Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow

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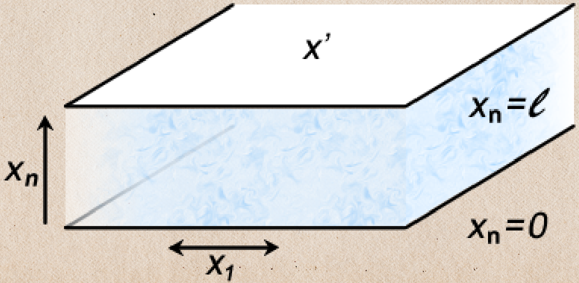
Compressible Navier-Stokes equation

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0 \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div}\, v + \nabla P(\rho) = \rho g \tag{1.2}$$

$$v|_{x_n=0} = V^1(t)\mathbf{e}_1, \quad v|_{x_n=\ell} = 0 \tag{1.3}$$

$$\Omega_\ell = \{x = (x', x_n)\,; x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1},\ 0 < x_n < \ell\}$$



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- $\rho = \rho(x, t)$ unknown density
- $v = (v^1(x,t), \cdots, v^n(x,t))$ unknown velocity
- $P = P(\rho)$ pressure, given smooth function of ρ, for given $\rho_* > 0$ we assume $P'(\rho_*) > 0$
- $\mathbf{g} = (g^1(x_n, t), 0, \cdots, 0, g^n(x_n))$ given function $\overline{T}$-periodic in t
- $V^1(t)$ given function $\overline{T}$-periodic in t

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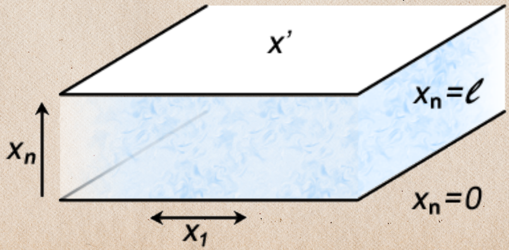

Existence of time-periodic parallel solution

If $\|g^n\| \ll 1$ then there exists

$$\overline{\rho}_p = \overline{\rho}_p(x_n) \qquad \overline{v}_p = (\overline{v}_p^1(x_n, t), 0, \dots, 0)$$

strong solution to (1.1)–(1.3) satisfying

$$\overline{v}_p^1(x_n, t + \overline{T}) = \overline{v}_p^1(x_n, t), \ \overline{T} > 0, \quad \rho_* = \frac{1}{\ell} \int_0^\ell \overline{\rho}_p(x_n)\, dx_n.$$

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Aim

Description of perturbations around time-periodic solution and their asymptotic properties.

Stability of parallel flows

Compressible Navier-Stokes equation (1.1)–(1.2)

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \operatorname{div} v + \nabla P(\rho) = \rho g \tag{1.2}$$

with

$$\mathbf{g} = (g^1(x_n), 0, \cdots, 0, g^n(x_n)), \;\; v|_{x_n=0} = V^1 \mathbf{e}_1, \;\; v|_{x_n=\ell} = 0. \tag{1.4}$$

If $\|g^n\| \ll 1$ then there exists

$$\overline{\rho}_s = \overline{\rho}_s(x_n) \qquad \overline{v}_s = (\overline{v}_s^1(x_n), 0, \ldots, 0)$$

stationary solution to (1.1)–(1.2) and (1.4).

Examples: Plane Couette flow, Poiseuille flow,..

Kagei, Y.

Asymptotic behavior of solutions of the compressible Navier-Stokes equation around parallel flows. *Arch. Rational Mech. Anal.* Vol. 205, pp.585–650.

For Reynolds and Mach numbers sufficiently small and

$$\|\rho_0 - \overline{\rho}_s\| \ll 1, \quad \|v_0 - \overline{v}_s\| \ll 1,$$

solutions are asymptotically stable.

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In the case $n \geq 3$, the disturbances behave in large time as solutions of the linearized problem, whose asymptotic leading parts are given by solutions of an $n-1$ dimensional linear heat equation with convective term.

In the case $n = 2$, the asymptotic behavior is no longer described by the linearized problem; and it is described by a nonlinear diffusion equation, namely, by a 1-dimensional viscous Burgers equation.

Setting $\rho = \overline{\rho}_p + \phi$ and $v = \overline{v}_p + w$ in (1.1)–(1.3):

$$\partial_t \phi + \overline{v}_p^1 \partial_{x_1} \phi + \mathrm{div}\,(\overline{\rho}_p w) = -\mathrm{div}\,(\phi w),$$

$$\partial_t w - \frac{\mu}{\overline{\rho}_p} \Delta w - \frac{\mu+\mu'}{\overline{\rho}_p} \nabla \mathrm{div}\, w + \nabla \left(\frac{P'(\overline{\rho}_p)}{\overline{\rho}_p} \phi \right)$$
$$+ \overline{v}_p^1 \partial_{x_1} w + \frac{\mu}{\overline{\rho}_p^2} (\partial_{x_n}^2 \overline{v}_p^1) \phi \mathbf{e}_1 + (\partial_{x_n} \overline{v}_p^1) w^n \mathbf{e}_1 = \mathbf{f},$$

$$w|_{\partial \Omega_l} = 0,$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0).$$

Dimensionless variables

$$V=\frac{\rho_*\ell^2}{\mu}\left\{|\partial_t V^1|_{C^0(\mathbb{R})}+|g^1|_{C^0(\mathbb{R}\times[0,\ell])}\right\}+|V^1|_{C^0(\mathbb{R})}.$$

$$x=\ell\widetilde{x},\quad t=\frac{\ell}{V}\widetilde{t},\quad w=V\widetilde{w},\quad \phi=\rho_*\gamma^{-2}\widetilde{\phi},\quad P=\rho_*V^2\widetilde{P},$$

$$\overline{v}_p^1=Vv_p^1,\quad \overline{\rho}_p=\rho_*\rho_p,\quad V^1=V\widetilde{V}^1,\quad \mathbf{g}=\frac{\mu V}{\rho_*\ell^2}\widetilde{\mathbf{g}},$$

Here,

$$\nu=\frac{\mu}{\rho_*\ell V},\qquad \nu'=\frac{\mu'}{\rho_*\ell V},\qquad \gamma=\frac{\sqrt{P'(\rho_*)}}{V},\qquad T=\frac{V}{\ell}\overline{T}.$$

Reynolds number $Re=\nu^{-1}$, Mach number $Ma=\gamma^{-1}$ and time period T.

Dimensionless variables

$$V = \frac{\rho_* \ell^2}{\mu} \left\{ |\partial_t V^1|_{C^0(\mathbb{R})} + |g^1|_{C^0(\mathbb{R}\times[0,\ell])} \right\} + |V^1|_{C^0(\mathbb{R})}.$$

$$x = \ell \widetilde{x}, \quad t = \frac{\ell}{V} \widetilde{t}, \quad w = V \widetilde{w}, \quad \phi = \rho_* \gamma^{-2} \widetilde{\phi}, \quad P = \rho_* V^2 \widetilde{P},$$

$$\overline{v}_p^1 = V v_p^1, \quad \overline{\rho}_p = \rho_* \rho_p, \quad V^1 = V \widetilde{V}^1, \quad \mathbf{g} = \frac{\mu V}{\rho_* \ell^2} \widetilde{\mathbf{g}},$$

Here,

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \gamma = \frac{\sqrt{P'(\rho_*)}}{V}, \quad T = \frac{V}{\ell} \overline{T}.$$

Let us write x, t, w and ϕ instead of $\widetilde{x}, \widetilde{t}, \widetilde{w}$ and $\widetilde{\phi}$.

Nondimensional form

On the layer $\Omega = \mathbb{R}^{n-1} \times (0,1)$:

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\,(\rho_p w) = -\mathrm{div}\,(\phi w), \tag{2.1}$$

$$\begin{aligned} &\partial_t w - \frac{\nu}{\rho_p} \Delta w - \frac{\widetilde{\nu}}{\rho_p} \nabla \mathrm{div}\, w + \nabla \left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p} \phi \right) \\ &\qquad + v_p^1 \partial_{x_1} w + \frac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n}^2 v_p^1) \phi \mathbf{e}_1 + (\partial_{x_n} v_p^1) w^n \mathbf{e}_1 = \mathbf{f}, \end{aligned} \tag{2.2}$$

$$w|_{\partial\Omega} = 0, \tag{2.3}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0). \tag{2.4}$$

Here, $\widetilde{\nu} = \nu + \nu'$.

Nondimensional form

On the layer $\Omega = \mathbb{R}^{n-1} \times (0,1)$:

$$\partial_t \phi + v_p^1(x_n, t)\partial_{x_1}\phi + \gamma^2 \mathrm{div}\,(\rho_p(x_n) w) = -\mathrm{div}\,(\phi w), \tag{2.1}$$

$$\partial_t w - \frac{\nu}{\rho_p(x_n)}\Delta w - \frac{\widetilde{\nu}}{\rho_p(x_n)}\nabla \mathrm{div}\, w + \nabla\left(\frac{\widetilde{P}'(\rho_p(x_n))}{\gamma^2 \rho_p(x_n)}\phi\right)$$
$$+ v_p^1(x_n, t)\partial_{x_1} w + \frac{\nu}{\gamma^2 \rho_p(x_n)^2}(\partial_{x_n}^2 v_p^1(x_n, t))\phi \mathbf{e}_1 + (\partial_{x_n} v_p^1(x_n, t)) w^n \mathbf{e}_1 = \mathbf{f}, \tag{2.2}$$

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Regularity assumptions

Let $m \geq [n/2] + 1$. We assume that:

$$g^1 \in \bigcap_{j=0}^{\left[\frac{m+1}{2}\right]} C^j_{per}([0, T]; H^{m+1-2j}(0,1)),$$

$$g^n \in C^{m+1}[0,1],$$

$$V^1 \in C_{per}^{\left[\frac{m+2}{2}\right]}([0, T]).$$

$$\widetilde{P}(\cdot) \in C^{m+2}(\mathbb{R}).$$

Properties of $u_p = {}^T(\rho_p(x_n), v_p(x_n, t))$

$$0 < \rho_1 \leq \rho_p(x_n) \leq \rho_2, \ \int_0^1 \rho_p(x_n) dx_n = 1, \ v_p(x_n, t) = {}^T(v_p^1(x_n, t), 0),$$

with

$$\widetilde{P}'(\rho) > 0 \text{ for } \rho_1 \leq \rho \leq \rho_2,$$

for some constants $0 < \rho_1 < 1 < \rho_2$.

Properties of $u_p = {}^T(\rho_p(x_n), v_p(x_n,t))$

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with

$$\widetilde{P}'(\rho) > 0 \text{ for } \rho_1 \le \rho \le \rho_2,$$

for some constants $0 < \rho_1 < 1 < \rho_2$.

Moreover,

$$|1-\rho_p|_{C^{m+1}([0,1])} \le \frac{C}{\gamma^2}\nu(|\widetilde{P}''|_{C^{m-1}(\rho_1,\rho_2)} + |g^n|_{C^m([0,1])}),$$

$$|\widetilde{P}'(\rho_p) - \gamma^2|_{C([0,1])} \le \frac{C}{\gamma^2}\nu|g^n|_{C([0,1])}.$$

Main results ([3])

Global existence and decay estimate

Suppose that $n \geq 2$. Let m be an integer satisfying $m \geq [n/2]+1$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ then the following assertions hold true.

There is a positive number ε_0 such that if $u_0 = {}^T(\phi_0, w_0) \in (H^m \cap L^1)(\Omega)$ satisfies suitable compatibility condition and $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of (2.1)–(2.4) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j}(\Omega))$ which satisfies

$$\|\partial_{x'}^k u(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{k}{2}}), \ k = 0, 1.$$

Main results ([3])

Asymptotic behavior $n = 2$

Moreover, there holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{3}{4}+\delta}), \ \forall \delta > 0,$$

as $t \to \infty$. *Here,* $u^{(0)} = u^{(0)}(x_2, t)$ *is a given function and* $\sigma = \sigma(x_1, t)$ *is a function satisfying*

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + a_0 \partial_{x_1}(\sigma^2) = 0, \ \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2) \, dx_2,$$

with given constants $\kappa_0, a_0 \in \mathbb{R}$, $\kappa_1 > 0$.

Main results ([3])

Asymptotic behavior $n \geq 3$

Furthermore, there holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)),$$

as $t \to \infty$*. Here,* $\sigma = \sigma(x', t)$ *is a function satisfying*

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \ \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n,$$

with given constants $\kappa_0 \in \mathbb{R}$, $\kappa_1, \kappa'' > 0$*; where* $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$*; and* $\eta_n(t) = \log(1+t)$ *when* $n = 3$ *and* $\eta_n(t) = 1$ *when* $n \geq 4$.

Sketch of the proof

Approach

(i) Spectral analysis of linearized problem (B.-Kagei [1,2]), i.e.,

$$\partial_t\phi + v_p^1\partial_{x_1}\phi + \operatorname{div}(\rho_p w) = 0,$$

$$\partial_t w - \frac{\nu}{\rho_p}\Delta w - \frac{\widetilde{\nu}}{\rho_p}\nabla \operatorname{div} w + \nabla\left(\frac{\widetilde{P}'(\rho_p)}{\rho_p}\phi\right)$$

$$+ v_p^1\partial_{x_1} w + \frac{\mu}{\rho_p^2}(\partial_{x_n}^2 v_p^1)\phi\mathbf{e}_1 + (\partial_{x_n} v_p^1) w^n\mathbf{e}_1 = \mathbf{0},$$

$$w|_{\partial\Omega} = 0,$$

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$$w|_{\partial\Omega} = 0,$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0).$$

(ii) Decomposition of solution and decay estimates based on the spectral analysis, energy method (B. [3]).

Fourier transform of linearized problem $x' \to \xi'$

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0, \ \ t > s, \quad \widehat{u}|_{t=s} = \widehat{u}_0,$$

on $X_0 = (H^1 \times L^2)(0,1)$. Here, $\widehat{L}_{\xi'}(t)$ is an operator on X_0 with domain

$$D(\widehat{L}_{\xi'}(t)) = H^1 \times (H^2 \cap H^1_0).$$

$$\widehat{L}_{\xi'}(t) =$$

$$\begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 \rho_p {}^T\xi' & \gamma^2 \partial_{x_n}(\rho_p \,\cdot\,) \\ i\xi' \frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \frac{\widetilde{\nu}}{\rho_p} \xi' {}^T\xi' & -i\frac{\widetilde{\nu}}{\rho_p}\xi' \partial_{x_n} \\ \partial_{x_n}\left(\frac{\widetilde{P}'(\rho_p)}{\gamma^2 \rho_p} \,\cdot\,\right) & -i\frac{\widetilde{\nu}}{\rho_p} {}^T\xi' \partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1(t))\mathbf{e}_1' & i\xi_1 v_p^1(t) I_{n-1} & \partial_{x_n}(v_p^1(t))\mathbf{e}_1' \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}.$$

By energy method the case $|\xi'| \geq r > 0$ has an exponential decay ([1]).

We treat the case $|\xi'| \ll 1$.

Floquet theory

We define operator $B_{\xi'}$ on space $Y^1_{per} = L^2_{per}([0,T]; X_0)$ with domain

$$D(B_{\xi'}) = H^1_{per}([0,T]; X_0) \cap L^2_{per}([0,T]; H^1 \times (H^2 \cap H^1_0)),$$

in the following way

$$B_{\xi'} v = \partial_t v + \widehat{L}_{\xi'}(\cdot) v,$$

for $v \in D(B_{\xi'})$. Moreover, we define formal adjoint operator $B^*_{\xi'}$ with respect to inner product $\frac{1}{T}\int_0^T \langle \cdot, \cdot \rangle dt$ as

$$B^*_{\xi'} v = -\partial_t v + \widehat{L}^*_{\xi'}(\cdot) v,$$

for $v \in D(B^*_{\xi'}) = D(B_{\xi'})$.

Spectral properties of $B_{\xi'}$

(i) *Let $1 \le l \le m+1$. There exists $q_1 > 0$ such that spectrum of operator $B_{\xi'}$ on Y^l_{per} satisfies*

$$\sigma(B_{\xi'}) \subset \bigcup_{k\in\mathbb{Z}} \{-\lambda_{\xi'} + i\frac{2k\pi}{T}\} \cup \{\lambda : \mathrm{Re}\,\lambda \ge q_1\},$$

with $0 \le -\mathrm{Re}\,\lambda_{\xi'} \le \frac{1}{2}q_1$ uniform for all l. Here, $-\lambda_{\xi'} + i\frac{2k\pi}{T}$ are simple eigenvalues of $B_{\xi'}$.

$-\lambda_{\xi'}$ has an expansion

$$-\lambda_{\xi'} = i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2 + O(|\xi'|^3),$$

where $\kappa_0 \in \mathbb{R}$ and $\kappa_1 > 0$, $\kappa'' > 0$.

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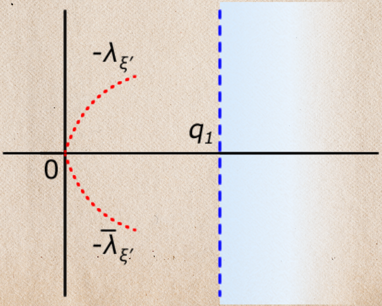

Spectral properties of $B_{\xi'}$

(ii) *There exist $u_{\xi'}$ and $u^*_{\xi'}$ eigenfunctions associated with $-\lambda_{\xi'}$ and $-\overline{\lambda}_{\xi'}$, respectively, with the following properties:*

$$\langle u_{\xi'}(t), u^*_{\xi'}(t)\rangle = 1,$$

$$u_{\xi'}(t) = u^{(0)}(t) + i\xi' \cdot u^{(1)}(t) + |\xi'|^2 u^{(2)}(\xi', t),$$

$$u^*_{\xi'}(t) = u^{*(0)} + i\xi' \cdot u^{*(1)}(t) + |\xi'|^2 u^{*(2)}(\xi', t),$$

for $t \in J_T$.

Floquet transform based on $u_{\xi'}$ and $u_{\xi'}^*$

We define operators $\mathscr{P}(t) : L^2(\Omega) \to L^2(\mathbb{R}^{n-1})$ by

$$\mathscr{P}(t)u = \mathscr{F}^{-1}\{\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u}\},$$

$$\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u} = \widehat{\chi}_1\langle \widehat{u}, u_{\xi'}^*(t)\rangle,$$

for $u \in L^2(\Omega)$ and $t \in [0, \infty)$.

Floquet transform based on $u_{\xi'}$ and $u^*_{\xi'}$

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for $u \in L^2(\Omega)$ and $t \in [0, \infty)$.

$\mathscr{P}(t)$ satisfies:

$$\mathscr{P}(t)(\partial_t + L(t))u(t) = (\partial_t - \Lambda)\mathscr{P}(t)u(t),$$

where multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ is defined by

$$\Lambda\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1\lambda_{\xi'}\widehat{\sigma}\},$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

Floquet transform based on $u_{\xi'}$ and $u^*_{\xi'}$

We define operators $\mathscr{Q}(t) : L^2(\mathbb{R}^{n-1}) \to L^2(\Omega)$ by

$$\mathscr{Q}(t)\sigma = \mathscr{F}^{-1}\{\widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma}\},$$

$$\widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma} = \widehat{\chi}_1 u_{\xi'}(\cdot, t)\widehat{\sigma},$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$ and $t \in [0, \infty)$.

Projections $\mathbb{P}(t)$

We define projections $\mathbb{P}(t)$ on $L^2(\Omega)$ as

$$\mathbb{P}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1 \langle \widehat{u}, u^*_{\xi'}(t)\rangle u_{\xi'}(\cdot, t)\},$$

for $t \in [0, \infty)$ and $u \in L^2(\Omega)$.

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for $t \in [0, \infty)$ and $u \in L^2(\Omega)$.

There holds

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))\mathbb{P}(t)u(t) = \mathscr{Q}(t)(\partial_t - \Lambda)\mathscr{P}(t)u(t).$$

Properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$

(i)

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l (\mathscr{Q}(t)\sigma)\|_{L^2(\Omega)} \le C\|\sigma\|_{L^2(\mathbb{R}^{n-1})},$$

for $0 \le 2j + l \le m + 1$, $k = 0, 1, \ldots$, and $\sigma \in L^2(\mathbb{R}^{n-1})$.

(ii)

$$\|\partial_t^j \partial_{x'}^k (\mathscr{P}(t)u)\|_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{L^2(\Omega)},$$

for $0 \le 2j \le m + 1$, $k = 0, 1, \ldots$, and $u \in L^2(\Omega)$.

(iii) $\mathscr{Q}(t)$ is decomposed as

$$\mathscr{Q}(t) = \mathscr{Q}^{(0)}(t) + \mathrm{div}'\mathscr{Q}^{(1)}(t) + \Delta'\mathscr{Q}^{(2)}(t).$$

Here,
$\mathscr{Q}^{(0)}(t)\sigma = (\mathscr{F}^{-1}\{\widehat{\chi}_1\widehat{\sigma}\})u^{(0)}(\cdot, t)$.

Properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$

(iv) $\mathscr{P}(t)$ is decomposed as

$$\mathscr{P}(t) = \mathscr{P}^{(0)} + \mathrm{div}'\mathscr{P}^{(1)}(t) + \Delta'\mathscr{P}^{(2)}(t).$$

For $u = {}^T(\phi, w)$,

$$\mathscr{P}^{(0)}u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle\widehat{u}, u^{*(0)}\rangle\} = \mathscr{F}^{-1}\{\widehat{\chi}_1\int_0^1 \widehat{\phi}(\cdot, x_n)dx_n\} = [\phi]_1,$$

$$\mathscr{P}^{(1)}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle\widehat{u}, u^{*(1)}(t)\rangle\},$$

$$\mathscr{P}^{(2)}(t)u = \mathscr{F}^{-1}\{-\widehat{\chi}_1\langle\widehat{u}, u^{*(2)}(\xi', t)\rangle\}.$$

$\mathscr{P}^{(p)}(t)$, $p = 0, 1, 2$, share the boundedness properties of $\mathscr{P}(t)$.

Nonlinear problem

Problem (2.1)–(2.4) is written in the form

$$\partial_t u + L(t)u = \mathbf{F},$$

$$w|_{\delta\Omega} = 0, \ \ u|_{t=0} = u_0.$$

Here, $u = {}^T(\phi, w)$; $\mathbf{F} = {}^T(-\mathrm{div}\,(\phi w), \mathbf{f})$ with $\mathbf{f} = {}^T(f^1, \cdots, f^n)$ is the nonlinearity.

Decomposition of $u(t)$

We decompose the solution $u(t)$ into

$$u(t) = \mathbb{P}(t)u(t) + (I - \mathbb{P}(t))u(t).$$

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(i) $\mathbb{P}(t)u(t)$ - Floquet theory

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = \mathbb{P}(t)\mathbf{F}(t).$$

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(ii) $(I - \mathbb{P}(t))u(t) = u_\infty(t)$ - Energy method

$$\partial_t u_\infty + L(t)u_\infty = (I - \mathbb{P}(t))\mathbf{F},$$

$$w_\infty|_{\partial\Omega} = 0, \quad u_\infty|_{t=0} = (I - \mathbb{P}(0))u_0.$$

A priori and decay estimates

For $\|u_0\|_{H^m \cap L^1} \ll 1$ we obtain a priori estimate

$$\sum_{j=0}^{[\frac{m}{2}]} \|\partial_t^j u(t)\|_{H^{m-2j}}^2 \le C\|u_0\|_{H^m \cap L^1}^2,$$

decay estimates

$$\|\partial_{x'}^k u(t)\|_2 \le C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{H^m \cap L^1}, \ \ k = 0, 1,$$

$$\|u(t) - \sigma_1(t)u^{(0)}(t)\|_2 \le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}}\|u_0\|_{H^m \cap L^1},$$

for $t \in [0, \infty)$ with constant $C > 0$ independent of t. Here,

$$\sigma_1(t) = \mathscr{P}(t)u(t).$$

Asymptotic behavior

Since

$$\mathscr{Q}(t)(\partial_t - \Lambda)\mathscr{P}(t)u(t) = \mathscr{Q}(t)\mathscr{P}(t)\mathbf{F}(t),$$

then

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$$(\partial_t - \Lambda)\sigma_1(t) = \mathscr{P}(t)\mathbf{F}(t),$$

and

$$\sigma_1(t) = e^{(t-s)\Lambda}\mathscr{P}(s)u_0 + \int_s^t e^{(t-z)\Lambda}\mathscr{P}(z)\mathbf{F}(z)dz.$$

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Asymptotic behavior $n = 2$

Since there holds

$$e^{(t-s)\lambda}\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1 e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+O(\xi_1^3))t}\widehat{\sigma}\},$$

and

$$\mathscr{P}(s)u_0 = [\phi_0]_1 + \partial_{x_1}\mathscr{P}^{(1)}(s)u_0 + \partial_{x_1}^2\mathscr{P}^{(0)}(s)u_0,$$

we obtain

$$e^{(t-s)\Lambda}\mathscr{P}(s)u_0 \asymp \mathscr{F}^{-1}\{e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2)t}[\widehat{\phi}_0]\}.$$

Asymptotic behavior $n = 2$

$$\sigma_1(t) = e^{(t-s)\Lambda}\mathscr{P}(s)u_0 + \int_s^t e^{(t-z)\Lambda}\mathscr{P}(z)\mathbf{F}(z)dz.$$

Asymptotic behavior $n = 2$

$$\sigma_1(t) = e^{(t-s)\Lambda}\mathscr{P}(s)u_0 + \int_s^t e^{(t-z)\Lambda}\mathscr{P}(z)\mathbf{F}(z)dz.$$

We have

$$\|\int_s^t e^{(t-z)\Lambda}\mathscr{P}(z)\mathbf{F}(z)dz\|_2 \leq C(1+t)^{-\frac{1}{4}}\|u_0\|_{H^m\cap L^1},$$

only. Further investigation necessary!

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only. Further investigation necessary!

Since $\sigma_1^2(t)$ is the slowest decaying term in $\mathbf{F}$, we write

$$\mathbf{F} = \sigma_1^2\mathbf{F}_1 + \mathbf{F}_2,$$

where $\mathbf{F}_2 = \mathbf{F} - \sigma_1^2\mathbf{F}_1$ contains terms involving u_∞, its derivatives and terms of order $O(\sigma_1\partial_{x'}\sigma_1)$ like $\sigma_1 u_1$, and $O(\sigma_1^3)$, but not just $O(\sigma_1^2)$.

Asymptotic behavior $n = 2$

Combining

$$\mathbf{F} = \sigma_1^2 \mathbf{F}_1 + \mathbf{F}_2 \text{ and } u = \sigma_1 u^{(0)} + u_1 + u_\infty,$$

with decomposition of $\mathscr{P}(z)$ we see that

$$\mathscr{P}(z)\mathbf{F}(z) = -\partial_{x_1}[\phi w^1]_1 + \partial_{x_1}\mathscr{P}^{(1)}(z)\mathbf{F}(z) + \partial_{x_1}^2 \mathscr{P}^{(2)}(z)\mathbf{F}(z)$$

$$= -\partial_{x_1}[\sigma_1^2 \phi^{(0)} w^{(0),1}]_1 - \partial_{x_1}[\phi w^1 - \sigma_1 \phi^{(0)} \sigma_1 w^{(0),1}]_1$$

$$+\partial_{x_1}\mathscr{P}^{(1)}(z)(\sigma_1^2 \mathbf{F}_1(z) + \mathbf{F}_2(z)) + \partial_{x_1}^2 \mathscr{P}^{(2)}(z)\mathbf{F}(z).$$

Asymptotic behavior $n = 2$

Therefore

$$\mathscr{P}(z)\mathbf{F}(z) = -a_1(z)\partial_{x_1}\sigma_1^2 + \text{fast terms.}$$

Here,

$$a_1(z) \equiv [\phi^{(0)} w^{(0),1}(z)] - \langle \mathbf{F}_1(z), u^{*(1)}(z) \rangle,$$

depends only on z and it is T-periodic in z.

Asymptotic behavior $n = 2$

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depends only on z and it is T-periodic in z.

We compute

$$\int_s^t e^{(t-z)\Lambda}\mathscr{P}(z)\mathbf{F}(z)dz = -\int_s^t e^{(t-z)\Lambda}a_1(z)\partial_{x_1}\sigma_1^2 dz + \text{ fast terms.}$$

Asymptotic behavior $n = 2$

$$\int_s^t e^{(t-z)\Lambda} a_1(z)\partial_{x_1}\sigma_1^2 dz = \int_s^t e^{(t-z)\Lambda} a_0 \partial_{x_1}\sigma_1^2 dz$$

$$+ \int_s^t e^{(t-z)\Lambda}(a_1(z) - a_0)\partial_{x_1}\sigma_1^2 dz.$$

Define

$$b(t) = \int_0^t a_1(z) - a_0 dz,$$

where

$$a_0 = \frac{1}{T}\int_0^T a_1(z) dz.$$

Then $\partial_t b(t) = a_1(t) - a_0$, $b(t+T) = b(t)$ and $b(0) = b(T) = 0$.

We calculate

$$\int_0^t (a_1(z) - a_0) e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2) dz = \int_0^t \partial_z b(z) e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2) dz$$

$$= \left[b(z) e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2(z)) \right]_0^t - \int_0^t b(z) \partial_z \left(e^{(t-z)\Lambda} \partial_{x_1}(\sigma_1^2(z)) \right) dz$$

$$= b(t) \partial_{x_1}(\sigma_1^2(t)) + \int_0^t b(z) e^{(t-z)\Lambda} \Lambda \partial_{x_1}(\sigma_1^2(z)) dz$$

$$- \int_0^t b(z) \partial_{x_1} e^{(t-z)\Lambda} \partial_z (\sigma_1^2(z)) dz.$$

Using

$$\| \partial_{x_1}^k e^{t\Lambda} \sigma \|_{L^2(\mathbb{R})} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \| \sigma \|_{L^1(\mathbb{R})},$$

for $\sigma \in L^1(\mathbb{R})$ and $k = 0, 1, \ldots,$ we obtain fast decay.

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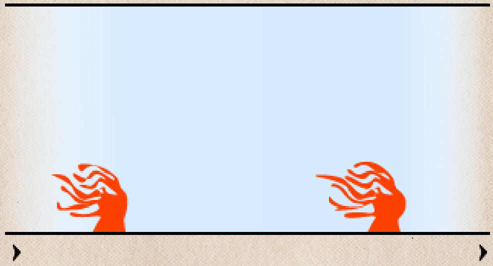
Thank you for your attention !



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