*

The uniqueness part is equivalent to saying that for the homogeneous problem (i.e., with $\theta = \sigma = 0$), there is only the trivial solution. And this is straightforward to prove: multiplying (5) by T and using Green's formula together with the boundary conditions, one deduces

$$0 = \int_M |\nabla T|^2 dX dY + \int_{\Gamma_3} \frac{T^2}{KX} dX \tag{9}$$

Thus T is constant in M and zero on Γ_1 and Γ_3, hence identically zero function. The computation is rigorous in the functional setup of Theorem 2.1, cf. Corollary 5.1 in the Appendix.

The content of the paper can be outlined as follows: without loss of generality, one can assume that $\theta = 0$ in (6). Indeed, it is possible to write $T = T_0 + \Theta$, where Θ is a suitable harmonic extension of θ into M, and T_0 solves the problem with zero on Γ_1, with appropriately modified boundary condition on Γ_3.

In Section 3, we solve an auxiliary problem (5)–(7) with $\theta = 0$ and together with

$$\frac{\partial T}{\partial Y} = \tau \qquad \text{on } \Gamma_3 \tag{10}$$

for a given Neumann boundary condition $\tau \in H^{-1/2}(\Gamma_3)$. Transforming conformally to the upper half-plane, we express the solution explicitly by means of a convolution of τ with a suitable logarithmic kernel.

In Section 4, we are thus able to rewrite (8) as an integral equation for the unknown value of $\tau = \frac{\partial T}{\partial Y}$. We identify the appropriate functional setup, in which the problem is reduced to a Fredholm-type operator equation. The existence of a unique solution τ is now a direct consequence of Fredholm's theorem.

The problem and its solution combine classical PDE analysis with modern tools from the theory of Sobolev spaces; most of this is well-known and can be found in various books. However, for the sake of completeness and readers convenience, we present in Appendix detailed treatment of these issues.

3 Problem with a given heat flux

In this Section, we solve an auxiliary problem

$$\Delta T = 0 \qquad \text{in } M \tag{11}$$

$$T = 0 \qquad \text{on } \Gamma_1 \tag{12}$$

$$\frac{\partial T}{\partial Y} = 0 \qquad \text{on } \Gamma_2 \tag{13}$$

$$\frac{\partial T}{\partial Y} = \tau \qquad \text{on } \Gamma_3 \tag{14}$$

for a given heat flux $\tau = \tau(X)$. Using the conformal mapping and the explicit form of the Green's function for the corresponding problem in the half plane, we will be able to write T explicitly in terms of a convolution with a suitable logarithmic kernel.

We start by observing that the mapping

$$F(Z) = \frac{1 - Z}{1 + Z} \tag{15}$$

or, writing $Z = X + iY$, $F = F_1 + iF_2$,

$$F_1(X,Y) = -1 + \frac{2(1+X)}{(1+X)^2 + Y^2} \qquad F_2(X,Y)) \;\; = \frac{-2Y}{(1+X)^2 + Y^2} \tag{16}$$

maps the lower circle to the first quadrant $\{x > 0,\ y > 0\}$. More specifically, boundary (12) goes to the positive imaginary axis $\{x = 0,\ y > 0\}$, boundary (13) goes to $\{x > 1,\ y = 0\}$ and boundary (14) goes to $\{0 < x < 1,\ y = 0\}$.

Hence, we want to solve

$$\Delta u = 0, \qquad x > 0,\ y > 0 \tag{17}$$

subject to boundary conditions

$$u = 0, \qquad x = 0,\ y > 0 \tag{18}$$

$$\frac{\partial u}{\partial y} = 0, \qquad x > 1,\ y = 0 \tag{19}$$

$$\frac{\partial u}{\partial y} = \psi, \qquad 0 < x < 1,\ y = 0, \tag{20}$$

for some given function $\psi = \psi(x)$. Note however that we have to be careful while coming from $\tau(X)$ to $\psi(x)$, since F is not isometry of the corresponding boundaries (20), (14). We will come to this problem later.

Recall that the function

$$V(x, y) = \frac{1}{2\pi} \ln\left(x^2 + y^2\right) \tag{21}$$

is the fundamental solution to the Neumann problem in the lower half-plane. More precisely, it solves

$$\Delta V = 0, \qquad -\infty < x < \infty,\ y > 0 \tag{22}$$

$$\frac{\partial V}{\partial y} = \delta_0(x), \qquad -\infty < x < \infty,\ y = 0- \tag{23}$$

Hence, we can solve the problem with arbitrary boundary condition $\psi = \psi(x)$

$$\Delta u = 0, \qquad -\infty < x < \infty,\ y > 0 \tag{24}$$

$$\frac{\partial u}{\partial y} = \psi, \qquad -\infty < x < \infty,\ y = 0 \tag{25}$$

via convolution

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln\left((x-\xi)^2 + y^2\right)\psi(\xi)d\xi \tag{26}$$

Coming back to problem (17) – (20), we extend the boundary conditions (19) – (20) as an odd function for $x < 0$; by symmetry this means that (18) is automatically satisfied. Thus we can eventually express the solution to (17) – (20) as

$$u(x, y) = \frac{1}{2\pi} \int_{0}^{1} \ln\left(\frac{(x-\xi)^2 + y^2}{(x+\xi)^2 + y^2)}\right)\psi(\xi)d\xi \tag{27}$$

We refer to Appendix, Lemma 5.7, for rigorous treatment in the appropriate function spaces. Now we come back to the problem (11) – (14). We set

$$T = u \circ F \tag{28}$$

Thanks to the properties of conformal mappings, T is a harmonic function with finite L^2-norm of the gradient, if and only u is such a function on the corresponding domain.

Concerning the boundary conditions, (12) is equivalent to (18). Furthermore,

$$\frac{\partial T}{\partial Y} = \left(\frac{\partial u}{\partial x} \circ F\right)\frac{\partial F_1}{\partial Y} + \left(\frac{\partial u}{\partial y} \circ F\right)\frac{\partial F_2}{\partial Y} \tag{29}$$

If $Y = 0$, then $\frac{\partial F_1}{\partial Y} = 0$, while $\frac{\partial F_2}{\partial Y} = -2/(1+X)^2$. Hence (13) is equivalent to (19) and the relation between (14) and (20) reads

$$\tau(X) = -\frac{2}{(1+X)^2}\psi\left(\frac{1-X}{1+X}\right) \tag{30}$$

or

$$\psi(x) = -\frac{2}{(1+x)^2}\,\tau\left(\frac{1-x}{1+x}\right) \tag{31}$$

for X respectively x in $[0,1]$ and extended by zero to $\mathbb{R}$. Clearly $\tau \in H^{-1/2}(\mathbb{R})$ with a support in $[0,1]$, if and only if ψ has the same property. In view of (27), solution to the original problem (11–14) can be written as

$$T = \frac{1}{\pi}\int_0^1 \ln\left(\frac{(F_1(X,Y)+\xi)^2 + (F_2(X,Y))^2}{(F_1(X,Y)-\xi)^2 + (F_2(X,Y))^2)}\right)\tau\left(\frac{1-\xi}{1+\xi}\right)\frac{d\xi}{(1+\xi)^2} \tag{32}$$

where F_1, F_2 are given in (16). We have established the following result. We refer to Appendix, Lemma 5.8 and Lemma 5.1 for the detailed proof.

Lemma 3.1. *Let $\tau \in H^{-1/2}(\mathbb{R})$ with support in $[0,1]$ be given. Then the (unique) solution $T \in H^1(M)$ to problem (11) – (14) is given by formula (32).*

4 Proof of Theorem 2.1

In view of the previous Section, we can reformulate our task as follows: find $\tau \in H^{-1/2}(0,1)$ such that

$$KX\tau(X) = -\gamma_{3D}(T_\tau)(X) - \sigma(X) \tag{33}$$

where $T_\tau \in H^1(M)$ is the solution to (11) – (14), given above by formula (32); and $\gamma_{3D} : H^1(M) \to H^{1/2}(\Gamma_3)$ is the Dirichlet trace operator for Γ_3. We set

$$F(X) = F_1(X,0) = \frac{1-X}{1+X}; \tag{34}$$

it is useful to note that $F = F^{-1}$. Making the substitution $\xi' = F(\xi)$ and setting $Y = 0$, we eventually obtain (at least for smooth functions τ)

$$\gamma_{3D}(T_\tau)(X) = \frac{1}{\pi}\int_0^1 \ln\left|\frac{F(X)-F(\xi)}{F(X)+F(\xi)}\right|\tau(\xi)d\xi = -\mathcal{K}_1\tau(X) + \mathcal{K}_2\tau(X), \tag{35}$$

where

$$\mathcal{K}_1\tau(X) = -\int_0^1 \frac{1}{\pi}\tau(\xi)\ln|X-\xi|d\xi \tag{36}$$

$$\mathcal{K}_2\tau(X) = -\int_0^1 \frac{1}{\pi}\tau(\xi)\ln|1-\xi X|d\xi. \tag{37}$$

However, operators $\mathcal{K}_1$, $\mathcal{K}_2$ are continuous from $H^{-1/2}(\Gamma_3)$ to $H^{1/2}(\Gamma_3)$ (see Lemma 5.9) and formula (35) holds for all $\tau \in H^{-1/2}(0,1)$ (see Lemma 5.8).

Our problem is reduced to an integral equation (writing henceforth X and K in lowercase)

$$kx\tau(x) + \mathcal{K}_1\tau(x) = \mathcal{K}_2\tau(x) - \sigma(x), \qquad x \in [0,1] \tag{38}$$

The key observation is that the operator on the left-hand side can be inverted in the appropriate functional setting. Let us remark that the weighted Lebesgue spaces L^2_{xdx} and $L^2_{dx/x}$ (see Appendix for definitions), which appear in the following lemma, are natural for this problem due to condition (8).

Lemma 4.1. *For any $f \in H^{1/2}(0,1) + L^2_{dx/x}(0,1)$, there exists a unique $\tau \in H^{-1/2}(0,1) \cap L^2_{xdx}(0,1)$ such that*

$$kx\tau(x) + \mathcal{K}_1\tau(x) = f(x), \qquad x \in [0,1]. \tag{39}$$

Moreover, the operator $f(x) \mapsto \tau(x)$ is continuous between the above-mentioned spaces.

Proof. Let us set $X = H^{-1/2}(0,1) \cap L^2_{xdx}(0,1)$. Then $X' := H^{1/2}(0,1) + L^2_{dx/x}(0,1)$ is dual to X (see Section 5.1.3). Since $\tau(x) \mapsto x\tau(x)$ is obviously bounded from L^2_{xdx} to $L^2_{dx/x}$ and $\mathcal{K}_1$ is bounded from $H^{-1/2}$ to $H^{1/2}$ by Lemma 5.9, the mapping

$$a(\tau,\phi) := k\int_0^1 x\tau(x)\phi(x) + \langle \mathcal{K}_1\tau, \phi\rangle_{(0,1)}$$

is a continuous bilinear form on $X \times X$. Moreover, we have $\int_0^1 x\tau^2(x) = \|\tau\|^2_{L^2_{xdx}}$ and

$$\langle \mathcal{K}_1\tau, \tau\rangle_{(0,1)} = \langle g * \tau, \tau\rangle_{(0,1)} \geq c\|\tau\|^2_{H^{-1/2}(0,1)}$$

since $g(x) = -\frac{1}{\pi}\ln x$ is a positive definite function on $(0,1)$ (see Lemma 5.9). Therefore, by the Lax–Milgram Theorem, for every $f \in X'$ there exists a unique solution $\tau \in X$ to

$$k\langle x\tau(x), \varphi(x)\rangle + \langle \mathcal{K}_1\tau(x), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle, \qquad \forall \varphi \in X \tag{40}$$

and the assertion is proved. □

Proof of Theorem 2.1. The solution T can clearly be written as $T = T_0 + \Theta$, where $\Theta \in H^1$ is an even and harmonic extension of θ to the unit disc, and T_0 satisfies the zero boundary condition in (6), while σ is replaced by $\sigma - \gamma_{3D}(\Theta) \in H^{1/2}(\Gamma_3)$ in (8). Note that the value of $\frac{\partial T}{\partial Y}$ for $Y = 0$ is not affected as the extension is even.

As explained above, our problem is now equivalent to finding $\tau \in X$ such that

$$\mathcal{B}\tau = \mathcal{K}_2\tau - \sigma + \gamma_{3D}(\Theta) \tag{41}$$

where $\mathcal{B}\tau$ denotes the left-hand side of (38); equivalently, in view of Lemma 4.1,

$$\mathcal{B}(I - \mathcal{B}_{-1}\mathcal{K}_2)\tau = -\sigma + \gamma_{3D}(\Theta) \tag{42}$$

Since $\mathcal{K}_2$ is compact from L^2_{xdx} into $L^2_{dx/x}$ (see Lemma 5.9) we obtain a Fredholm type equation, which is uniquely solvable for any right-hand side if and only if the left-hand side has a trivial kernel.

However, this is equivalent to saying that the homogeneous problem, i.e., with $\theta = \sigma = 0$ only has a trivial solution. And this has been established above, cf. (9) and the following discussion. □

Remark 4.1. *If can be shown that if $k \gtrsim 1$, the first term on the left-hand side of (38) dominates the operator $\mathcal{K}_2$, making the proof considerably simpler. We recall however that k, i.e., K in (8), is small in our setting.*

5 Appendix

Here we summarize several auxiliary results and facts of technical character. In particular, we provide rigorous treatment of Dirichlet and Neumann traces of H^1 functions. We show that (locally) the traces are characterized as limit of u from inside. This enables to identify the trace on (relatively open) parts of the boundary.

We also show that various integrals, which are somehow abusively used in the text, are well-defined in a sense of (unique) bounded extension of densely defined linear mappings; this is consistent with the way the traces are understood. The inverse problem of the Green operator is treated in this functional setting, too.

5.1 Function spaces

5.1.1 Spaces $H^{1/2}$ and $H^{-1/2}$

Definitions and results of this section are taken from Grisvard [5], Runst and Sickel [14] and Franke [4].

Let us define the spaces $H^s = H^s(\mathbb{R})$ for an arbitrary $s \in \mathbb{R}$ as the set of all distributions u on $\mathbb{R}$ satisfying

$$\|u\|_{H^s} := \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < +\infty \tag{43}$$

with the norm given by (43) ($\hat{u}$ is the Fourier transform of u).

Let $I \subset \mathbb{R}$ be an open bounded interval. For $s \in (0,1)$ we define $H^s(I)$ as the set of functions with an extension belonging to $H^s(\mathbb{R})$ with

$$\|u\|_{H^s(I)} := \inf\{\|v\|_{H^s} : \ u = v|_I\}.$$

It is known that $\mathcal{D}(I)$ is dense in $H^s(I)$ and

$$\|u\|^2_{L^2(I)} + \iint\limits_{I \times I} \frac{|u(x) - u(\tilde{x})|^2}{|x - \tilde{x}|^{1+2s}} \, dx d\tilde{x} \tag{44}$$

is an equivalent norm on $H^s(I)$. For $s \in (-1,0)$ we set $H^s(I) := (H^{-s}(I))'$.

Further, for $s \in (0,1)$ we define $\tilde{H}^s(I)$ to be the space of all functions in $H^s(I)$ that belong to $H^s(\mathbb{R})$ when extended by zero. It is known that $\tilde{H}^s(I)$ is a Banach space with the norm

$$\|u\|_{s,\sim} := \|u\|_{L^2(I)} + \int_I \frac{|u(s)|^2}{d(s, \partial I)} ds,$$

where $d(s, A)$ is the distance of s and the set A. Moreover, $\mathcal{D}(I)$ is dense in $\tilde{H}^s(I)$ and the restriction of any $T \in H^{-s}(\mathbb{R})$ to I belongs to $(\tilde{H}^s(I))'$.

Let $\Gamma \subset \mathbb{R}^2$ be a Lipschitz 1-manifold, i.e., it is locally a graph of a Lipschitz function $\phi : I \to \mathbb{R}^2$. For $s \in (-1,1)$, we define the space $H^s(\Gamma)$ via parametrization and partition of unity. In particular, we say that a distribution u on Γ belongs to $H^s(\Gamma)$, if $(u\theta) \circ \phi \in H^s$

for every Lipschitz continuous $\phi : I \to \Gamma$ and smooth θ supported in $\phi(I)$. It is known that an equivalent norm on $H^s(\Gamma)$ is given by (44) where we replace I by Γ and integrate with respect to the Hausdorff measure on Γ.

Let us prove several Lemmas (we denote the duality between $H^{-1/2}(J)$ and $H^{1/2}(J)$ by $\langle \cdot, \cdot \rangle_J$).

Lemma 5.1. *Let $a \in (0, 1)$. Then the spaces $H^{-a}(I)$ and*

$$H^{-a}(\mathbb{R}) \cap \{\operatorname{supp} \subset \overline{I}\} \tag{45}$$

are isomorphic to each other.

Remark 5.1. *Second condition in* (45) *is understood in the sense of $\mathcal{D}'(\mathbb{R}) \supset H^{-a}(\mathbb{R})$. Equivalently, it means all the elements $\tau \in H^{-a}(\mathbb{R})$ that vanish on test functions $\phi \in H^a(\mathbb{R})$ with* $\operatorname{supp} \phi \cap \overline{I} = \emptyset$*. By a simple shifting argument, it is the same as to require that $\langle \tau, \phi \rangle_{\mathbb{R}}$ vanishes whenever* $\operatorname{supp} \phi \cap \operatorname{int} I = \emptyset$*.*

Proof of Lemma 5.1. For $\tau \in H^{-a}(I)$, we define $I_1\tau$ by the formula

$$\langle I_1\tau, \phi \rangle_{\mathbb{R}} := \langle \tau, R\phi \rangle_I \qquad \phi \in H^a(\mathbb{R}) \tag{46}$$

where R is the restriction to I. Clearly, $R : H^a(\mathbb{R}) \to H^a(I)$ is continuous, hence $I_1\tau \in H^{-a}(\mathbb{R})$. Moreover, if $\operatorname{supp} \phi \cap \overline{I} = \emptyset$, then $R\phi = 0$ in I, hence $I_1\tau$ indeed belongs to the space (45).

Conversely, let τ belongs to the space (45). We then define $I_2\tau$ by the formula

$$\langle I_2\tau, \phi \rangle_I := \langle \tau, E\phi \rangle_{\mathbb{R}} \qquad \phi \in H^a(I)$$

where E is an (arbitrary) extension operator $E : H^a(I) \to H^a(\mathbb{R})$. Now $I_2\tau$ belongs to $H^{-a}(I)$ in view of the continuity of E. Moreover, $I_2\tau$ does not depend on the particular choice of E, since $\operatorname{supp}(E\phi - \tilde{E}\phi) \cap I = \emptyset$ for arbitrary choice of extension operators E, $\tilde{E}$.

Observe finally that $I_1 I_2 \tau = \tau$, $I_2 I_1 \tau = \tau$ for any τ in (45) or $\tau \in H^{-a}(I)$, respectively. Indeed, for τ in (45),

$$\langle I_1 I_2 \tau, \phi \rangle_{\mathbb{R}} = \langle \tau, ER\phi \rangle_{\mathbb{R}} = \langle \tau, \phi \rangle_{\mathbb{R}} \qquad \phi \in H^a(\mathbb{R});$$

the second equality follows since $ER\phi$ and ϕ can only differ outside I. Similarly,

$$\langle I_2 I_1 \tau, \phi \rangle_I = \langle \tau, RE\phi \rangle_I = \langle \tau, \phi \rangle_I \qquad \phi \in H^a(I)$$

since obviously $RE\phi = \phi$. □

Lemma 5.2. *The space $\mathcal{D}(\operatorname{int} I)$ is dense in $H^{-a}(I)$.*

Proof. Let $\tau \in H^{-a}(I)$ be given. Then $I_1\tau$ belongs to (45). By scaling, there exist $\psi_n \to I_1\tau$ in $H^{-a}(\mathbb{R})$ with support strictly inside I; these can be further approximated by smooth functions (which we identify with the corresponding regular distributions) with slightly larger support still inside I.

Now $\tau_n = I_2\psi_n$ is the desired approximation. Indeed, $I_2\psi_n \to I_2I_1\tau = \tau$. On the other hand,

$$\langle \tau_n, \phi\rangle_I = \langle I_2\psi_n, \phi\rangle_I = \langle \psi_n, E\phi\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \psi_n E\phi = \int_I \psi_n\phi \qquad \phi \in H^a(I) \tag{47}$$

In other words, τ_n is represented by $R\psi_n$ and we are done. □

Lemma 5.3. *Let ϕ be a Lipschitz mapping from I onto I with Lipschitz continuous inverse. Then the following holds*

1. *$u \in H^{1/2}(I)$ if and only if $u \circ \phi \in H^{1/2}(I)$,*
2. *$u \in H^{-1/2}(I)$ if and only if $(u \circ \phi)\phi' \in H^{-1/2}(I)$.*

Proof. The first assertion is obvious using the norm (44), the second assertion follows from the first one using the duality and substitution. □

5.1.2 Spaces L^2_{xdx} and $L^2_{dx/x}$

We will also work with weighted L^2 spaces on $[0,1]$ with the norms

$$\|u(x)\|^2_{xdx} = \int_0^1 |u(x)|^2 x dx, \tag{48}$$

$$\|v(x)\|^2_{dx/x} = \int_0^1 |v(x)|^2 \frac{dx}{x}. \tag{49}$$

The spaces will be denoted L^2_{xdx} and $L^2_{dx/x}$, respectively. These spaces are separable and (in view of being dual to each other), reflexive. The following lemma shows how compact subsets of $L^2_{dx/x}$ look like.

Lemma 5.4. *The set $K \subset L^2_{dx/x}$ is precompact, if for every $\delta \in (0,1)$ the set $\{f|_{(\delta,1)} : f \in K\}$ is precompact in $L^2(\delta,1)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f|_{(0,\delta)}\|_{dx/x} < \varepsilon$ for all $f \in K$.*

Proof. Consider $\Phi : L^2_{dx/x}(0,1) \to L^2(-\infty,0)$ defined by $(\Phi f)(t) := f(e^t)$. Then Φ is an isometric isomorphism between these spaces. The assertion follows from Theorem 2.33 in Adams [1]. □

5.1.3 Sums and intersections of spaces

Let us define the following spaces

$$X := H^{-1/2}(0,1) \cap L^2_{xdx}(0,1), \quad X' := H^{1/2}(0,1) + L^2_{dx/x}(0,1).$$

These spaces are dual to each other (see Liu and Rooij [7]) with the duality defined as follows

$$\langle f, g\rangle := \langle f, g_1\rangle_{H^{-1/2},H^{1/2}} + \int_0^1 f g_2,$$

where $f \in X$, $g \in X'$, $g = g_1 + g_2$, $g_1 \in H^{1/2}(0,1)$, $g_2 \in L^2_{dx/x}(0,1)$.

5.2 Dirichlet and Neumann trace

Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary. Then there exists a bounded, linear operator $\gamma_D : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ such that $\gamma_D(u) = u|_{\partial\Omega}$ if $u \in C^1(\overline{\Omega})$.

If u is smooth (in particular, if u is harmonic) in Ω, then γ_D can be (locally) characterized as a limit for $x \to \partial\Omega$. More precisely: let $x_0 \in \partial\Omega$, let $w \in \mathbb{R}^d$ be "outward" direction so that $x - hw \in \Omega$ for all $x \in U(x_0, 2\delta) \cap \partial\Omega$ and $0 < h < \delta$. Then $u(\cdot - hw)$ are smooth functions that converge to u in $H^1(U(x_0, 2\delta) \cap \Omega)$. Hence $u(\cdot - hw)_{U(x_0,\delta)\cap\partial\Omega} \to \gamma_D(u)|_{U(x_0,\delta)}$. Note that this enables to identify the trace locally, i.e. in a neighborhood of a given point, or on a relatively open part of the boundary.

If $u \in H^1(\Omega)$, $\nabla u \in L^2(\Omega)$ is not defined on the boundary in the above sense. However, its normal component $\nabla u \cdot n$ can be identified provided some estimates of Δu are available.

Lemma 5.5. *Let Ω be bounded domain with Lipschitz boundary. Then there exists a bounded, linear operator $\gamma_N : H^1(\Omega) \cap \{\Delta u \in H^{-1}(\Omega)\} \to H^{-1/2}(\partial\Omega)$ such that $\gamma_N(u) = \nabla u \cdot n$ if $u \in C^1(\overline{\Omega})$.*

Proof. Let $\tilde{v} \in H^1(\Omega)$ be such that $\gamma_D(\tilde{v}) = v$. Then we set

$$\langle \gamma_N(u), v\rangle = \int_\Omega \nabla u \cdot \nabla \tilde{v} + \langle \Delta u, \tilde{v}\rangle \tag{50}$$

The continuity is clear since the extension mapping $v \mapsto \tilde{v}$ can be chosen in a continuous way. Linearity is also obvious, but we need to verify that $\gamma_N(u)$ is independent of the particular choice of $\tilde{v}$. However, this amounts to say that

$$\int_\Omega \nabla u \cdot \nabla \tilde{w} = -\langle \Delta u, \tilde{w}\rangle \tag{51}$$

for any $\tilde{w} \in H^1_0(\Omega)$. But this is true for $w \in C^\infty_0(\Omega)$ and extends to the desired conclusion by continuity on the both sides. □

Note that if $u \in H^1(\Omega)$ is (weakly) harmonic, and $v \in H^1(\Omega)$, then

$$\langle \gamma_N(u), \gamma_D(v) \rangle = \int_\Omega \nabla u \cdot \nabla v$$

which can be seen as a generalized Green's formula. We observe that the general point behind Lemma 5.5 is that one can define the normal boundary component $u \cdot n$ for $u \in L^2(\Omega, \mathbb{R}^d)$ provided that $\operatorname{div} u \in H^{-1}(\Omega)$.

For the (relatively open) parts Γ_1, Γ_2, Γ_3 (defined in Section 2) of the boundary of our half-disc M we will define $\gamma_{1D}(u)$, $\gamma_{3N}(u)$, etc. as restrictions of $\gamma_D(u)$, $\gamma_N(u)$ to the respective parts of the boundary Γ_1, Γ_3. We will also use γ_{23N}, etc. for the restriction to $\Gamma_2 \cup \{[0,0]\} \cup \Gamma_3$, etc. Since for $u \in H^1(M)$ with $\Delta u = 0$ we have $\gamma_{iD}(u) \in H^{1/2}(\Gamma_i)$ and $\gamma_{iN}(u) \notin H^{-1/2}(\Gamma_i)$ (in general, see e.g. Proposition 1.4.2.3 in Grisvard [5]), the duality $\langle \gamma_{iN}(u), \gamma_{iD}(v) \rangle$ has no sense unless we know that one of the distribution is in a better space as shown in the following lemma.

Lemma 5.6. *Let $\phi \in H^{1/2}(\mathbb{R})$, $\tau \in H^{-1/2}(\mathbb{R})$ and supports of ϕ and τ consist of a finite number of bounded intervals. Assume that on the intersection I of their supports' interiors it holds that $\phi \in L^2(I, \mu)$ and $\tau \in L^2(I, 1/\mu)$, where $\mu(x) \le d(x, \partial(\operatorname{supp} \phi))^{-1}$. Then*

$$\langle \phi, \tau \rangle = \int_I \tau \phi.$$

Proof. Let us first prove the assertion for ϕ bounded. Let A be the union of the boundaries of $\operatorname{supp} \phi$ and $\operatorname{supp} \tau$ (finite set), then $\mathbb{R} \setminus A$ is a union of intervals $I_1, \dots I_n$. Then we take $\varepsilon > 0$ arbitrary and $\tilde{\phi}$ according Lemma 5.11, so that $\|\tilde{\phi}\|_{H^{1/2}} < \varepsilon$ and $\operatorname{supp}(\phi - \tilde{\phi}) \cap A = \emptyset$. Then $\psi := \phi - \tilde{\phi}$ is a sum of functions ψ_k with compact supports in the intervals I_k. Clearly, only intervals where both τ and ψ are nonzero are interesting. So,

$$\langle \psi, \tau \rangle = \sum_{I_k \subset I} \langle \psi_k, \tau \rangle = \sum_{I_k \subset I} \langle \psi_k|_{I_k}, \tau|_{I_k} \rangle.$$

In fact, for each k we have $\tau|_{I_k} \in (\tilde{H}^{1/2}(I_k))'$ and $\psi_k|_{I_k} \in \tilde{H}^{1/2}(I_k)$, so the last equality holds. Moreover, since both dualities $\tilde{H}^{1/2} - (\tilde{H}^{1/2})'$ and $L^2(I, \mu) - L^2(I, 1/\mu)$ extend the scalar product in L^2, they are equal for such pairs $\tau|_{I_k}$, $\psi_k|_{I_k}$ that both of them have sense. Therefore

$$\langle \psi_k|_{I_k}, \tau|_{I_k} \rangle = \int_{I_k} \tau \psi_k.$$

Then we have

$$\left| \langle \phi, \tau \rangle - \int_I \tau \phi \right| \le \varepsilon \|\tau\|_{H^{-1/2}} + \left| \langle \psi, \tau \rangle - \int_I \tau \psi \right| + \|\tau\|_{L^2(I,1/\mu)} \|\tilde{\phi}\|_{L^2(I,\mu)}.$$

On a neighborhood of $a \in \partial(\operatorname{supp} \phi)$, the function ϕ is extendable by zero, so

$$\int \frac{\phi(x)^2}{|x-a|} < +\infty.$$

Therefore, replacing ϕ with $\tilde{\phi}$ the integral can be made smaller than ε by Lemma 5.11. On a neighborhood of $a \in A \setminus \partial(\operatorname{supp}\phi)$, $L^2(I,\mu)$-norm is equivalent to L^2-norm, which is smaller than $H^{1/2}$ norm and therefore less than ε. So, $\|\tilde{\phi}\|_{L^2(I,\mu)}$ is smaller than ε. Together, we have

$$\left|\langle \phi, \tau\rangle - \int_I \tau\phi\right| \le \varepsilon(\|\tau\|_{H^{-1/2}} + \|\tau\|_{L^2(I,1/\mu)}).$$

Since $\varepsilon > 0$ was arbitrary, the assertion is proved for any bounded ϕ.

For a general $\phi \in H^{1/2}(\mathbb{R})$ we consider

$$\phi_L(x) = \begin{cases} L, & \phi(x) > L \\ \phi(x), & \phi(x) \in [-L, L] \\ -L, & \phi(x) < -L \end{cases} \tag{52}$$

It is easy to see that $\|f_L\|_{H^{1/2}} \le \|f\|_{H^{1/2}}$ (since the norm is given by (44)) and by the Lebesgue theorem $f_L \to f$ in $H^{1/2}$ and also in $L^2_{x/dx}$ for $L \to \infty$. Then for every $\varepsilon > 0$ we can find $L > 0$ such that

$$\left|\langle \phi, \tau\rangle - \int_I \tau\phi\right| \le |\langle \phi, \tau\rangle - \langle \phi_L, \tau\rangle| + \left|\langle \phi_L, \tau\rangle - \int_I \tau\phi_L\right| + \left|\int_I \tau\phi_L - \int_I \tau\phi\right| < 2\varepsilon$$

and the proof is completed. □

Clearly, this lemma can be reformulated for $\partial\Omega$ instead of $\mathbb{R}$ and we have the following corollary ((54) follows from (50) and (53)).

Corollary 5.1. *Let $\tau \in H^{-1/2}(\partial M)$, let $\varphi \in H^{1/2}(\partial M)$. Let $\tau = 0$ on Γ_2, $\varphi = 0$ on Γ_1. Let there exist $g(x) \in L^2_{xdx}(0,1)$, $f(x) \in L^2_{dx/x}(0,1)$ such that $\tau = g$, $\varphi = f$ in $(0,1)$. Then*

$$\langle \tau, \varphi\rangle_{H^{-1/2},H^{1/2}} = \int_0^1 g(x)f(x)dx. \tag{53}$$

Moreover, if $\tau = \gamma_N(T)$ and $\varphi = \gamma_D(T)$ for some $T \in H^1(M)$, $\Delta T = 0$ in M, then

$$\int_M |\nabla T|^2 = -\int_0^1 g(x)f(x)dx. \tag{54}$$

5.3 Green operator for half-space and half-circle

For the sake of the following lemma, let us define

$$\|\psi\|_W := \int_{\mathbb{R}} |\xi|^{-1}|\hat{\psi}(\xi)|^2\,d\xi \quad \text{for} \quad \psi \in W_0 := \left\{\psi \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} \psi = 0\right\}$$

and

$$W(\mathbb{R}) = \overline{W_0}^{\,\|\cdot\|_W}.$$

Note that the compact support of ψ implies that $\hat{\psi}$ is smooth and $\hat{\psi}(0) = \int \psi = 0$ then yields $\|\psi\|_W < +\infty$. Note further that $W(\mathbb{R})$ with the corresponding norm is a closed subspace of $H^{-1/2}(\mathbb{R})$.

Lemma 5.7. *The integral*

$$u(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \ln\big((x-\xi)^2+y^2\big)\psi(\xi)\,d\xi \qquad (x,y)\in P_+ \tag{55}$$

can be uniquely extended to $W(\mathbb{R})$ *and*

$$\|\nabla u\|^2_{L^2(P_+)} \le c\|\psi\|^2_{W(\mathbb{R})} \tag{56}$$

holds with $c>0$ *independent of* ψ.

Proof. Assume first that $\psi \in W_0$. By direct computation one has

$$\nabla u(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{(x-\xi,y)}{(x-\xi)^2+y^2}\psi(\xi)\ d\xi = K_y * \psi(x), \tag{57}$$

where

$$K_y(x) = \frac{1}{\pi}\frac{(x,y)}{x^2+y^2}. \tag{58}$$

One further computes that

$$\mathcal{F}[K_y(x)](\xi) = e^{-2\pi y|\xi|}(-i\,\mathrm{sgn}(\xi),1) \tag{59}$$

By Plancherel's an Fubini's Theorem we eventually obtain

$$\|\nabla u\|^2_{L^2(P_+)} = \int_{P_+} |\hat{K}_y(\xi)|^2|\hat{\psi}(\xi)|^2\,d\xi dy \le \int_{P_+} \sqrt{2}e^{-2\pi y|\xi|}|\hat{\psi}(\xi)|^2\,d\xi dy = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{|\xi|}|\hat{\psi}(\xi)|^2\,d\xi \tag{60}$$

This proves (56) for smooth functions. A general $\psi \in W$ is approximated by a sequence $\psi_n \in W_0$. ☐

Lemma 5.8. *The integral* (32) *can be uniquely extended to any* $\tau \in H^{-1/2}(0,1)$*, such that*

$$\|T\|_{H^1} \le c\|\tau\|_{H^{-1/2}(\mathbb{R})} \tag{61}$$

$$\gamma_{1D}(T) = 0 \tag{62}$$

$$\gamma_{2N}(T) = 0 \tag{63}$$

$$\gamma_{3N}(T) = \tau \tag{64}$$

$$\gamma_{3D}(T) = \mathcal{K}_1(\tau) - \mathcal{K}_2(\tau). \tag{65}$$

Proof. Take any $\tau \in \mathcal{D}(0,1)$. Then (62) – (65) hold due to computations in Section 3. Let us consider the following sequence of mappings $\tau \mapsto \psi \mapsto \tilde{\psi} \mapsto u \mapsto T$. Here ψ is given by (31), $\tilde{\psi}$ is the extension of ψ by zero for $x>1$ and then the odd extension to $x<0$. Function u is given by (55) and T by (28) and restriction to M. The first mapping is bounded from $H^{-1/2}(0,1)$ to $H^{-1/2}(0,1)$ by Lemma 5.3, the extension is bounded from $H^{-1/2}(0,1)$ to $W(\mathbb{R})$ by remark before Lemma 5.7, $\tilde{\psi} \to \nabla u$ is bounded by Lemma 5.7 and the last mapping is conformal, so it preserves the L^2-norm of the gradient. Therefore, we have $\|\nabla T\|_{L^2} \le c\|\tau\|_{H^{-1/2}(0,1)}$ and due to Poincaré inequality and (62) we obtain (61).

Since $\mathcal{D}(0,1)$ is dense in $H^{-1/2}(0,1)$ (Lemma 5.2), the assertion follows. ☐

5.4 Integral operator estimates.

In this section we show boundedness and positive definiteness of $\mathcal{K}_1$ and boundedness and compactness of $\mathcal{K}_2$.

Lemma 5.9. *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be integral operators defined in* (36), (37).

1. *Operator $\mathcal{K}_1$ is continuous from $H^{-1/2}(0,1)$ to $H^{1/2}(0,1)$. Moreover,*
$$\langle \mathcal{K}_1\tau, \tau\rangle_{(0,1)} \geq c\|\tau\|^2_{H^{-1/2}(0,1)} \tag{66}$$
on this space.

2. *Operator $\mathcal{K}_2$ is continuous from $H^{-1/2}(0,1)$ to $H^{1/2}(\mathbb{R})$.*

3. *Operator $\mathcal{K}_2$ is compact from $L^2_{xdx}(0,1)$ into $L^2_{dx/x}(0,1)$.*

Proof. 1. Assume first that $\tau \in \mathcal{D}(0,1)$ extended by zero to $\mathbb{R}$. The key observation is that
$$\mathcal{K}_1\tau = \frac{1}{\pi}R\big(g * \tau\big) \tag{67}$$
where R is the restriction from to $(0,1)$ and $g(x) = -\ln|x|\mathbf{1}_{[-1,1]}(x)$. One computes that
$$-\hat{g}(\xi) = 2\int_0^1 \ln|x|\cos(2\pi\xi x)\,dx = -2\int_0^1 \frac{\sin(2\pi\xi x)}{2\pi\xi x}\,dx = -\frac{1}{\pi\xi}\int_0^{2\pi\xi}\frac{\sin y}{y}\,dy \tag{68}$$
from which we easily obtain
$$\frac{c_1}{1+|\xi|} \leq |\hat{g}(\xi)| \leq \frac{c_2}{1+|\xi|} \tag{69}$$
Hence
$$\|\mathcal{K}_1\tau\|^2_{H^{1/2}(0,1)} \leq \|g*\tau\|^2_{H^{1/2}(\mathbb{R})} = \int_{\mathbb{R}}(1+|\xi|)|\hat{g}(\xi)|^2|\hat{\tau}(\xi)|^2\,d\xi \leq c\|\tau\|^2_{H^{-1/2}(\mathbb{R})} \leq c\|\tau\|^2_{H^{-1/2}(0,1)} \tag{70}$$
where the last estimate follows from the fact that $\operatorname{supp}\tau \subset [0,1]$ and Lemma 5.1.

We show the estimate (66). We can see from (68) that $\hat{g}$ is positive, so (69) holds without the modulus. Then we can estimate for $\tau \in \mathcal{D}(0,1)$
$$\langle \mathcal{K}_1\tau, \tau\rangle_{(0,1)} = \int_{\mathbb{R}}(g*\tau)(x)\tau(x)dx = \int_{\mathbb{R}}\hat{g}(\xi)|\tau(\xi)|^2 d\xi \geq \int_{\mathbb{R}}\frac{c_1}{1+|\xi|}|\tau(\xi)|^2 d\xi = c_1\|\tau\|_{H^{-1/2}(\mathbb{R})}$$
which is equivalent to the norm in $H^{-1/2}(0,1)$ by Lemma 5.1.

For a general τ, the conclusion (in fact, the very definition of $\mathcal{K}_1\tau$) is obtained by a standard limiting argument, in view of Lemma 5.2.

2. Assuming that $\tau \in \mathcal{D}(0,1)$, we use the Taylor expansion

$$\ln|1-\xi x| = -\ln(1-\xi x) = \sum_{k=1}^{\infty} \frac{1}{k}\xi^k x^k \tag{71}$$

to write

$$\mathcal{K}_2\tau(x) = \sum_{k=1}^{\infty} \frac{1}{k}\langle \xi^k, \tau(\xi)\rangle x^k \tag{72}$$

Using the simple estimate

$$\|x^k\|_{H^{1/2}(0,1)} \le \|x^k\|_{H^1(0,1)} \le c\left(\int_0^1 x^{2k} + kx^{2(k-1)}\,dx\right)^{1/2} \le c\,k^{-1/2} \tag{73}$$

we conclude

$$\|\mathcal{K}_2\tau(x)\|_{H^{1/2}(0,1)} \le \sum_{k=1}^{\infty} \frac{1}{k}|\langle \xi^k, \tau(\xi)\rangle x^k| \|x^k\|_{H^{1/2}(0,1)} \le c\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)\|\tau(\xi)\|_{H^{-1/2}} \tag{74}$$

3. We estimate

$$|\mathcal{K}_2\tau(x)| \le \int_0^1 \tau(\xi)\xi^{1/2}\xi^{-1/2}|\ln(1-\xi x)|d\xi \le d(x)\|\tau\|_{L^2_{xdx}} \tag{75}$$

where

$$d^2(x) = \int_0^1 |\ln^2(1-\xi x)|\frac{d\xi}{\xi}$$

Since $|\ln(1-\xi x)| \le |\ln(1-\xi)|$ for $x \in (0,1)$, the function $d(x)$ is bounded. Moreover, if $x \in (0,1/2)$, then $|\ln(1-\xi x)| \le c\xi x$, hence we have $d(x) \le \hat{c}x$. It follows that

$$\|\mathcal{K}_2\tau\|_{L^2(0,1)} + \|\mathcal{K}_2\tau\|_{L^2_{dx/x^{2-\epsilon}}} \le K\|\tau\|_{L^2_{xdx}(0,1)} \tag{76}$$

In particular, it is enough to show that $\mathcal{K}_2$ is compact into $L^2(0,1)$. To this end we observe that

$$\frac{d}{dx}\mathcal{K}_2\tau(x) = \frac{1}{\pi}\int_0^1 \tau(\xi)\frac{\xi}{1-\xi x}d\xi \tag{77}$$

hence

$$\left|\frac{d}{dx}\mathcal{K}_2\tau(x)\right| \le e(x)\|\tau\|_{L^2_{xdx}(0,1)} \tag{78}$$

where

$$\begin{aligned} e^2(x) &= \int_0^1 \frac{\xi}{(1-\xi x)^2}d\xi = \frac{d}{dx}\int_0^1 \frac{d\xi}{1-\xi x} = \frac{d}{dx}\frac{\ln(1-x)}{-x} \\ &= \frac{1}{x^2}\Big(\ln(1-x) + \frac{x}{1-x}\Big) \end{aligned} \tag{79}$$

Thus $e(x)$ is bounded for $x \to 0+$, and behaves like $(1-x)^{-1/2}$ for $x \to 1-$; hence (say) $e(x) \in L^1(0,1)$. Thus $\mathcal{K}_2$ is continuous from $L^2_{xdx}(0,1)$ into $W^{1,1}(0,1) \hookrightarrow\hookrightarrow L^2(0,1)$ and we are done. $\square$

5.5 Further properties of $H^{1/2}(\mathbb{R})$

Lemma 5.10. *For a given y and $\varepsilon > 0$, there exists $\theta \in H^{1/2}(\mathbb{R})$ with support in $(-\varepsilon, \varepsilon)$ such that $\|\theta\|_{H^{1/2}(\mathbb{R})} < \varepsilon$. Moreover, there exists $\tilde{\varepsilon} \in (0, \varepsilon)$ such that $\theta = y$ on $[-\tilde{\varepsilon}, \tilde{\varepsilon}]$.*

Proof. We verify first by a direct computation that the function

$$\theta_n(x) = \begin{cases} 1 - |x|^{1/n}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \tag{80}$$

has an arbitrarily small $H^{1/2}$-norm for n large. Obviously, $\|\theta_n\|_{L^2} \to 0$ as $n \to \infty$. It remains to estimate the $H^{1/2}$-seminorm

$$\|f\|^2_{\tilde{H}^{1/2}(\mathbb{R})} = \int_{\mathbb{R}\times\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x-y|^2}\, dxdy = c_{1/2} \int_{\mathbb{R}} |\xi||\hat{f}(\xi)|^2\, d\xi \tag{81}$$

We have

$$\hat{\theta}_n(\xi) = 2\int_0^1 \left(1 - x^{1/n}\right) \cos(2\pi\xi x)\, dx = \frac{1}{\pi\xi n} \underbrace{\int_0^1 x^{1/n-1} \sin(2\pi\xi x)\, dx}_{=:I_n(\xi)} \tag{82}$$

Here, on the one hand, by means of a simple estimate $|\sin y| \leq |y|$, we have $|I_n(\xi)| \leq 2\pi|\xi|$, hence $|\hat{\theta}_n(\xi)| \leq c_1/n$ for any ξ.

On the other hand, we can express

$$I_n(\xi) = \left(2\pi\xi\right)^{1/n} \underbrace{\int_0^{2\pi\xi} \frac{\sin(y)}{y^{1-1/n}}\, dy}_{=:h_n(\xi)} \tag{83}$$

One observes that

$$0 \leq h_n(\xi) \leq h_\infty(1/2\pi) < \infty \tag{84}$$

Hence $|\hat{\theta}_n(\xi)| \leq c_2|\xi|^{-1/n-1}/n$. It follows that

$$\int_{\mathbb{R}} |\xi||\hat{\theta}_n(\xi)|^2\, d\xi = 2\int_0^1 |\xi||\hat{\theta}_n(\xi)|^2\, d\xi + 2\int_1^\infty |\xi||\hat{\theta}_n(\xi)|^2\, d\xi \leq \frac{2c_1^2}{n^2} + \frac{2c_2^2}{n^2} \underbrace{\int_1^\infty |\xi|^{-2/n-1}\, d\xi}_{=n/2}$$

Observing further that the scaling $\theta_n(x/\varepsilon)$ does not increase the L^2-norm (if $\varepsilon < 1$), while leaving the $H^{1/2}$-seminorm (81) unaltered, it is clear that

$$\theta(x) = 2y \min\{1/2, \theta_n(x/\varepsilon)\} \tag{85}$$

for sufficiently large n, is the sought-for function. ☐

Let us remark that it is not difficult to observe (see formula (44)) that the $H^{1/2}$-norm of f_+, f_- and $|f|$ is estimated by the corresponding norm of f.

Lemma 5.11. *Let $\phi \in H^{1/2}(\mathbb{R})$ be bounded. Let $\varepsilon > 0$ be given. Then there exists $\tilde{\phi} \in H^{1/2}(\mathbb{R})$ such that $\|\tilde{\phi}\|_{H^{1/2}} < \varepsilon$, $\operatorname{supp} \tilde{\phi} \subset (-\varepsilon, \varepsilon)$, and there exists $\tilde{\varepsilon} \in (0, \varepsilon)$ such that $\phi = \tilde{\phi}$ on $(-\tilde{\varepsilon}, \tilde{\varepsilon})$.*

Moreover, if $\phi = 0$ on some left neighborhood of 0, then so does $\tilde{\phi}$ and

$$\int_0^\varepsilon \frac{|\tilde{\phi}(x)|^2}{x}\, dx < 2\varepsilon \tag{86}$$

Proof. In view of the above remarks, we can write $\phi = \phi_+ - \phi_-$ and assume henceforth that $0 \leq \phi \leq L$. Then the function

$$\tilde{\phi} = \min\{\phi, \theta\}, \tag{87}$$

where θ comes from lemma 5.10 with some $y > L$, has the desired properties. In fact, the smallness of its norm follows from the fact that

$$\min\{\phi, \theta\} = \frac{1}{2}(\phi + \theta) - \frac{1}{2}|\phi - \theta| \to \frac{1}{2}(\phi - |\phi|) = 0 \tag{88}$$

as $\theta \to 0$ in $H^{1/2}$ and the other properties are obvious.

The second part follows immediately from the definition (87) and the estimate

$$\varepsilon > \int_{y \in (-\varepsilon, 0)} \int_{x \in (0, \varepsilon)} \frac{|\tilde{\phi}(x) - \tilde{\phi}(y)|^2}{|x - y|^2}\, dx dy = \int_0^\varepsilon \frac{\varepsilon |\tilde{\phi}(x)|^2}{(x + \varepsilon)x}\, dx \geq \frac{1}{2} \int_0^\varepsilon \frac{|\tilde{\phi}(x)|^2}{x}\, dx. \tag{89}$$

□

References

[1] R.A. Adams, Sobolev Spaces, Second edition, Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] D. M. Anderson and S. H. Davis, Local fluid and heat flow near contact lines, J. Fluid Mech., 268 (1994), 231–265.

[3] D. M. Anderson and S.H. Davis, The spreading of volatile liquid droplets on heated surfaces, Phys. Fluids 7, (1995), 248–265

[4] J. Franke, On the spaces $F^s_{p,q}$ of Triebel–Lizorkin type: pointwise multipliers and spaces on domains. Math. Nachr. 125 (1986), 29–68.

[5] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[6] V. Janeček, B. Andreotti, D. Pražák, T. Bárta, V. S. Nikolayev, Moving contact line of a volatile fluid, Phys. Rev. E, 88 (2013).

[7] T. Liu and A. Van Rooij, Sums and Intersections of Normed Linear Spaces, Math. Nachr. 42 (1969), 29–42.

[8] M. Mann, K. Stephan, P. Stephan, Influence of heat conduction in the wall on nucleate boiling heat transfer, Int. J. Heat Mass Transfer, 43 (2000), 2193 - 2203.

[9] B. Mathieu, Etudes physiques, expérimentale et numérique des mécanismes de base intervenant dans les écoulements diphasiques, Polytech Marseille, Université de Provence, CNRS - UMR 6595, 2003.

[10] S.J.S. Morris, A phenomenological model for the contact region of an evaporating meniscus on a superheated slab, J. Fluid Mech. 411 (2000), 59–89.

[11] S.J.S. Morris, Contact angles for evaporating liquids predicted and compared with existing experiments, J. Fluid Mech. 432 (2001), 1 - 30.

[12] V.S. Nikolayev, D.A. Beysens, G.L. Lagier, J. Hegseth, Growth of a dry spot under a vapor bubble at high heat flux and high pressure, Int. J. Heat Mass Transfer, 44 (2001), 3499 - 3511.

[13] M. Potash, P.C. Wayner, Evaporation from a two-dimensional extended meniscus, Int. J. Heat Mass Transfer, 15 (1972), 1851 - 1863.

[14] T. Runst and W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, De Gruyter Series in Nonlinear Analysis and Applications 3, Walter de Gruyter, Berlin, 1996

[15] P.C. Stephan, C.A. Busse, Analysis of the heat transfer coefficient of grooved heat pipe evaporator walls, Int. J. Heat Mass Transfer, 35 (1992), 383 - 391.