On generalized Stokes' and Brinkman's equations with a pressure- and shear-dependent viscosity and drag coefficient

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Abstract

We study generalizations of the Darcy, Forchheimer, Brinkman and Stokes problem in which the viscosity and the drag coefficient depend on the shear rate and the pressure. We focus on existence of weak solutions to the problem, with the chief aim to capture as wide a group of viscosities and drag coefficients as mathematically feasible and to provide a theory that holds under minimal, not very restrictive conditions. Even in the case of generalized Stokes system, the established result answers a question on existence of weak solutions that has been open so far.

Keywords

Existence theory, weak solution, incompressible fluid, pressure-dependent viscosity, shear-dependent viscosity, Dirichlet boundary condition, Lipschitz truncation of Sobolev functions, flow through porous media, Stokes problem, Darcy problem, Brinkman problem, Forchheimer problem, Div-Curl lemma, biting lemma

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1 Introduction

In this work we study a boundary value problem associated with a system of nonlinear partial differential equations (PDEs) that generalize the classical fluid flow models of Stokes, Darcy, Forchheimer and Brinkman. The problem considered takes the form

$$\left.\begin{aligned} -\operatorname{div}[2\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}] + \beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v} = 0 \quad & \text{in } \Omega,\\ \boldsymbol{v} = \boldsymbol{0} \quad & \text{on } \partial\Omega,\\ \frac{1}{|\Omega|}\int_\Omega p\,dx = p_0. \quad & \end{aligned}\right\} \tag{1}$$

We focus on existence of its (generalized) solutions, pursuing the goal to cover as large a class of functions ν and β as possible and to provide a theory that holds under minimal, not very restrictive

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conditions. In the PDE problem (1) the set $\Omega \subset \mathbb{R}^d$ is an open, bounded, connected domain with a Lipschitz boundary and the sought-after quantities $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ correspond to the velocity and the pressure fields, respectively. The symbol $\boldsymbol{D} = \frac{1}{2}(\nabla + \nabla^T)$ stands for the symmetric part of the gradient. The external body forces $\boldsymbol{f}$ are for the sake of convenience supposed to be of the form

$$\boldsymbol{f} = -\operatorname{div} \boldsymbol{F},$$

where $\boldsymbol{F}$ is a given tensor-valued function. A prescribed value of the integral average of the pressure is given by $p_0 \in \mathbb{R}$. The PDE system (1) arises in the field of flows through porous media and non-Newtonian fluid mechanics. We provide more information below.

Linear examples. Consider first one of the primitive cases $\beta \equiv 0$ and the viscosity ν being a positive constant. Then the problem (1) reduces to the classical Stokes equation, describing a steady (slow) flow of an incompressible fluid adhering to the boundary (by the no-slip boundary condition $(1)_3$) and where the pressure p is determined up to a constant specified by $(1)_4$. Conversely, if $\nu \equiv 0$ and β is a positive constant, the PDE system (1) simplifies to the standard Darcy's equation [17]. The number β is then the *drag coefficient*. Thirdly, if both ν and β are positive constants, (1) simplifies to Brinkman's equation [10, 11], representing another popular model capable of describing certain flows through porous media.

Note that each of the three aforementioned PDE systems is linear. Non-linear models have also been proposed to describe the flow that takes place through rigid porous solids. For instance, taking pressure-dependent viscosity and drag coefficient in (1) leads to a *ceiling flux* (a saturation phenomenon; see [43]), while approaches based on classical Darcy's and Brinkman's models result in a flux that is linearly increasing with the pressure.

Our principal interest in the present study is to analyze flows in which the material moduli—the generalized drag coefficient and viscosity—depend on the pressure and the shear rate, where dependence on the latter quantity is usually confined to $|\boldsymbol{D}\boldsymbol{v}|^2 = \boldsymbol{D}\boldsymbol{v} \cdot \boldsymbol{D}\boldsymbol{v} = \mathrm{Tr}(\boldsymbol{D}\boldsymbol{v})^2$.

Dependence on the shear rate and the pressure. It has been convincingly documented in multiple studies (see e.g. [2, 9, 40, 34, 26, 45]) that the viscosity of a fluid can vary by several orders of magnitude with the pressure. Since the friction due to fluid–(rigid) solid interaction usually dominates the friction between layers of the fluid itself, the relation between the drag coefficient and the pressure is even more substantial. Likewise, the viscosity of many fluids varies with the shear rate. See for example [6] and [31] for illustrative lists of areas where incompressible fluids with shear (rate)-dependent viscosity are extensively used, ranging from geophysics, chemical engineering and bio-material science up to the food industry. Both phenomena can also play an important role in understanding the problems of enhanced oil recovery, carbon dioxide sequestration or extraction of unconventional oil deposits.

Compatibility with the second law of thermodynamics. A thermodynamic basis for the derivation of the Darcy, Forchheimer and Brinkman models and their generalizations falling within the class given by $(1)_{1,2}$ was developed in a recent work by Srinivasan and Rajagopal [44]. The authors of that study set out from the theory of interacting continua as developed in [36, 38, 47]. Following a systematic derivation based on clearly articulated simplifications (as presented earlier in [35]), they arrive at a general reduced thermodynamical system describing steady (slow) flows of a single liquid through a rigid porous solid that takes form[2]

$$\left.\begin{aligned} -\operatorname{div}\boldsymbol{S}+\boldsymbol{m} &= -\nabla p+\rho\boldsymbol{f},\\ \operatorname{div}\boldsymbol{v} &= 0,\\ \xi &= \boldsymbol{S}\cdot\boldsymbol{D}\boldsymbol{v}+\boldsymbol{m}\cdot\boldsymbol{v}. \end{aligned}\right\} \tag{2}$$

Here $\boldsymbol{S}$ stands for the *deviatoric part* of the Cauchy stress $\boldsymbol{T}$ and p for the *mean normal stress*, i.e. the pressure. In other words

$$\boldsymbol{S}=\boldsymbol{T}-\frac{1}{d}\operatorname{Tr}(\boldsymbol{T})\boldsymbol{I} \qquad \text{and} \qquad p=-\frac{1}{d}\operatorname{Tr}(\boldsymbol{T}),$$

so that $\boldsymbol{T}=\boldsymbol{S}-p\boldsymbol{I}$ with $\boldsymbol{I}$ being the identity tensor. The symbol $\boldsymbol{m}$ signifies the force acting on the fluid due to its interaction with the rigid solid and ξ stands for the rate of dissipation, which should be non-negative by the second law of thermodynamics. Note that the choice

$$\boldsymbol{S}=2\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v} \quad \text{with } \nu\geq 0, \tag{3}$$

$$\boldsymbol{m}=\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \quad \text{with } \beta\geq 0 \tag{4}$$

entails $\xi\geq 0$. Consequently, the model considered in our study is thermodynamically compatible. Srinivasan and Rajagopal were actually interested in more delicate issues, namely how to derive (3) and (4) purely from the knowledge of appropriately chosen constitutive equations for ξ. Towards this objective they apply the *criterion of maximal rate of entropy production*; see [44] for details.

More involved examples. The constitutive equations (3) and (4) (and subsequently also the PDE problem (1)) include the following nonlinear models as particular cases[3]:

(i) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)=\beta_0+\beta_1|\boldsymbol{v}|$ begets the so called Darcy-Forchheimer model [21]. Variants can be obtained by considering $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)=\beta_0+\beta_1|\boldsymbol{v}|^q$ for $q>0$.

(ii) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\nu_0\exp(\nu_1 p)$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ leads to the Barus model [5].

(iii) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\nu_0(\varepsilon+|\boldsymbol{D}\boldsymbol{v}|^2)^{\frac{r-2}{2}}$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ with $\varepsilon\geq 0$ produces the power-law fluid models (see e.g. [41, 42] and many further references listed in [14]).

(iv) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\nu_0(\varepsilon+(1+\exp(\nu_1 p))^{-q}+|\boldsymbol{D}\boldsymbol{v}|^2)^{\frac{r-2}{2}}$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ with $r\in(1,2)$, $\varepsilon>0$ and $q\in(0,\frac{r-1}{2\nu_1(2-r)}\varepsilon^{\frac{2-r}{2}})$ exemplifies a model for which the global-in-time existence of weak solutions was established in [29].

(v) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)=\dfrac{2\nu_0 p}{|\boldsymbol{D}\boldsymbol{v}|}$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ leads to the Schaeffer model [39], proposed to describe flowing granular materials.

(vi) $\nu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\equiv 0$ and $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)=\beta(p,|\boldsymbol{v}|)$ brings a generalized Darcy-Forchheimer model that in the special case $\beta(p,|\boldsymbol{v}|)=\beta_0\exp(\beta_1 p)(1+\beta_2|\boldsymbol{v}|^q)\boldsymbol{v}$ has recently been successfully analyzed by the authors of this work. The original references concerning the physical context, solutions of semi-inverse problems and some computational results may be found in [33].

The list is meant for illustrative purposes only with no aim to be exhaustive.

[2]The velocity $\boldsymbol{v}$ in the product $\boldsymbol{m}\cdot\boldsymbol{v}$ in $(2)_3$ should be understood as the difference of the velocity of the fluid (which is $\boldsymbol{v}$) and the velocity of the rigid solid (which is zero).

[3]The constants ν_0, ν_1, β_0, β_1, β_2 are always assumed to be greater than zero.

Structure of the paper. In Section 2, we formulate assumptions specifying the admissible structure of the functions $\nu(p, |\boldsymbol{Dv}|^2)$ and $\beta(p, |\boldsymbol{v}|, |\boldsymbol{Dv}|^2)$. Then in Section 3 we state the main result, set it within earlier works and highlight the novel features. Section 4 surveys auxiliary mathematical tools used in the proof of the main result. The complete proof is then to be found in Section 5.

2 Preliminaries

Notation. We utilize the standard symbolism with a few perhaps non-obvious exceptions: If $X(\Omega)$ is a Lebesgue or Sobolev space, we denote

$$\mathring{X}(\Omega) := \Big\{ f \in X(\Omega) \ \Big|\ \int_\Omega f(x)\, dx = 0 \Big\}.$$

No explicit distinction between spaces of scalar- and vector-valued functions will be made. Confusion should never come to pass as we employ small boldfaced letters to denote vectors and bolded capitals for tensors. Accordingly, for $r > 1$ we set

$$\begin{aligned} W^{1,r}_{0,\mathrm{div}}(\Omega) &:= \big\{ \boldsymbol{f} \in W^{1,r}_0(\Omega) \ \big|\ \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega \big\}, \\ W^{-1,r'}(\Omega) &:= \big(W^{1,r}_0(\Omega) \big)^*, \\ \mathcal{C}^\infty_c(\Omega) &:= \big\{ f \in \mathcal{C}^\infty(\Omega) \ \big|\ f \text{ is compactly supported in } \Omega \big\}. \end{aligned}$$

For $f \in L^1(\Omega)$ we denote

$$f_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x)\, dx.$$

For any $K > 0$ we introduce the cut-off function $T_K : \mathbb{R} \to \mathbb{R}$ as

$$T_K(x) := \begin{cases} x & \text{for } |x| < K, \\ K \dfrac{x}{|x|} & \text{for } |x| \geq K. \end{cases}$$

Completely analogously we define the cut-off function $\boldsymbol{T}_K : \mathbb{R}^d \to \mathbb{R}^d$. If $U, V \subset \mathbb{R}^d$, we say V is compactly contained in U, symbolically $V \Subset U$, if V is bounded and $\overline{V} \subset U$. The symbol $\cdot$ stands for the dot product and $\otimes$ signifies the tensor product. When an integral norm misses the set over which the integral is being taken, always Ω is implicitly considered. For $r \in (1, \infty)$ we denote $r' = r/(r-1)$ and $r^* = dr/(d-r)$, provided further $r < d$. If $r = d$, let r^* be an arbitrary number from $[1, \infty)$. The generic constants are denoted simply by C.

Assumptions on nonlinearities. For the purpose of brevity, we introduce

$$\boldsymbol{S}(p, \boldsymbol{Dv}) := 2\nu(p, |\boldsymbol{Dv}|^2)\boldsymbol{Dv}, \tag{5}$$

which will be used widely throughout the paper. Let $r \in (1, 2]$ be a fixed number and $d \geq 2$. Inspired by [30], below we reproduce assumptions on the smooth nonlinearity $\nu(p, |\boldsymbol{Dv}|^2)$.

Assumption 2.1 *Let there be positive constants C_1 and C_2 such that for all $\boldsymbol{B}, \boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and all $p \in \mathbb{R}$*

$$C_1(1 + |\boldsymbol{D}|^2)^{(r-2)/2} |\boldsymbol{B}|^2 \leq \frac{\partial \boldsymbol{S}(p, \boldsymbol{D})}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \leq C_2(1 + |\boldsymbol{D}|^2)^{(r-2)/2} |\boldsymbol{B}|^2.$$

Assumption 2.2 *Let for all $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$*

$$\left|\frac{\partial \boldsymbol{S}(p,\boldsymbol{D})}{\partial p}\right| \le \gamma_0(1+|\boldsymbol{D}|^2)^{(r-2)/4}, \quad \text{with } 0<\gamma_0<\frac{C_1}{C_1+C_2}.$$

As for the drag term $\beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)$, not considered in [29], we will assume it meets the following requirements:

Assumption 2.3 *Let $\beta : \mathbb{R}\times[0,\infty)\times[0,\infty) \longrightarrow [0,\infty)$ be a continuous function for which there exist $c>0$, $q_0 \in [1,d')$, $q_1 \in [1,r^*)$ and $q_2 \in [1,r)$ such that for all $(p,\boldsymbol{v},\boldsymbol{D}) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}$*

$$0 \le \beta(p,|\boldsymbol{v}|,|\boldsymbol{D}|^2) \le c(1+|p|^{q_0}+|\boldsymbol{v}|^{q_1}+|\boldsymbol{D}|^{q_2}).$$

3 Main result

Without loss of generality we will suppose that $p_0 = 0$ in $(1)_4$, thus getting rid of an expendable symbol and making the presentation neater overall. Our paper is devoted to the justification of the following assertion:

Theorem 3.1 *Let $d \ge 2$ and $\Omega \subset \mathbb{R}^d$ be an open, bounded, connected set with a Lipschitz boundary. Consider $\boldsymbol{F} \in L^{r'}(\Omega)$, $r \in (1,2]$ and suppose that Assumptions 2.1–2.3 hold. Then there exists a weak solution to the equation (1), i.e. a pair $(\boldsymbol{v},p) \in W^{1,r}_{0,\mathrm{div}}(\Omega)\times \mathring{L}^{d'}(\Omega)$ satisfying $\beta(p,\boldsymbol{v},|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \in L^1(\Omega)$ and*

$$\int_\Omega \left[\boldsymbol{S}(p,\boldsymbol{D}\boldsymbol{v})\cdot\boldsymbol{D}\boldsymbol{\varphi} + \beta(p,|\boldsymbol{v}|,|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v}\cdot\boldsymbol{\varphi} - p\operatorname{div}\boldsymbol{\varphi}\right]dx = \int_\Omega \boldsymbol{F}\cdot\nabla\boldsymbol{\varphi}\,dx \quad \text{for every } \boldsymbol{\varphi} \in W^{1,\infty}_0(\Omega).$$

Importance of and comparison to past results. The present paper may be deemed a spiritual descendant of Bulíček and Fišerová [13] who practically further developed the work of Franta et al. [22]. These researchers investigated the model of ours, but without the drag β. On the other hand, their model contains an additional convective term $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})$, which imposes a restriction on the exponent r to be strictly greater than $2d/(d+2)$ at best. In [22], the case $r > 3d/(d+2)$ was investigated and the proof hinged on the fact that the solution velocity field $\boldsymbol{v}$ was an admissible test function. When $r > 2d/(d+2)$, as improved in [13], this is no longer the case and one has to resort to certain additional measures, namely the *Lipschitz approximation of Sobolev functions.* This powerful tool has since its inception in the paper of Acerbi and Fusco [1] been built upon and applied in numerous works (see its evolution e.g. [18], [19] and [8]).

Here we abstain from incorporating the convective term (but see Theorem 6.1). The point is that handling it requires a slightly stronger tool than the drag β alone, namely the Lipschitz approximation lemma from [18] instead of the primordial [1]. Bear in mind that only due to dropping the convective term are we able to make use of that fact that $r > 1$, otherwise $r > 2d/(d+2)$ would have been necessary. Had we kept the convective term, for $r \in (2d/(d+2), 2)$ it would have been sufficient to copy the proof from [13], yet again at the cost of obscuring issues related to the β-term. Another reason for avoiding the convective term is usability of the PDE system (1), as it stands, to real world applications.

Under our assumptions, the solution $\boldsymbol{v}$ is still generally an inadmissible test function. However, as gradients of the test functions do not need to possess a very high integrability (unlike the case with the convective term present), the Lipschitz truncation method might be for $r < 2$ replaced with the L^∞-*truncation* (see [7], [23] or [37]), which may be regarded technically simpler than the Lipschitz truncation. The approach based on the L^∞-truncation method turns out insufficient when trying to cover the case $r = 2$. Interestingly enough, such a situation has been uniformly avoided in the past works ever since the results established in [29] and [30]. In [15] the case $r = 2$ was treated only due to additional assumptions on the viscosity ν.

In this paper, we are able to consider the case $r = 2$, using a combination of the primeval version of the Lipschitz truncation from [1] with the well known Chacon's biting lemma [12] and the Div-Curl lemma [32, 46]. These tools are summarized in Lemmas 4.5–4.7. It is worth noting that when $\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2) \equiv \nu_0$ for some $\nu_0 > 0$, the proof of Theorem 3.1 could be simplified considerably, although even there we would need certain nontrivial bits, specifically local regularity results (11) from Lemma 4.2. As an illustration of what specific model the case $r = 2$ covers, consider for example

$$\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2) = \nu_0 + \frac{\alpha(p)}{1 + |\boldsymbol{D}\boldsymbol{v}|},$$

where $\nu_0 > 0$ is a constant and $\alpha(\cdot)$ is a smooth function satisfying

$$0 \leq \alpha(\cdot) \leq \alpha_0 \quad \text{for some } \alpha_0 > 0 \text{ and } \quad |\alpha'(\cdot)| \leq \frac{\nu_0}{2\nu_0 + \alpha_0}.$$

It is not difficult to observe that such a situation, similar to Schaeffer's model [39] mentioned in the introductory part, falls within the framework of Theorem 3.1.

A natural question arises and that is whether the case $r = 2$ would admit the reintroduction of the convective term back into the equation. Without going much into details, the answer is positive. One would only have to combine our approach (based on the Biting and Div-Curl lemmas) with the procedure from [13]. We state the corresponding assertion in Theorem 6.1 at the end of the paper even though we do not delve into its proof.

As intimated a few lines above, the principal aim of this paper is the inclusion of the drag term β into the PDE analysis, the first such an attempt as far as we can tell. This term allows for a superlinear growth in the pressure, while add to that, possesses *almost critical growth*. More precisely, under Assumption 2.3 with $r \in (1, 2]$, we have $\beta(p, \boldsymbol{v}, |\boldsymbol{D}\boldsymbol{v}|^2) \in L^{1+\delta}(\Omega)$ for some $\delta > 0$, provided $(\boldsymbol{v}, p) \in W^{1,r}(\Omega) \times L^{d'}(\Omega)$. Note then $\beta(p, \boldsymbol{v}, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v}$ a priori need not even be integrable, making our investigation of particular interest.

Incidentally, we might replace the requirement on β to be non-negative with

$$\beta(p, |\boldsymbol{v}|, |\boldsymbol{D}|^2)\boldsymbol{v} \cdot \boldsymbol{v} \geq \beta_0 |\boldsymbol{v}|^2 + \beta_1 |\boldsymbol{v}|^q \quad \text{in } \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}_{\text{sym}} \tag{6}$$

for certain $q > 2$, $\beta_0 \in \mathbb{R}$ and $\beta_1 > 0$. This would be quite useful to embrace drag coefficients of the form

$$\beta(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2) = \beta(|\boldsymbol{v}|) = \beta_0 + \beta_1 |\boldsymbol{v}|^{q-2}.$$

The number β_0 is then called the Darcy coefficient and β_1 the Forchheimer coefficient (see [25]). We will not investigate such a digression for the difference from Assumption 2.3 is minimal, at least in terms of the existence theory analysis. The point is that coercivity (6) guarantees $\boldsymbol{v}$ to belong in $L^q(\Omega)$, which in turn may allow to slacken the growth conditions on the drag coefficient in Assumption 2.3.

Unlike, for instance, the classical Navier-Stokes equations, some kind of a pressure anchorage in (1) is necessary, hence $(1)_4$. The reason is that in our model, not only the pressure gradient is present but there is dependance also on the pressure itself. From the practical viewpoint it would make more sense to prescribe values of the pressure pointwise, for example along a part of the boundary (the so called *accessible boundary* [3]). Unfortunately, the pressure constructed here is only an integrable function so we cannot refer to its point values. In this case one could take for instance the integral average over a (possibly small) set $\Omega_0 \subset \Omega$, thus approximating the pointwise prescription (see [16]). In this paper we chose fixing p_Ω in the spirit of [22], as the generalization p_{Ω_0} could easily be made but it is not the gist of this paper. A reader requiring more information on this topic should address for example [13] and the references given there. A similar argument applies to our choice of the boundary condition.

We are aware of the fact that for problems connected with flows through porous media, the boundary condition $(1)_3$ is rather crude as one usually prescribes e.g. the inflow/outflow velocity along parts of the boundary. The no-slip condition could well be generalized but we picked this one as it makes the analysis easy to follow. For more information concerning alternative boundary conditions for the velocity and the pressure alike consult e.g. [27].

Lastly, it is worth remarking that the upper bound on the value of γ_0 in Assumption 2.2 has since [13, 22] been improved. In other words, our viscosity ν allows a faster growth rate in the pressure variable, albeit still a sublinear one. Aside from C_1 and C_2, the bound γ_0 also used to detrimentally depend on geometry of the set Ω through the Bogovskiĭ operator on Ω (for more information about the constant see [24, Lemma III.3.1]). The idea behind the enhancement in our work is to replace the Bogovskiĭ operator with the Newtonian potential at some point. We recall the key properties of the Newtonian potential in Lemma 4.4.

Highlights. We want to conclude this part listing the principal contributions of this paper:

1. We establish large-data existence theory for a generalized Brinkman problem with the viscosity and drag coefficients depending on the pressure and the shear rate; see Theorem 3.1. To the best of our knowledge, a PDE analysis for similar problems with a pressure- and shear-dependent drag satisfying Assumption 2.3 has not been carried out yet.
2. Within the setting considered, even for a generalized Stokes problem (i.e. $\beta = 0$) we establish new results when $r = 2$, thus improving the works [13] and [22]; see Theorem 6.1.
3. The earlier studies concerning the PDE analysis of a generalized Stokes' problem with $\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2)$ in general bounded domains suffered a serious drawback. The parameter γ_0 appearing in Assumption 2.2 used to be restricted by a constant depending on the geometry of the set Ω. This severe constraint has been removed here. The theory presented in this work thus holds under the same restrictions as the theory developed for an (idealized) spatially periodic problem in [29].

4 Auxiliary tools

In this section we survey a couple of results exploited in the proof of Theorem 3.1. First off, we state what one might call a *compensated monotonicity* of the nonlinearity $\boldsymbol{S}$, as well as coercivity and boundedness thereof.

Lemma 4.1 ([22], Lemmas 3.3, 3.4) *Let Assumptions 2.1 and 2.2 hold. For arbitrary $\boldsymbol{D}^1, \boldsymbol{D}^2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p^1, p^2 \in \mathbb{R}$ we set*

$$I^{1,2} := \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2|^2 \, ds,$$

with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2)$. Then

$$\frac{1}{2} C_1 I^{1,2} \le (\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2)) \cdot (\boldsymbol{D}^1 - \boldsymbol{D}^2) + \frac{\gamma_0^2}{2C_1} |p^1 - p^2|^2. \tag{7}$$

Furthermore

$$|(\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2)| \le \gamma_0 |p^1 - p^2| + C_2 \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2| \, ds. \tag{8}$$

Finally, for all $p \in \mathbb{R}$, $r \in (1, 2]$ and $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\boldsymbol{S}(p, \boldsymbol{D}) \cdot \boldsymbol{D} \ge \frac{C_1}{2r} (|\boldsymbol{D}|^r - 1) \tag{9}$$

and

$$|\boldsymbol{S}(p,\boldsymbol{D})| \le \frac{C_2}{r-1}(1+|\boldsymbol{D}|)^{r-1}. \tag{10}$$

The corresponding statement in [22] does not include (8). However, it is only an easy observation stemming from

$$\boldsymbol{S}(p^1,\boldsymbol{D}^1) - \boldsymbol{S}(p^2,\boldsymbol{D}^2) = \int_0^1 \frac{d}{ds}\boldsymbol{S}(p^2 + s(p^1-p^2), \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2))\, ds$$

and Assumptions 2.1 and 2.2.

On occasion, we will use the theory for the Stokes problem. All necessary ingredients are compiled in the lemma below. Beware of our extracting only what is to be needed for purposes of this paper, as we deem stating these theorems in their full form rather distracting.

Lemma 4.2 ([24], Theorems IV.1.1, IV.4.1, IV.4.4) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d \ge 2$. There exists a continuous linear operator*

$$\mathcal{H} : W^{-1,2}(\Omega) \longrightarrow W^{1,2}_{0,\mathrm{div}}(\Omega) \times \mathring{L}^2(\Omega)$$

assigning to any $\boldsymbol{f} \in W^{-1,2}(\Omega)$ the unique weak solution $(\boldsymbol{v}, p)$ of the Stokes problem

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \\ p_\Omega &= 0. \end{aligned}$$

Moreover, if $\boldsymbol{f} \in W^{-1,2}(\Omega) \cap W^{k,q}_{loc}(\Omega)$ for certain $1 < q < \infty$ and $k \ge -1$, then $\mathcal{H}(\boldsymbol{f}) \in W^{k+2,q}_{loc}(\Omega) \times W^{k+1,q}_{loc}(\Omega)$ and one has the estimate

$$\left\|\nabla^{k+2}\boldsymbol{v}\right\|_{q;\Omega''} + \left\|\nabla^{k+1}p\right\|_{q;\Omega''} \le c\big(\|\boldsymbol{f}\|_{k,q;\Omega'} + \|\boldsymbol{v}\|_{k+1,q;\Omega'} + \|p\|_{k,q;\Omega'}\big). \tag{11}$$

for any $\Omega'' \Subset \Omega' \Subset \Omega$, where $c = c(d, q, k, \Omega', \Omega'')$.

Aside from the Stokes problem, we will have to be capable of dealing effectively with the divergence equation. The following statement about the Bogovskiĭ operator provides us with a necessary tool.

Lemma 4.3 ([24], Theorem III.3.3) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d \ge 2$ and $1 < q < \infty$. There is a continuous linear operator*

$$\mathcal{B} : \mathring{L}^q(\Omega) \longrightarrow W^{1,q}_0(\Omega)$$

assigning to any $f \in \mathring{L}^q(\Omega)$ a weak solution $\boldsymbol{v}$ of the divergence equation

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= f \quad \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned}$$

The Bogovskiĭ operator will at times be replaced with the Newtonian potential. Then the following result will be used:

Lemma 4.4 *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and $f \in L^q(\Omega)$, $q \in (1,\infty)$. Denote $\widetilde{f}$ the zero extension of f on the whole space $\mathbb{R}^d$ and Γ the Newtonian kernel in R^d, i.e.*

$$\Gamma(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| & \text{for } d = 2, \\ \dfrac{1}{d(d-2)\omega_d}|x|^{2-d} & \text{for } d > 2, \end{cases}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Define

$$\mathcal{N}(f) := (\widetilde{f} * \Gamma)\big|_{\Omega}.$$

Then $\mathcal{N}$ is continuous from $L^q(\Omega)$ into $W^{2,q}(\Omega)$ and for $q = 2$ one has $\left\|\nabla^2\mathcal{N}(f)\right\|_2 \le \|f\|_2$.

Proof. We only sketch out the proof as the result is standard. Continuity from $L^q(\Omega)$ into $W^{1,q}(\Omega)$ follows from Young's inequality for convolutions and boundedness of Ω. In order to bound the second gradients, employ the Calderón-Zygmund theory for singular operators; see [20, Theorem 10.10].

As for the last inequality, we have $-\Delta(\widetilde{f} * \Gamma) = \widetilde{f}$ a.e. in $\mathbb{R}^d$ and $\left\|\nabla^2 g\right\|_{2;\mathbb{R}^d} = \|\Delta g\|_{2;\mathbb{R}^d}$ holding for any $g \in W^{2,2}(\mathbb{R}^d)$. Hence

$$\left\|\nabla^2\mathcal{N}(f)\right\|_2 \le \left\|\nabla^2(\widetilde{f} * \Gamma)\right\|_{2;\mathbb{R}^d} = \left\|\Delta(\widetilde{f} * \Gamma)\right\|_{2;\mathbb{R}^d} = \left\|\widetilde{f}\right\|_{2;\mathbb{R}^d} = \left\|f\right\|_2.$$

□

For the sake of completeness, we explicitly formulate yet three classical results here, namely Chacon's biting lemma [12], Murat's and Tartar's Div-Curl lemma [32, 46] and Acerbi's and Fusco's Lipschitz approximation of Sobolev functions [1]:

Lemma 4.5 (Biting lemma, [4]) *Let $\Omega \subset \mathbb{R}^d$ have a finite Lebesgue measure and $\{f^k\}$ be a bounded sequence in $L^1(\Omega)$. Then there exist a function $f \in L^1(\Omega)$, a subsequence $\{f^j\}$ of $\{f^k\}$ and a non-increasing sequence of measurable sets $E_n \subset \Omega$ with $\lim_{n\to\infty}|E^n| = 0$, such that $f^j \to f$ weakly in $L^1(\Omega \setminus E_n)$ for every fixed n.*

Lemma 4.6 (Div-Curl lemma, [20], Theorem 10.21) *Let $\Omega \subset \mathbb{R}^d$ be open. Assume $\boldsymbol{u}^n \to \boldsymbol{u}$ weakly in $L^p(\Omega)$ and $\boldsymbol{v}^n \to \boldsymbol{v}$ weakly in $L^q(\Omega)$, where $1/p + 1/q = 1/r < 1$. In addition, let $\{\operatorname{div}\boldsymbol{u}^n\}$ be relatively compact in $W^{-1,s}(\Omega)$ and $\{\operatorname{curl}\boldsymbol{v}^n\}$[4] be relatively compact in $W^{-1,s}(\Omega)$ for a certain $s > 1$. Then $\boldsymbol{u}^n \cdot \boldsymbol{v}^n \to \boldsymbol{u} \cdot \boldsymbol{v}$ weakly in $L^r(\Omega)$.*

Lemma 4.7 (Lipschitz approximation of Sobolev functions, [1]) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz open set and $p \ge 1$. There exists a constant c such that, for every $u \in W^{1,p}(\Omega)$ and every $\lambda > 0$ there exists $u_\lambda \in W^{1,\infty}(\Omega)$ satisfying*

$$\|u_\lambda\|_{1,\infty} \le \lambda, \tag{12}$$

$$|\{u \ne u_\lambda\}| \le c\frac{\|u\|_{1,p}^p}{\lambda^p}, \tag{13}$$

$$\|u_\lambda\|_{1,p} \le c\,\|u\|_{1,p}\,. \tag{14}$$

Strictly speaking, the bound (14) does not appear in [1]. It is, however, a trivial consequence of (12) and (13). Secondly, the original result [1] mentions only a *regular* set Ω. Since this regularity is required for a $W^{1,p}$-continuous extension operator, Lipschitz sets are perfectly acceptable.

[4] $\operatorname{curl} = \frac{1}{2}(\nabla - \nabla^T)$

5 Proof of the existence theorem

Solutions asserted by Theorem 3.1 will be found as a weak limit of a twofold approximation scheme: One is the so called *quasicompressible approximation* (see [22]), which serves to construct at least some kind of a pressure as a solution to an elliptic problem featuring divergence of the velocity field. The term *quasicompressible* is motivated by the fact that the resultant velocity is only *almost* solenoidal (see below). In our exposition we identify this modification with the parameter ε and the goal is to perform $\varepsilon \to 0_+$. The second level is an *L^∞-truncation* of the β-term, the necessity of which is attributable to quite draconian growth conditions in Assumption 2.3. This level is associated with the parameter K and our plan is to justify $K \to \infty$. Nonetheless, we have to show any such an approximation exists for each ε and K in the first place.

Lemma 5.1 *Under the assumptions of Theorem 3.1, for every $\varepsilon, K > 0$ there exist $(\boldsymbol{v}^{\varepsilon,K}, p^{\varepsilon,K}) \in W_0^{1,r}(\Omega) \times \big(\mathring{W}^{1,2}(\Omega) \cap \mathring{L}^{r'}(\Omega)\big)$ satisfying*

$$\varepsilon \int_\Omega \nabla p^{\varepsilon,K} \cdot \nabla \varphi \, dx + \int_\Omega \varphi \operatorname{div} \boldsymbol{v}^{\varepsilon,K} \, dx = 0 \quad \textit{for all } \varphi \in W^{1,2}(\Omega) \cap L^{r'}(\Omega) \tag{15}$$

and

$$\int_\Omega \big[S(p^{\varepsilon,K}, \boldsymbol{D}\boldsymbol{v}^{\varepsilon,K}) \cdot \boldsymbol{D}\boldsymbol{\varphi} + T_K\beta(p^{\varepsilon,K}, |\boldsymbol{v}^{\varepsilon,K}|, |\boldsymbol{D}\boldsymbol{v}^{\varepsilon,K}|^2)\boldsymbol{T}_K\boldsymbol{v}^{\varepsilon} \cdot \boldsymbol{\varphi} - p^{\varepsilon,K} \operatorname{div} \boldsymbol{\varphi}\big] dx$$
$$= \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi} \, dx \quad \textit{for all } \boldsymbol{\varphi} \in W_0^{1,r}(\Omega). \tag{16}$$

Proof. We will drop the ε, K-indices for the sake of a neater notation. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$ and $\{z_i\}_{i\in\mathbb{N}} \subset \mathring{W}^{1,2}(\Omega) \cap \mathring{L}^{r'}(\Omega)$ be linearly independent, with linear spans dense in the respective spaces. To begin with, we will deduce existence of solutions to an approximate problem, i.e. for $n \in \mathbb{N}$ we seek

$$\boldsymbol{v}^n(x) = \sum_{i=1}^n a_i^n \boldsymbol{w}_i(x),$$
$$p^n(x) = \sum_{i=1}^n b_i^n z_i(x),$$

satisfying

$$\varepsilon \int_\Omega \nabla p^n \cdot \nabla z_i \, dx + \int_\Omega z_i \operatorname{div} \boldsymbol{v}^n \, dx = 0, \tag{17}$$

$$\int_\Omega \boldsymbol{S}(p^n, \boldsymbol{D}\boldsymbol{v}^n) \cdot \boldsymbol{D}\boldsymbol{w}_i \, dx + \int_\Omega \beta^n \boldsymbol{T}_K\boldsymbol{v}^n \cdot \boldsymbol{w}_i \, dx - \int_\Omega p^n \operatorname{div} \boldsymbol{w}_i \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{w}_i \, dx \tag{18}$$

L^∞-truncation for $i = 1, \ldots, n$, recalling (5) and setting $\beta^n := T_K\beta(p^n, |\boldsymbol{v}^n|, |\boldsymbol{D}\boldsymbol{v}^n|^2)$.

Towards showing the existence of $\{a_i^n\}_{i=1}^n$ and $\{b_i^n\}_{i=1}^n$, we employ the standard corollary of Brouwer's fixed point theorem [28, Lemme 4.3]. Its applicability follows from the oncoming lines and will not be discussed in detail. Our undivided attention is zoomed in on the limit passage $n \to \infty$.

Multiplying eq. (17) by b_i^n and eq. (18) by a_i^n and summing the resultant $2n$ equalities, we obtain

$$\varepsilon \left\|\nabla p^n\right\|_2^2 + \int_\Omega \boldsymbol{S}(p^n, \boldsymbol{D}\boldsymbol{v}^n) \cdot \boldsymbol{D}\boldsymbol{v}^n \, dx + \int_\Omega \beta^n \boldsymbol{T}_K\boldsymbol{v}^n \cdot \boldsymbol{v}^n \, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{v}^n \, dx.$$

Now we recall the coercivity condition (9), Korn's, Young's and Hölder's inequalities, non-negativity of β^n and the fact that $\boldsymbol{T}_K\boldsymbol{v}^n \cdot \boldsymbol{v}^n \geq 0$, deducing

$$\sup_n \big(\varepsilon \left\|\nabla p^n\right\|_2^2 + \left\|\boldsymbol{D}\boldsymbol{v}^n\right\|_r^r\big) < \infty.$$

By Korn's and Poincaré's inequalities and the bound (10), we may select a subsequence (labelled again $(p^n, \boldsymbol{v}^n)$) such that for $n \to \infty$[5]

$$\begin{aligned}
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{weakly in } W_0^{1,r}(\Omega),\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{a.e. in } \Omega,\\
p^n &\to p && \text{weakly in } \mathring{W}^{1,2}(\Omega),\\
p^n &\to p && \text{strongly in } L^2(\Omega),\\
p^n &\to p && \text{a.e. in } \Omega,\\
\boldsymbol{S}(p^n, \boldsymbol{D}\boldsymbol{v}^n) &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(\Omega),\\
\beta^n \boldsymbol{T}_K \boldsymbol{v}^n &\to \overline{\beta \boldsymbol{v}} && \text{weakly in } L^q(\Omega) \text{ for any } q \in [1,\infty).
\end{aligned} \tag{19}$$

Letting $n \to \infty$ in the approximate eq. (17) and the density of z_i in $\mathring{W}^{1,2}(\Omega)$ guarantee (15). Similarly, letting $n \to \infty$ in the approximation (18) implies

$$\int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi}\, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi}\, dx - \int_\Omega p \operatorname{div} \boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}\, dx \quad \text{for all } \boldsymbol{\varphi} \in W_0^{1,2}(\Omega). \tag{20}$$

Now we need to show $p \in L^{r'}(\Omega)$. To this end, let $L > 0$ and define ξ_L as the indicator function of $\{|p| < L\}$. Recalling Lemma 4.3 on the Bogovskiĭ operator, we set

$$\boldsymbol{\varphi} := \mathcal{B}\big(|p|^{r'-2} p \xi_L - (|p|^{r'-2} p \xi_L)_\Omega\big).$$

In particular, such a $\boldsymbol{\varphi}$ can be used in (20) and by the continuity of $\mathcal{B}$

$$\|\boldsymbol{\varphi}\|_{1,r} \le C \big\| |p|^{r'-1} \xi_L \big\|_r = C \, \|p \xi_L\|_{r'}^{r'-1} .$$

As $p_\Omega = 0$, plugging $\boldsymbol{\varphi}$ into eq. (20) and recalling (19) leads to

$$\|p\xi_L\|_{r'}^{r'} = \int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi}\, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi}\, dx - \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}\, dx \le C \, \|\boldsymbol{\varphi}\|_{1,r} \le C \, \|p\xi_L\|_{r'}^{r'-1} .$$

Since C is independent of L, we obtain $p \in L^{r'}(\Omega)$. Thus the equation (20) holds for any $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)$.

What remains is the identification of the nonlinear terms $\overline{\boldsymbol{S}}$ and $\overline{\beta \boldsymbol{v}}$. Considering the continuity of ν and β and the convergences $(19)_2$ and $(19)_5$, it is sufficient to verify the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^n$ a.e. in Ω. Then $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ and $\overline{\beta \boldsymbol{v}} = T_K \beta\big(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2\big) \boldsymbol{T}_K \boldsymbol{v}$ by Vitali's theorem. We will, however, take these identities for granted and skip the derivation of the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^n$, as it will once again be reiterated in the following section under more inimical conditions, that time in detail. □

5.1 Vanishing artificial compressibility ($\varepsilon \to 0_+$)

Now we justify the limit passage $\varepsilon \to 0_+$ for solutions yielded by Lemma 5.1. Let us again drop the index K and denote the solutions at hand simply $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$.

Uniform estimates. Taking $\varphi = p^\varepsilon$ in (15), $\boldsymbol{\varphi} = \boldsymbol{v}^\varepsilon$ in (16) and summing up the resultant identities, we obtain

$$\varepsilon \big\|\nabla p^\varepsilon\big\|_2^2 + \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \boldsymbol{D}\boldsymbol{v}^\varepsilon\, dx + \int_\Omega \beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon \cdot \boldsymbol{v}^\varepsilon\, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{v}^\varepsilon\, dx,$$

[5]We employ bars for unidentified weak limits.

where $\beta^\varepsilon := T_K\beta(p^\varepsilon, |\boldsymbol{v}^\varepsilon|, |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2)$.

Using $\beta^\varepsilon \boldsymbol{T}_K\boldsymbol{v}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \geq 0$, the property (9), Poincaré's, Young's and Korn's inequalities, we observe

$$\sup_\varepsilon \sqrt{\varepsilon}\big\|\nabla p^\varepsilon\big\|_2 < \infty, \tag{21}$$

$$\sup_\varepsilon \|\boldsymbol{v}^\varepsilon\|_{1,r} < \infty, \tag{22}$$

the latter of which we further combine with (10), deducing

$$\sup_\varepsilon \big\|\boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)\big\|_{r'} < \infty. \tag{23}$$

As for bounds on the pressure, we can copy the procedure from the previous proof. Setting

$$\boldsymbol{\varphi} := \mathcal{B}\big(|p^\varepsilon|^{r'-2}p^\varepsilon - (|p^\varepsilon|^{r'-2}p^\varepsilon)_\Omega\big),$$

we observe that $\boldsymbol{\varphi} \in W^{1,r}_0(\Omega)$ and furthermore $\big\|\boldsymbol{\varphi}\big\|_{1,r} \leq C\big\||p^\varepsilon|^{r'-1}\big\|_r = C\big\|p^\varepsilon\big\|^{r'-1}_{r'}$ due to continuity of $\mathcal{B}$, with C independent of ε. Recalling that $(p^\varepsilon)_\Omega = 0$, the insertion of $\boldsymbol{\varphi}$ into (16) hence produces

$$\|p^\varepsilon\|^{r'}_{r'} = \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)\cdot \boldsymbol{D}\boldsymbol{\varphi}\,dx + \int_\Omega \beta^\varepsilon \boldsymbol{T}_K\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}\,dx - \int_\Omega \boldsymbol{F}\cdot\nabla\boldsymbol{\varphi}\,dx \leq C\,\|\boldsymbol{\varphi}\|_{1,r} \leq C\,\|p^\varepsilon\|^{r'-1}_{r'}$$

and thus we infer

$$\sup_\varepsilon \|p^\varepsilon\|_{r'} < \infty. \tag{24}$$

The bounds (21)–(24) imply that we may assume the following convergences as $\varepsilon \to 0_+$:

$$\begin{aligned}
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{weakly in } W^{1,r}_0(\Omega),\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{a.e. in } \Omega,\\
p^\varepsilon &\to p && \text{weakly in } L^{r'}(\Omega),\\
\varepsilon\nabla p^\varepsilon &\to \boldsymbol{0} && \text{strongly in } L^2(\Omega),\\
\boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(\Omega),\\
\beta^\varepsilon\boldsymbol{T}_K\boldsymbol{v}^\varepsilon &\to \overline{\beta\boldsymbol{v}} && \text{weakly in } L^q(\Omega) \text{ for any } q\in[1,\infty).
\end{aligned} \tag{25}$$

The limit $\varepsilon \to 0_+$ applied to eq. (15) then guarantees $\operatorname{div}\boldsymbol{v} = 0$ a.e. in Ω and eq. (16) yields

$$\int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{D}\boldsymbol{\varphi}\,dx + \int_\Omega \overline{\beta\boldsymbol{v}}\cdot\boldsymbol{\varphi}\,dx - \int_\Omega p\operatorname{div}\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{F}\cdot\nabla\boldsymbol{\varphi}\,dx \quad \text{for all } \boldsymbol{\varphi}\in W^{1,r}_0(\Omega).$$

Furthermore, since $\mathring{L}^{r'}(\Omega)$ is a weakly closed subset of $L^{r'}(\Omega)$, the property $p_\Omega = 0$ has retained.

We have yet to identify the nonlinear terms $\overline{\boldsymbol{S}}$ and $\overline{\beta\boldsymbol{v}}$. The objective is to verify the pointwise convergence of p^ε and $\boldsymbol{D}\boldsymbol{v}^\varepsilon$. Then $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ and $\overline{\beta\boldsymbol{v}} = T_K\beta\big(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2\big)\boldsymbol{T}_K\boldsymbol{v}$ by $(25)_5$, $(25)_6$ and Vitali's theorem. It suffices to prove these pointwise convergences in an arbitrary compactly contained subdomain $\Omega' \Subset \Omega$.

Convergence of p^ε. Let $\eta \in \mathcal{C}^\infty_c(\Omega)$ be such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in Ω'. Recall the operator $\mathcal{N}$ from Lemma 4.4 and set $u^\varepsilon = \mathcal{N}((p^\varepsilon - p)\eta)$. Note

$$u^\varepsilon \to 0 \quad \text{weakly in } W^{2,2}(\Omega), \tag{26}$$

$$u^\varepsilon \to 0 \quad \text{strongly in } W^{1,2}(\Omega) \tag{27}$$

by the continuity of $\mathcal{N}$ and a compact embedding, respectively. Now

$$\left\|(p^\varepsilon - p)\eta\right\|_2^2 = -\int_\Omega p^\varepsilon \eta \Delta u^\varepsilon \, dx + \int_\Omega p\eta\Delta u^\varepsilon \, dx, \tag{28}$$

and the second integral tends to zero as $\varepsilon \to 0_+$ by (26). We develop the first term:

$$-\int_\Omega p^\varepsilon \eta \Delta u^\varepsilon \, dx = -\int_\Omega p^\varepsilon \operatorname{div}(\eta \nabla u^\varepsilon) \, dx + \int_\Omega p^\varepsilon \nabla \eta \cdot \nabla u^\varepsilon \, dx.$$

As $\varepsilon \to 0_+$ the second term again approaches zero by (27). At this moment the reason for adding η is becoming apparent, namely to ensure the zero trace of $\eta\nabla u^\varepsilon$. Onwards, by (16) we have

$$-\int_\Omega p^\varepsilon \operatorname{div}(\eta\nabla u^\varepsilon) \, dx = -\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \nabla(\eta\nabla u^\varepsilon) \, dx - \int_\Omega \beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon \cdot \eta \nabla u^\varepsilon \, dx + \int_\Omega \boldsymbol{F} \cdot \nabla(\eta\nabla u^\varepsilon) \, dx.$$

The latter two terms tend to zero by (26) and (27), as $\{\beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon\}$ is still bounded in $L^\infty(\Omega)$. Further

$$-\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \nabla(\eta\nabla u^\varepsilon) \, dx = -\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \eta \nabla^2 u^\varepsilon \, dx - \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot (\nabla u^\varepsilon \otimes \nabla \eta) \, dx$$

and the last term converges to zero by (27). Lastly

$$-\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \eta \nabla^2 u^\varepsilon \, dx = -\int_\Omega \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) \cdot \eta \nabla^2 u^\varepsilon \, dx + \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \eta \nabla^2 u^\varepsilon \, dx.$$

The first integral on the right-hand side vanishes for $\varepsilon \to 0_+$ by (26). The second one will be handled by means of the pointwise estimate (8) as

$$\begin{aligned}&\int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \eta \nabla^2 u^\varepsilon \, dx \\ &\qquad \le \gamma_0 \int_\Omega |(p - p^\varepsilon)\eta||\nabla^2 u^\varepsilon| \, dx + C_2 \int_\Omega \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)||\nabla^2 u^\varepsilon| \eta \, ds \, dx,\end{aligned} \tag{29}$$

with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v}^\varepsilon + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}^\varepsilon)$. Denote

$$I^\varepsilon = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2 ds.$$

Since $(1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \le (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$ and $\eta \le \sqrt{\eta}$, Hölder's inequality and Lemma 4.4 applied on (29) yield

$$\begin{aligned}\int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \eta \nabla^2 u^\varepsilon \, dx &\le \gamma_0 \left\|(p^\varepsilon - p)\eta\right\|_2^2 + C_2 \Big(\int_\Omega I^\varepsilon \eta \, dx\Big)^{1/2} \left\|(p^\varepsilon - p)\eta\right\|_2 \\ &\le \frac{1 + \gamma_0}{2} \left\|(p^\varepsilon - p)\eta\right\|_2^2 + \frac{C_2^2}{2(1 - \gamma_0)} \left\|I^\varepsilon \eta\right\|_1.\end{aligned} \tag{30}$$

It remains to estimate $\left\|I^\varepsilon \eta\right\|_1$. Using (7), we have

$$\frac{C_1}{2} \left\|I^\varepsilon \eta\right\|_1 \le \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx + \frac{\gamma_0^2}{2C_1} \left\|(p^\varepsilon - p)\eta\right\|_2^2. \tag{31}$$

The property (10) and the convergence $(25)_1$ yield

$$\lim_{\varepsilon \to 0_+} \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx = 0.$$

Towards handling the other integral in (31), we set $\boldsymbol{\varphi}_\varepsilon = (\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta$ and write

$$\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx = \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \boldsymbol{D}\boldsymbol{\varphi}_\varepsilon \, dx - \int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot (\nabla\eta \otimes (\boldsymbol{v} - \boldsymbol{v}^\varepsilon)) \, dx.$$

The latter integral vanishes for $\varepsilon \to 0_+$ by (25). As for the former, we employ the weak formulation (16) tested with $\boldsymbol{\varphi}_\varepsilon = (\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta$:

$$\int_\Omega \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon) \cdot \boldsymbol{D}\boldsymbol{\varphi}_\varepsilon \, dx = \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi}_\varepsilon \, dx - \int_\Omega \beta^\varepsilon \boldsymbol{T}_K \boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}_\varepsilon \, dx + \int_\Omega \boldsymbol{F} \cdot \nabla\boldsymbol{\varphi}_\varepsilon \, dx. \tag{32}$$

The last two terms vanish for $\varepsilon \to 0_+$ by (25). As for the first one, we recall eq. (15) and write

$$\begin{aligned}
\int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi}_\varepsilon \, dx &= -\int_\Omega p^\varepsilon \eta \operatorname{div} \boldsymbol{v}^\varepsilon \, dx + \int_\Omega p^\varepsilon (\boldsymbol{v} - \boldsymbol{v}^\varepsilon) \cdot \nabla\eta \, dx \\
&= \varepsilon \int_\Omega \nabla(p^\varepsilon \eta) \cdot \nabla p^\varepsilon \, dx + \int_\Omega p^\varepsilon (\boldsymbol{v} - \boldsymbol{v}^\varepsilon) \cdot \nabla\eta \, dx \\
&= \varepsilon \int_\Omega |\nabla p^\varepsilon|^2 \eta \, dx + \varepsilon \int_\Omega p^\varepsilon \nabla p^\varepsilon \cdot \nabla\eta \, dx + \int_\Omega p^\varepsilon (\boldsymbol{v} - \boldsymbol{v}^\varepsilon) \cdot \nabla\eta \, dx.
\end{aligned}$$

From the convergences (25), we hence elicit

$$\liminf_{\varepsilon \to 0_+} \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi}_\varepsilon \, dx \geq 0.$$

Plugging this result into (32), the entire first integral on the right in (31) therefore satisfies

$$\limsup_{\varepsilon \to 0_+} \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx \leq 0. \tag{33}$$

Inserting this information back into (30) and recalling the steps starting from (28), we conclude

$$\limsup_{\varepsilon \to 0_+} \big\|(p^\varepsilon - p)\eta\big\|_2^2 \leq \left(\frac{1 + \gamma_0}{2} + \frac{C_2^2 \gamma_0^2}{2C_1^2(1 - \gamma_0)}\right) \limsup_{\varepsilon \to 0_+} \big\|(p^\varepsilon - p)\eta\big\|_2^2.$$

Hence

$$\lim_{\varepsilon \to 0_+} \big\|(p^\varepsilon - p)\eta\big\|_2 = 0 \tag{34}$$

as long as

$$\frac{1 + \gamma_0}{2} + \frac{C_2^2 \gamma_0^2}{2C_1^2(1 - \gamma_0)} < 1,$$

which corresponds to the condition $\gamma_0 < C_1/(C_1 + C_2)$; see Assumption 2.2.

Convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$. What remains is to prove the strong convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ (for a subsequence at least). For $r \in (1, 2]$ we may invoke Hölder's inequality and calculate

$$\begin{aligned}
\big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\eta\big\|_r^r &= \int_\Omega \Big(\int_0^1 (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2)^{(r-2)/2} \\
&\qquad \times (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2)^{(2-r)/2} |\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|^2 \, ds\Big)^{r/2} \eta^r \, dx \\
&\leq \int_\Omega \Big(\int_0^1 (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|^2 \, ds\Big)^{r/2} \eta^{r/2} \\
&\qquad \times (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r(2-r)/4} \, dx \\
&\leq \Big(\int_\Omega I^\varepsilon \eta \, dx\Big)^{r/2} \Big(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2} dx\Big)^{(2-r)/2}.
\end{aligned} \tag{35}$$

Recalling (7) and (22), we have thus deduced a useful (though standard) estimate

$$C\big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\eta\big\|_r^2 \le \int_\Omega (\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(p^\varepsilon, \boldsymbol{D}\boldsymbol{v}^\varepsilon)) \cdot \boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)\eta \, dx + \frac{\gamma_0^2}{2C_1}\big\|(p^\varepsilon - p)\eta\big\|_2^2,$$

which, together with (33) and (34), implies the required convergence

$$\lim_{\varepsilon\to 0_+} \big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\eta\big\|_r = 0.$$

Hence we have obtained $(\boldsymbol{v}, p) = (\boldsymbol{v}^K, p^K) \in W^{1,r}_{0,\mathrm{div}}(\Omega) \times \mathring{L}^{r'}(\Omega)$, satisfying

$$\int_\Omega \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) \cdot \boldsymbol{D}\boldsymbol{\varphi} \, dx + \int_\Omega T_K\beta(p^K, |\boldsymbol{v}^K|, |\boldsymbol{D}\boldsymbol{v}^K|^2)\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{\varphi} \, dx - \int_\Omega p^K \operatorname{div} \boldsymbol{\varphi} \, dx$$
$$= \int_\Omega \boldsymbol{F} \cdot \nabla\boldsymbol{\varphi} \, dx \quad \text{for every } \boldsymbol{\varphi} \in W^{1,r}_0(\Omega). \tag{36}$$

5.2 Truncation removal ($K \to \infty$)

The final and key part concerns the limit $K \to \infty$. The essential procedures at this phase will lie in a decomposition of the pressures p^K, followed by an interesting application of the Div-Curl lemma.

Uniform estimates. Let us pick $\boldsymbol{\varphi} = \boldsymbol{v}^K$ in the relation (36), as in the previous step. Exactly like in (22) and (23), we obtain bounds

$$\sup_K \left(\big\|\boldsymbol{v}^K\big\|_{1,r} + \big\|\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)\big\|_{r'}\right) < \infty \tag{37}$$

and, denoting $\beta^K := T_K\beta\big(p^K, |\boldsymbol{v}^K|, |\boldsymbol{D}\boldsymbol{v}^K|^2\big)$, now by non-negativity of β also

$$\sup_K \big\|\beta^K|\boldsymbol{T}_K\boldsymbol{v}^K|^2\big\|_1 \le \sup_K \big\|\beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{v}^K\big\|_1 < \infty. \tag{38}$$

Recall that for each $K \in \mathbb{N}$, $p^K \in L^{r'}(\Omega) \hookrightarrow L^{d'}(\Omega)$. The pressure will be uniformly estimated in the latter space, once again by dint of the Bogovskiĭ operator. This is where we finally give reason for the growth conditions in Assumption 2.3. Set

$$\boldsymbol{\varphi} = \mathcal{B}\Big(|p^K|^{d'-2}p^K - \big(|p^K|^{d'-2}p^K\big)_\Omega\Big).$$

Note that $\boldsymbol{\varphi} \in W^{1,d}_0(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1,\infty)$ and $\big\|\boldsymbol{\varphi}\big\|_{1,d} \le C\big\|p^K\big\|_{d'}^{d'-1}$ due to the continuity of $\mathcal{B}$. Using $\boldsymbol{\varphi}$ as a test function in (36) yields

$$\big\|p^K\big\|_{d'}^{d'} = \int_\Omega \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) : \boldsymbol{D}\boldsymbol{\varphi} \, dx + \int_\Omega \beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{\varphi} \, dx - \int_\Omega \boldsymbol{F} \cdot \nabla\boldsymbol{\varphi} \, dx. \tag{39}$$

Next,

$$\int_\Omega \big|\beta^K\boldsymbol{T}_K\boldsymbol{v}^K \cdot \boldsymbol{\varphi}\big| \, dx \le \Big(\int_\Omega \beta^K\big|\boldsymbol{T}_K\boldsymbol{v}^K\big|^2 dx\Big)^{1/2}\Big(\int_\Omega \beta^K|\boldsymbol{\varphi}|^2 dx\Big)^{1/2} \tag{40}$$

and owing to (38) the first term is bounded. Hence we can focus purely on the last integral in (40). Recalling Assumption 2.3 (with $q_0 < d'$, $q_1 < r^*$ and $q_2 < r$) and the classical Sobolev embedding, we estimate it as follows:

$$\begin{aligned}\int_\Omega \beta^K|\boldsymbol{\varphi}|^2 dx &\le c\int_\Omega |\boldsymbol{\varphi}|^2(1 + |p^K|^{q_0} + |\boldsymbol{v}^K|^{q_1} + |\boldsymbol{D}\boldsymbol{v}^{\boldsymbol{K}}|^{q_2}) \, dx \\ &\le C\big(1 + \big\|p^K\big\|_{d'}^{q_0} + \big\|\boldsymbol{v}^K\big\|_{r^*}^{q_1} + \big\|\boldsymbol{D}\boldsymbol{v}^K\big\|_r^{q_2}\big) \, \|\boldsymbol{\varphi}\|_{1,d}^2 \,.\end{aligned} \tag{41}$$

Combining with (37), the above computation amounts to

$$\int_\Omega \left|\beta^K \boldsymbol{T}_K \boldsymbol{v}^K \cdot \boldsymbol{\varphi}\right| dx \le C\|\boldsymbol{\varphi}\|_{1,d}\big(1 + \|p^K\|_{d'}^{q_0/2}\big),$$

with C independent of K. In light of $W_0^{1,d}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ and (37), eq. (39) gives rise to

$$\|p^K\|_{d'}^{d'} \le C\|\boldsymbol{\varphi}\|_{1,d}\big(1 + \|p^K\|_{d'}^{q_0/2}\big) \le C\|p^K\|_{d'}^{d'-1}\big(1 + \|p^K\|_{d'}^{q_0/2}\big).$$

Since $q_0 < d' \le 2$, we have arrived at

$$\sup_K \|p^K\|_{d'} < \infty. \tag{42}$$

We also observe that (40) and (41) would just as well work with $\boldsymbol{\varphi} \in L^{1+1/\delta}(\Omega)$ for some small $\delta > 0$, whence

$$\sup_K \left\|\beta^K \boldsymbol{T}_K \boldsymbol{v}^K\right\|_{1+\delta} < \infty. \tag{43}$$

Notice that the right-open intervals for q_0, q_1 and q_2 from Assumption 2.3 are indispensable for such a claim. It is hence possible by the estimates (37), (42) and (43), to let $K \to \infty$ and presuppose (after a pertinent relabelling of the sequence) that

$$\begin{aligned}
\boldsymbol{v}^K &\to \boldsymbol{v} && \text{weakly in } W_{0,\mathrm{div}}^{1,r}(\Omega),\\
\boldsymbol{v}^K &\to \boldsymbol{v} && \text{a.e. in } \Omega,\\
p^K &\to p && \text{weakly in } \mathring{L}^{d'}(\Omega),\\
\boldsymbol{S}(p^K, \boldsymbol{D}(\boldsymbol{v}^K)) &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(\Omega),\\
\beta^K \boldsymbol{T}_K \boldsymbol{v}^K &\to \overline{\beta \boldsymbol{v}} && \text{weakly in } L^{1+\delta}(\Omega).
\end{aligned} \tag{44}$$

From (36) we have moved on to

$$\int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{\varphi}\, dx + \int_\Omega \overline{\beta \boldsymbol{v}} \cdot \boldsymbol{\varphi}\, dx - \int_\Omega p \operatorname{div} \boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}\, dx \quad \text{for any } \boldsymbol{\varphi} \in W_0^{1,r}(\Omega) \cap L^\infty(\Omega)$$

with $\operatorname{div} \boldsymbol{\varphi} \in L^d(\Omega)$. Not unlike the limit $\varepsilon \to 0_+$, the identification of the weak limits $\overline{\boldsymbol{S}}$ and $\overline{\beta \boldsymbol{v}}$ can and will be performed via the pointwise convergence of p^K and $\boldsymbol{D}\boldsymbol{v}^K$.

Decomposition of p^K. Beginning with the pressure, for which we would like to utilize the monotonicity relation (7), we run into trouble as $\{p^K\}$ need not be bounded in $L^2(\Omega)$. This is why we decompose p^K into two parts: one being pointwise convergent and the other still converging only weakly, though now in $L^{r'}(\Omega)$, whence the monotonicity property may be used. It is again sufficient to prove the convergence in an arbitrary compactly contained subdomain $\Omega' \Subset \Omega$.

Referring back to Lemma 4.2 and noticing that both $\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \operatorname{div} \boldsymbol{F}$ and $\beta^K \boldsymbol{T}_K \boldsymbol{v}^K$ belong to $W^{-1,2}(\Omega)$, we may define

$$\begin{aligned}
(\boldsymbol{v}_1^K, p_1^K) &:= \mathcal{H}(\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \operatorname{div} \boldsymbol{F}),\\
(\boldsymbol{v}_2^K, p_2^K) &:= \mathcal{H}(-\beta^K \boldsymbol{T}_K \boldsymbol{v}^K).
\end{aligned} \tag{45}$$

The uniqueness of solutions to the Stokes problem and (36) imply

$$\begin{aligned}
\boldsymbol{v}_1^K + \boldsymbol{v}_2^K &= 0,\\
p_1^K + p_2^K &= p^K.
\end{aligned} \tag{46}$$

From (37) and the continuity of $\mathcal{H}$ we observe

$$\sup_K \left(\|\boldsymbol{v}_1^K\|_{1,2} + \|p_1^K\|_2\right) < \infty. \tag{47}$$

Further, tacitly assuming $\delta \le 1/(d-1)$, we may apply (11) to $(45)_2$ with $k = 0$ and deduce

$$\begin{aligned}\|\nabla^2 \boldsymbol{v}_2^K\|_{1+\delta;\Omega'} + \|\nabla p_2^K\|_{1+\delta;\Omega'} &\le c\left(\|\beta^K \boldsymbol{T}_K \boldsymbol{v}^K\|_{1+\delta} + \|\boldsymbol{v}_2^K\|_{1,1+\delta} + \|p_2^K\|_{1+\delta}\right)\\ &\le c\left(\|\beta^K \boldsymbol{T}_K \boldsymbol{v}^K\|_{1+\delta} + \|\boldsymbol{v}_1^K\|_{1,2} + \|p^K - p_1^K\|_{d'}\right)\\ &\le C,\end{aligned}$$

where C is independent of K. Now consider $r < 2$ and Ω'', Ω''' satisfying $\Omega' \Subset \Omega'' \Subset \Omega''' \Subset \Omega$. Since $r' > 2$ we elicit the existence of a $\sigma > 0$ such that (11) may be employed again, this time with $k = -1$, leading to

$$\begin{aligned}\|\boldsymbol{v}_1^K\|_{1,2+\sigma;\Omega''} + \|p_1^K\|_{2+\sigma;\Omega''} &\le c\left(\|\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \operatorname{div} \boldsymbol{F}\|_{-1,2+\sigma} + \|\boldsymbol{v}_1^K\|_{2+\sigma;\Omega'''} + \|p_1^K\|_{-1,2+\sigma;\Omega'''}\right)\\ &\le c\left(\|\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)\|_{-1,r'} + \|\boldsymbol{F}\|_{r'} + \|\boldsymbol{v}_1^K\|_{1,2;\Omega'''} + \|p_1^K\|_{2;\Omega'''}\right)\\ &\le C;\end{aligned}$$

the last estimate is due to (47). Utilizing the bootstrap argument, the above estimate yields

$$\|\boldsymbol{v}_1^K\|_{1,r';\Omega'} + \|p_1^K\|_{r';\Omega'} \le c\left(\|\operatorname{div} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)\|_{-1,r'} + \|\boldsymbol{F}\|_{r'} + \|\boldsymbol{v}_1^K\|_{1,2} + \|p_1^K\|_2\right) \le C.$$

In other words, using (46) we observe,

$$\sup_K \left(\|\boldsymbol{v}_1^K\|_{1,r';\Omega'} + \|\boldsymbol{v}_1^K\|_{2,1;\Omega'} + \|p_1^K\|_{r';\Omega'} + \|p_2^K\|_{1,1;\Omega'}\right) < \infty, \tag{48}$$

for any $r \in (1, 2]$, as the case $r = 2$ is covered directly by (47). Hence we may assume

$$\begin{aligned} p_1^K &\to p_1 \quad \text{weakly in } \mathring{L}^{r'}(\Omega'),\\ p_2^K &\to p_2 \quad \text{a.e. in } \Omega'.\end{aligned} \tag{49}$$

Note that (46) yields trivially $p_1 + p_2 = p$. What we are left with is thus to show the pointwise convergence of p_1^K.

Convergence of p_1^K. We first notice that (45) and (48) imply

$$\left\|\operatorname{div}\left(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I} - \boldsymbol{F}\right)\right\|_{1;\Omega'} \le C.$$

As $L^1(\Omega') \hookrightarrow\hookrightarrow W^{-1,q'}(\Omega')$ for $q > d$, this estimate together with (44) and (49) allows us to use Div-Curl lemma 4.6. Indeed, let $s > r$ and

$$\boldsymbol{\varphi}^K \to \boldsymbol{\varphi} \quad \text{weakly in } W^{1,s}(\Omega'). \tag{50}$$

Then $1/r' + 1/s < 1$, $\operatorname{curl} \nabla \boldsymbol{\varphi}^K = 0$ and Div-Curl lemma 4.6 implies

$$\left(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\right) \cdot \nabla \boldsymbol{\varphi}^K \to \left(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\right) \cdot \nabla \boldsymbol{\varphi} \quad \text{weakly in } L^1(\Omega'). \tag{51}$$

Note we also tacitly used $\boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}^K \to \boldsymbol{F} \cdot \nabla \boldsymbol{\varphi}$ weakly in $L^1(\Omega')$.

Let $L > 0$. We shall first consider (51) with $\boldsymbol{\varphi}^K = \nabla \psi_L^K$, where (see Lemma 4.4 for notation)

$$\psi_L^K = \mathcal{N}(T_L(p_1^K - p_1)). \tag{52}$$

Note that due to the truncation, we have (for a subsequence if need be)

$$T_L(p_1^K - p_1) \to \overline{T}_L \quad \text{weakly in } L^q(\Omega) \text{ for all } q \in [1, \infty),$$

and hence by the continuity of $\mathcal{N}$ (see Lemma 4.4) also

$$\psi_L^K \to \psi_L = \mathcal{N}(\overline{T}_L) \quad \text{weakly in } W^{2,q}(\Omega) \text{ for all } q \in [1, \infty). \tag{53}$$

Therefore (51) yields

$$\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla^2 \psi_L^K \to \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla^2 \psi_L \quad \text{weakly in } L^1(\Omega'),$$

which, after a simple rearrangement and using the pointwise convergence of p_2^K from (49), leads to

$$\begin{aligned}\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v}) - (p_1^K - p_1)\boldsymbol{I}\big) \cdot \nabla^2 \psi_L^K& \\ \to \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p_1 + p_2, \boldsymbol{D}\boldsymbol{v})\big) \cdot \nabla^2 \psi_L \quad &\text{weakly in } L^1(\Omega').\end{aligned}$$

As a result, recalling also the definition of ψ_L^K, we find that for an arbitrary measurable $\Omega'' \subset \Omega'$

$$\begin{aligned}&\limsup_{K\to\infty} \int_{\Omega''} |p_1^K - p_1||T_L(p_1^K - p_1)|\, dx = \limsup_{K\to\infty} \int_{\Omega''} (p_1^K - p_1)\, \boldsymbol{I} \cdot \nabla^2 \psi_L^K\, dx \\ &\le \limsup_{K\to\infty} \int_{\Omega''} |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})||\nabla^2 \psi_L^K|\, dx + \Big|\int_{\Omega''} \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\big) \cdot \nabla^2 \psi_L\, dx\Big|. \end{aligned}\tag{54}$$

Towards estimating the first term on the right-hand side, the relation (8) implies that

$$\begin{aligned}&\int_{\Omega''} |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})||\nabla^2 \psi_L^K|\, dx \\ &\quad\le \gamma_0 \int_{\Omega''} |p_1^K - p_1||\nabla^2 \psi_L^K|\, dx + C_2 \int_{\Omega''} \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})||\nabla^2 \psi_L^K|\, ds\, dx, \end{aligned}\tag{55}$$

where $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v}^K + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}^K)$. Denoting

$$I^K = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})|^2 ds,$$

and using Hölder's inequality and $(1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \le (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$, we turn (55) into

$$\begin{aligned}&\int_{\Omega''} |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})||\nabla^2 \psi_L^K|\, dx \\ &\qquad\le \gamma_0 \left\|p_1^K - p_1\right\|_{2;\Omega''} \left\|\nabla^2 \psi_L^K\right\|_{2;\Omega''} + C_2 \left\|I^K\right\|_{1;\Omega''}^{1/2} \left\|\nabla^2 \psi_L^K\right\|_{2;\Omega''}.\end{aligned}$$

Hence we are able to develop (54) as

$$\begin{aligned}&\limsup_{K\to\infty} \int_{\Omega''} |p_1^K - p_1||T_L(p_1^K - p_1)|\, dx \\ &\quad\le \limsup_{K\to\infty} \Big(\gamma_0 \left\|p_1^K - p_1\right\|_{2;\Omega''} \left\|\nabla^2 \psi_L^K\right\|_{2;\Omega''} + C_2 \left\|I^K\right\|_{1;\Omega''}^{1/2} \left\|\nabla^2 \psi_L^K\right\|_{2;\Omega''}\Big) \\ &\qquad + \Big|\int_{\Omega''} \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\big) \cdot \nabla^2 \psi_L\, dx\Big|. \end{aligned}\tag{56}$$

Next, due to (53) we may assume without loss of generality that both $|\nabla^2\psi_L^K|^2$ and $|\Delta\psi_L^K|^2$ converge weakly in $L^q(\Omega')$ for any $q \in [1,\infty)$ as $k \to \infty$. To compare these weak limits, it suffices to investigate

$$\lim_{K\to\infty} \int_{\Omega'} \left(|\nabla^2\psi_L^K|^2 - |\Delta\psi_L^K|^2\right) \varphi \, dx$$

for arbitrary $\varphi \in \mathcal{C}_c^\infty(\Omega')$. Using the integration by parts, we find that

$$\begin{aligned}
&\lim_{K\to\infty} \int_{\Omega'} \left(|\nabla^2\psi_L^K|^2 - |\Delta\psi_L^K|^2\right)\varphi \, dx \\
&\qquad = \lim_{K\to\infty} \int_{\Omega'} \left(\nabla^2\psi_L^K \cdot \nabla^2\psi_L^K \varphi - |\Delta\psi_L^K|^2\varphi\right) dx \\
&\qquad = \lim_{K\to\infty} \int_{\Omega'} \left(-\nabla\psi_L^K \cdot \nabla\Delta\psi_L^K \varphi - (\nabla\psi_L^K \otimes \nabla\varphi) \cdot \nabla^2\psi_L^K - |\Delta\psi_L^K|^2\varphi\right) dx \\
&\qquad = \lim_{K\to\infty} \int_{\Omega'} \left(\nabla\psi_L^K \cdot \nabla\varphi\Delta\psi_L^K - (\nabla\psi_L^K \otimes \nabla\varphi) \cdot \nabla^2\psi_L^K\right) dx \\
&\qquad = \int_{\Omega'} \left(\nabla\psi_L \cdot \nabla\varphi\Delta\psi_L - (\nabla\psi_L \otimes \nabla\varphi) \cdot \nabla^2\psi_L\right) dx \\
&\qquad = \int_{\Omega'} \left(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\right)\varphi \, dx.
\end{aligned}$$

By the density argument, we therefore get for all measurable $\Omega'' \subset \Omega'$

$$\lim_{K\to\infty} \int_{\Omega''} \left(|\nabla^2\psi_L^K|^2 - |\Delta\psi_L^K|^2\right) dx = \int_{\Omega''} \left(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\right) dx,$$

in particular then

$$\limsup_{K\to\infty} \int_{\Omega''} |\nabla^2\psi_L^K|^2 \, dx \le \limsup_{K\to\infty} \int_{\Omega''} |\Delta\psi_L^K|^2 \, dx + \int_{\Omega''} \left(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\right) dx.$$

Hence, substituting this relation into (56), using the pointwise estimate

$$|\Delta\psi_L^K|^2 = |T_L(p_1^K - p_1)|^2 \le |p_1^K - p_1|^2$$

and the a priori estimates (44) and (47), we find out

$$\begin{aligned}
\limsup_{K\to\infty} \int_{\Omega''} |p_1^K - p_1||T_L(p_1^K - p_1)| \, dx &\le \limsup_{K\to\infty} \left(\gamma_0 \left\|p_1^K - p_1\right\|_{2;\Omega''} + C_2 \left\|I^K\right\|_{1;\Omega''}^{1/2}\right) \\
&\times \left(\left\|p_1^K - p_1\right\|_{2;\Omega''}^2 + \int_{\Omega''} \left(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\right) dx\right)^{1/2} + \left|\int_{\Omega''} \left(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\right) \cdot \nabla^2\psi_L \, dx\right| \\
&\le \limsup_{K\to\infty} \left(\gamma_0 \left\|p_1^K - p_1\right\|_{2;\Omega''}^2 + C_2 \left\|I^K\right\|_{1;\Omega''}^{1/2} \left\|p_1^K - p_1\right\|_{2;\Omega''}\right) \\
&+ C\left|\int_{\Omega''} \left(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\right) dx\right|^{1/2} + \left|\int_{\Omega''} \left(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})\right) \cdot \nabla^2\psi_L \, dx\right|.
\end{aligned} \tag{57}$$

Finally, we choose Ω'' so that the truncator T_L could be disregarded. For this sake recall the Biting lemma 4.5 that we are going to apply to

$$f^K = |p_1^K|^{r'} + |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)|^{r'}. \tag{58}$$

Note that $\{f^K\}$ form a bounded sequence in $L^1(\Omega')$ by (44) and (48). Hence, the Biting lemma guarantees the existence of a nonincreasing sequence of measurable sets $E_n \subset \Omega'$ fulfilling $\lim_{n\to\infty} |E_n| = 0$ such that (modulo a subsequence) for each $n \in \mathbb{N}$ the sequence $\{f^K\}$ is uniformly equi-integrable in $\Omega_n := \Omega' \setminus E_n$.

The estimate (57) with $\Omega'' = \Omega_n$ entails for each $n \in \mathbb{N}$

$$
\begin{aligned}
\limsup_{K\to\infty} \|p_1^K - p_1\|_{2;\Omega_n}^2 \le \limsup_{K\to\infty} \int_{\Omega_n} |p_1^K - p_1||p_1^K - p_1 - T_L(p_1^K - p_1)|\,dx \\
+ \limsup_{K\to\infty} \Big(\gamma_0 \|p_1^K - p_1\|_{2;\Omega_n}^2 + C_2 \|I^K\|_{1;\Omega_n}^{1/2} \|p_1^K - p_1\|_{2;\Omega_n}\Big) \qquad (59) \\
+ C\Big|\int_{\Omega_n} \big(|\nabla^2\psi_L|^2 - |\Delta\psi_L|^2\big)\,dx\Big|^{1/2} + C\Big|\int_{\Omega_n} \big(\overline{\boldsymbol{S}} - \boldsymbol{S}(p, \boldsymbol{Dv})\big)\cdot\nabla^2\psi_L\,dx\Big|.
\end{aligned}
$$

We further let $L \to \infty$ in order to eliminate the terms depending on L. Denoting $\Omega_L^K = \{|p_1^K - p_1| > L\}$, we observe from (47) that $|\Omega_L^K| \le C/L^2$ whence, using the uniform equi-integrability of $|p_1^K|^{r'}$ in Ω_n,

$$
\limsup_{L\to\infty}\limsup_{K\to\infty} \int_{\Omega_n} |p_1^K - p_1||p_1^K - p_1 - T_L(p_1^K - p_1)|\,dx \le \limsup_{L\to\infty}\limsup_{K\to\infty} \int_{\Omega_n\cap\Omega_L^K} |p_1^K - p_1|^2\,dx = 0.
$$

At this point we wish to highlight the importance of the uniform equi-integrability of $|p_1^K|^2$. For $r < 2$ it would be trivial from (49), whereas when $r = 2$, the Biting lemma seems to be essential.

The remaining L-dependent terms in (59) tend with $L \to \infty$ likewise to zero, towards which it is evidently enough to prove

$$
\psi_L \to 0 \quad \text{strongly in } W^{2,2}(\Omega). \qquad (60)
$$

Due to continuity of the Newtonian potential $\mathcal{N}$ (see Lemma 4.4), the problem (52) implies that (60) holds so long as (see (53))

$$
\overline{T}_L \to 0 \quad \text{strongly in } L^2(\Omega). \qquad (61)
$$

To achieve this, we first draw from (47) that

$$
T_L(p_1^K - p_1) - (p_1^K - p_1) \to \overline{T}_L \quad \text{weakly in } L^2(\Omega)
$$

and therefore from the weak lower semicontinuity of the L^1-norm we find that

$$
\|\overline{T}_L\|_1 \le \liminf_{K\to\infty} \|T_L(p_1^K - p_1) - (p_1^K - p_1)\|_1 \le 2\limsup_{K\to\infty} \int_{\Omega_L^K} |p_1^K - p_1|\,dx \le C/L,
$$

whence

$$
\overline{T}_L \to 0 \quad \text{strongly in } L^1(\Omega).
$$

To strenghten the strong convergence from $L^1(\Omega)$ into $L^2(\Omega)$, it is enough to find a dominating function belonging to $L^2(\Omega)$. However, denoting $\nu \in L^2(\Omega)$ the weak limit

$$
|p_1^K - p_1| \to \nu \quad \text{weakly in } L^2(\Omega),
$$

a simple estimate $|T_L(p_1^K - p_1)| \le |p_1^K - p_1|$ implies $|\overline{T}_L| \le \nu$ and Lebesgue's dominated convergence theorem finishes the proof of (61). As a consequence, we conclude from (59) that

$$
\limsup_{K\to\infty} \|p_1^K - p_1\|_{2;\Omega_n}^2 \le \limsup_{K\to\infty} \Big(\gamma_0 \|p_1^K - p_1\|_{2;\Omega_n}^2 + C_2 \|I^K\|_{1;\Omega_n}^{1/2} \|p_1^K - p_1\|_{2;\Omega_n}\Big),
$$

ultimately implying for each n (note that $\gamma_0 < 1$)

$$\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \le \frac{C_2}{1-\gamma_0} \limsup_{K\to\infty} \left\|I^K\right\|_{1;\Omega_n}^{1/2}. \tag{62}$$

We want to develop (62) into

$$\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \le \alpha \limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \quad \text{for some } \alpha \in (0,1).$$

We are again going to utilize the observation based on the Div-Curl lemma (51). To this end, take a fixed $\lambda > 0$, recall Lemma 4.7 about Lipschitz approximations of Sobolev functions and set $\boldsymbol{\varphi}^K := \boldsymbol{v}_\lambda^K$ in (51), where $\boldsymbol{v}_\lambda^K$ denotes the Lipschitz approximation of $\boldsymbol{v}^K$. Note that due to (12), fulfillment of the condition (50) may be taken for granted. Hence

$$\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}_\lambda^K \to \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \quad \text{weakly in } L^1(\Omega'), \tag{63}$$

where

$$\boldsymbol{v}_\lambda^K \to \overline{\boldsymbol{v}}_\lambda \quad \text{weakly in } W^{1,q}(\Omega) \text{ for all } q \in [1,\infty).$$

Note that (63) directly implies

$$\lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}_\lambda^K \, dx = \int_{\Omega_n} \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \, dx \tag{64}$$

for each n, where Ω_n are still the subsets specified above, when we applied the Biting lemma to the sequence given in (58). By (64), we have

$$\begin{aligned}
&\lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}^K \, dx \\
&\quad = \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla (\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K) \, dx + \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}_\lambda^K \, dx \\
&\quad = \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla (\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K) \, dx + \int_{\Omega_n} \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \, dx,
\end{aligned}$$

and consequently, as the left-hand side is independent of λ,

$$\begin{aligned}
&\lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}^K \, dx \\
&\qquad = \lim_{\lambda\to\infty} \lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla (\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K) \, dx + \lim_{\lambda\to\infty} \int_{\Omega_n} \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \overline{\boldsymbol{v}}_\lambda \, dx.
\end{aligned} \tag{65}$$

First we notice the first term on the right vanishes. Indeed, thanks to (14) we have

$$\int_{\Omega_n} \left|\big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla (\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K)\right| \, dx \le C \left\|\boldsymbol{v}^K\right\|_{1,r} \left\|\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\right\|_{r';\Omega_n \cap \{\boldsymbol{v}^K \neq \boldsymbol{v}_\lambda^K\}}$$

and the claim follows from the uniform equi-integrability of

$$|p_1^K|^{r'} + |\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K)|^{r'}$$

in Ω_n (see (58)) and (13), i.e. $|\{\boldsymbol{v}^K \neq \boldsymbol{v}_\lambda^K\}| \le C/\lambda^r$.

The second term on the right-hand side of (65) can easily be identified: The weak lower semicontinuity of a norm and (14) bring about

$$\|\overline{\boldsymbol{v}}_\lambda\|_{1,r} \leq \liminf_{K\to\infty} \left\|\boldsymbol{v}_\lambda^K\right\|_{1,r} \leq C \limsup_{K\to\infty} \left\|\boldsymbol{v}^K\right\|_{1,r} \leq C.$$

Accordingly, we may safely assume

$$\overline{\boldsymbol{v}}_\lambda \to \overline{\boldsymbol{v}} \quad \text{weakly in } W^{1,r}(\Omega).$$

On the other hand, it follows from the compact embedding, (13) and (14) that

$$\|\boldsymbol{v} - \overline{\boldsymbol{v}}_\lambda\|_1 = \lim_{K\to\infty} \int_\Omega |\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K|\, dx = \lim_{K\to\infty} \int_{\{\boldsymbol{v}^K \neq \boldsymbol{v}_\lambda^K\}} |\boldsymbol{v}^K - \boldsymbol{v}_\lambda^K|\, dx \leq C/\lambda^{r-1},$$

meaning

$$\overline{\boldsymbol{v}}_\lambda \to \boldsymbol{v} \quad \text{strongly in } L^1(\Omega),$$

and finally, due to uniqueness of weak limits

$$\overline{\boldsymbol{v}}_\lambda \to \boldsymbol{v} \quad \text{weakly in } W^{1,r}(\Omega).$$

Thus we are able to pass $\lambda \to \infty$ on the right-hand side of (65), obtaining

$$\lim_{K\to\infty} \int_{\Omega_n} \big(\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - p_1^K \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}^K dx = \int_{\Omega_n} \big(\overline{\boldsymbol{S}} - p_1 \boldsymbol{I}\big) \cdot \nabla \boldsymbol{v}\, dx.$$

Since $\boldsymbol{v}^K$ and $\boldsymbol{v}$ are both divergence-free, this is actualy tantamount to

$$\lim_{K\to\infty} \int_{\Omega_n} \boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) \cdot \boldsymbol{D}\boldsymbol{v}^K dx = \int_{\Omega_n} \overline{\boldsymbol{S}} \cdot \boldsymbol{D}\boldsymbol{v}\, dx$$

and thanks to the strong convergence of p_2^K (see (49)) also

$$\lim_{K\to\infty} \int_{\Omega_n} (\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})) \cdot \boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})\, dx = 0. \tag{66}$$

At long last, recalling (7) we see that (66) implies

$$\limsup_{K\to\infty} \left\|I^K\right\|_{1;\Omega_n} \leq \frac{\gamma_0^2}{C_1^2} \limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n}^2. \tag{67}$$

Substituting (67) into (62), we have

$$\limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} \leq \frac{C_2\gamma_0}{C_1(1-\gamma_0)} \limsup_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n}.$$

Considering

$$\frac{C_2\gamma_0}{C_1(1-\gamma_0)} < 1 \iff \gamma_0 < \frac{C_1}{C_1 + C_2},$$

by the Assumption 2.2 we have

$$\lim_{K\to\infty} \left\|p_1^K - p_1\right\|_{2;\Omega_n} = 0. \tag{68}$$

Since $\lim_{n\to\infty} |\Omega' \setminus \Omega_n| = 0$, we may suppose $p_1^K \to p_1$ a.e. in Ω'.

Convergence of $\boldsymbol{D}\boldsymbol{v}^K$. We have yet to affirm the strong convergence of $\boldsymbol{D}\boldsymbol{v}^K$. A simple task, in fact, for we can proceed just as in the ε-passage. Exactly like in (35), we would use Hölder's inequality to show

$$\|\boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})\|_{r;\Omega_n}^2 \leq C \int_{\Omega_n} \boldsymbol{I}^K dx,$$

whence by (7) also

$$C\|\boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})\|_{r;\Omega_n}^2 \leq \int_{\Omega_n} (\boldsymbol{S}(p^K, \boldsymbol{D}\boldsymbol{v}^K) - \boldsymbol{S}(p_1 + p_2^K, \boldsymbol{D}\boldsymbol{v})) \cdot \boldsymbol{D}(\boldsymbol{v}^K - \boldsymbol{v})\, dx + \frac{\gamma_0^2}{2C_1}\|p_1^K - p_1\|_{2;\Omega_n}^2.$$

The right-hand side tends to zero as $K \to \infty$ by (66) and (68). This fact implies we may assume $\boldsymbol{D}\boldsymbol{v}^K \to \boldsymbol{D}\boldsymbol{v}$ a.e. in Ω' and also eventually finishes the proof. □

6 Closing remarks

We would like to finish this paper with a theorem directly improving the results of [13] and [22]:

Theorem 6.1 *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Consider $\boldsymbol{f} \in W^{-1,r'}(\Omega)$, $p_0 \in \mathbb{R}$, $r \in (2d/(d+2), 2]$ and let Assumptions 2.1 and 2.2 hold. Enforcing a slightly strenghtened Assumption 2.3, namely let $q_0 \in \left[1, \min\left\{d', \frac{dr}{2(d-r)}\right\}\right)$, there exists a pair $(\boldsymbol{v}, p) \in W^{1,r}_{0,\mathrm{div}}(\Omega) \times L^{\min\left\{d', \frac{dr}{2(d-r)}\right\}}(\Omega)$ satisfying $p_\Omega = p_0$, $\beta(p, \boldsymbol{v}, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \in L^1(\Omega)$ and*

$$\int_\Omega \left[2\nu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v} \cdot \boldsymbol{D}\boldsymbol{\varphi} - (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla\boldsymbol{\varphi} + \beta(p, |\boldsymbol{v}|, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{v} \cdot \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi}\right] dx = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle$$

for every $\boldsymbol{\varphi} \in W_0^{1,\infty}(\Omega)$.

As insinuated in Section 2, we are not going to establish this result in detail. The proof would lie in a straightforward combination of the procedure implemented here and steps used in [13] to control the convective term. To be more specific, one would need a stronger version of the Lipschitz approximation lemma than Lemma 4.7, namely that from [18]. The second change would be in the decomposition of the pressure (45). Informally speaking, we would add one more partial pressure corresponding to the convective term, i.e. $(\boldsymbol{v}_3^K, p_3^K) := \mathcal{H}(-\operatorname{div}(\boldsymbol{v}^K \otimes \boldsymbol{v}^K))$. The new pressure would, like p_2^K, also converge pointwise due to estimates based on the regularity theory for the Stokes problem. In reality however, there would have to appear an additional regularizing term in the argument of $\mathcal{H}$; see [13] for details.

As far as the possible deterioration of the pressure integrability is concerned, the culprit is again the convective term. Note that $\frac{dr}{2(d-r)} < d'$ for $r < \frac{2d}{d+1}$, so that the exponent of integrability becomes worse for low values of r. If $\frac{dr}{2(d-r)} < d'$ then $p \in L^{\frac{dr}{2(d-r)}}(\Omega)$ and it is necessary to restrict growth of the drag β in the pressure accordingly, as the original $q_0 < d'$ from Assumption 2.3 requires being able to bound the pressure in $L^{d'}(\Omega)$. It is an easy exercise to perform variants of the estimates (39)–(42) again with the convective term present.

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