A mixed finite element method for Darcy's equations with pressure dependent porosity

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Introduction

Some applications of understanding behavior of fluid through porous medium

- Ground water pollution
- Mesoscale blood flows
- Filter design
- Enhanced oil recovery
- Carbon dioxide sequestration
- ...

Introduction

General formulation of (steady) fluid through porous media

$$\begin{cases} -\nabla \cdot \mathbb{T} + \boldsymbol{I} = \rho \boldsymbol{b}\,, \\ \nabla \cdot \boldsymbol{u} = 0\,, \end{cases}$$

where

- $\mathbb{T}$ Cauchy stress
- $\boldsymbol{I}$ Interaction of the liquid with the porous media
- $\boldsymbol{u}$ velocity of the fluid
- b Body forces

Remark

In the case of $\boldsymbol{I} = 0$ and depending on the relation between $\mathbb{T}$ and $\mathbb{D}$ ($\mathbb{D} = 1/2(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t)$) we can arrive to different well-known models for fluids: Navier-Stokes, Power Law, Bingham, ...

Remark

Assuming $\mathbb{T} = -p\mathbb{I}$ (dissipation due to drag much larger than due to shear), depending on the choice of $\boldsymbol{I}$ we can arrive to different well-known models: Darcy, Barus, Forchheimer, ...

Model problem

The balance of linear momentum for the (steady) fluid, is given by

$$-\,\mathbf{div}\,\mathbb{T} + \boldsymbol{I} = \rho\,\boldsymbol{f},$$

where $\boldsymbol{I}$ is the frictional resistance at the pores of the solid on the fluid that is flowing. Assume

$$\boldsymbol{I} = \alpha(P)\,\boldsymbol{U},$$

Simplify the Cauchy stress assuming dissipation due to the drag at the pores is much larger than the dissipation due to the shear in the bulk fluid,

$$\mathbb{T} = -\,P\,\mathbb{I}\,.$$

Model problem

The model

$$\begin{cases} \alpha(P)\,\boldsymbol{U} + \nabla P = \boldsymbol{f} & \text{in } \Omega\,, \\ \operatorname{div} \boldsymbol{U} = 0 & \text{in } \Omega\,, \\ P = 0 & \text{on } \Gamma_D\,, \\ \boldsymbol{U}\cdot\boldsymbol{n} = g & \text{on } \Gamma_N\,, \end{cases} \tag{1}$$

where $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in H_{00}^{-1/2}(\Gamma_N)$. Furthermore, we recall that the Sobolev space to which the Neumann datum g belongs, that is $H_{00}^{-1/2}(\Gamma_N)$, is the dual of $H_{00}^{1/2}(\Gamma_N)$, where

$$H_{00}^{1/2}(\Gamma_N) := \left\{ v|_{\Gamma_N} : \quad v \in H^1(\Omega)\,, \quad v = 0 \quad \text{on} \quad \Gamma_D \right\}.$$

We consider there exist constants $\alpha_0, \gamma > 0$ such that

$$\alpha(s) = \alpha_0\, e^{\gamma\, s} \qquad \forall\, s \in \boldsymbol{R}\,.$$

[Srinivasan,Rajagopal'14] Thermodynamic basis for the derivation of models for flows through porous media and their generalizations

Model problem

Hence, we rewrite the first equation of (1) as:

$$\boldsymbol{U} = \frac{1}{\alpha(P)}(\boldsymbol{f} - \nabla P) = \frac{1}{\alpha_0}(e^{-\gamma P}\boldsymbol{f} - e^{-\gamma P}\nabla P) = \frac{1}{\alpha_0}\left(e^{-\gamma P}\boldsymbol{f} + \frac{1}{\gamma}\nabla(e^{-\gamma P})\right),$$

so that, assuming heuristically that (1) has at least one solution, and defining the new unknowns

$$\boldsymbol{u} := \boldsymbol{U} \quad \text{and} \quad p := e^{-\gamma P} - 1 \quad \text{in } \Omega,$$

we can recast (1) in the form

$$\begin{cases} \boldsymbol{u} = \dfrac{1}{\alpha_0}(p+1)\boldsymbol{f} + \dfrac{1}{\alpha_0\,\gamma}\nabla p & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g & \text{on } \Gamma_N, \end{cases} \tag{2}$$

Introducing the additional unknown $\lambda := -p|_{\Gamma_N} \in H^{1/2}_{00}(\Gamma_N)$, we get
Find $(\boldsymbol{u}, (p, \lambda)) \in \boldsymbol{H} \times \boldsymbol{Q}$ such that

$$\left\{ \begin{array}{rcll} \boldsymbol{a}(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{b}(\boldsymbol{v}, (p, \lambda)) & = & \gamma \int_\Omega p\, \boldsymbol{f} \cdot \boldsymbol{v} + \gamma \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} & \forall\, \boldsymbol{v} \in \boldsymbol{H}, \\ \boldsymbol{b}(\boldsymbol{u}, (q, \xi)) & = & \langle g, \xi \rangle_{\Gamma_N} & \forall\, (q, \xi) \in \boldsymbol{Q}, \end{array} \right. \tag{3}$$

where $\boldsymbol{H} := \boldsymbol{H}(\mathrm{div}; \Omega)$, $\boldsymbol{Q} := L^2(\Omega) \times H^{1/2}_{00}(\Gamma_N)$, and $\boldsymbol{a} : \boldsymbol{H} \times \boldsymbol{H} \to R$ and $\boldsymbol{b} : \boldsymbol{H} \times \boldsymbol{Q} \to R$ are the bounded bilinear forms defined by

$$\begin{array}{rcll} \boldsymbol{a}(\boldsymbol{u}, \boldsymbol{v}) & := & \alpha_0\, \gamma \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} & \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}, \\ \boldsymbol{b}(\boldsymbol{v}, (q, \xi)) & := & \int_\Omega q\, \mathrm{div}\, \boldsymbol{v} + \langle \boldsymbol{v} \cdot \boldsymbol{n}, \xi \rangle_{\Gamma_N} & \forall\, (\boldsymbol{v}, (q, \xi)) \in \boldsymbol{H} \times \boldsymbol{Q}. \end{array} \tag{4}$$

Another formulation: Find $(\boldsymbol{u},(p,\lambda)) \in \boldsymbol{H} \times \boldsymbol{Q}$ such that

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{u},\boldsymbol{v}) + \boldsymbol{b}_1(\boldsymbol{v},(p,\lambda)) &= \gamma \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} && \forall\, \boldsymbol{v} \in \boldsymbol{H}\,, \\ \boldsymbol{b}_2(\boldsymbol{u},(q,\xi)) &= \langle g,\xi\rangle_{\Gamma_N} && \forall\,(q,\xi) \in \boldsymbol{Q}\,, \end{aligned} \tag{5}$$

where $\boldsymbol{b}_1 : \boldsymbol{H} \times \boldsymbol{Q} \to \boldsymbol{R}$ is the bounded bilinear form defined by

$$\boldsymbol{b}_1(\boldsymbol{v},(q,\xi)) := \int_\Omega q \operatorname{div} \boldsymbol{v} + \langle \boldsymbol{v}\cdot\boldsymbol{n},\xi\rangle_{\Gamma_N} - \gamma \int_\Omega q\, \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall\,(\boldsymbol{v},(q,\xi)) \in \boldsymbol{H}\times\boldsymbol{Q}\,, \tag{6}$$

and $\boldsymbol{b}_2 = \boldsymbol{b}$, which constitutes a particular example of the generalized Babuska-Brezzi theory.

Remark

Nevertheless, for easiness of the analysis, in what follows we do not adopt this approach, but rather apply a combination of the classical Babuska-Brezzi theory and the Banach fixed point Theorem.

Variational formulation

Proposition

There clearly holds

$$|\boldsymbol{a}(\boldsymbol{u},\boldsymbol{v})| \le \|\boldsymbol{a}\|\,\|\boldsymbol{u}\|_{\boldsymbol{H}}\,\|\boldsymbol{v}\|_{\boldsymbol{H}} \quad \forall\, \boldsymbol{u},\, \boldsymbol{v}\in \boldsymbol{H}\,,$$

and

$$|\boldsymbol{b}(\boldsymbol{v},(q,\xi))| \le \|\boldsymbol{b}\|\,\|\boldsymbol{v}\|_{\boldsymbol{H}}\,\|(q,\xi)\|_{\boldsymbol{Q}} \quad \forall\,(\boldsymbol{v},(q,\xi))\in \boldsymbol{H}\times\boldsymbol{Q}\,,$$

with $\|\boldsymbol{a}\| = \alpha_0\,\gamma$ *and* $\|\boldsymbol{b}\|$ *depending on the normal trace operator in* $H(\mathrm{div};\Omega)$*. Moreover,* $\boldsymbol{a}(\cdot,\cdot)$ *is* $\boldsymbol{V}$*-elliptic*

$$\boldsymbol{a}(\boldsymbol{v},\boldsymbol{v}) = \alpha_0\,\gamma\,\|\boldsymbol{v}\|^2_{0,\Omega} = \alpha\,\|\boldsymbol{v}\|^2_{\boldsymbol{H}} \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}\,,$$

with $\alpha := \alpha_0\,\gamma$ *and the kernel of* $\boldsymbol{b}$ *characterized as:*

$$\boldsymbol{V} := \boldsymbol{N}(\boldsymbol{b}) = \Big\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega):\quad \mathrm{div}\,\boldsymbol{v}=0 \quad in\,\Omega\,,\quad and \quad \boldsymbol{v}\cdot\boldsymbol{\nu}=0 \quad on \quad \Gamma_N\Big\}\,,$$

Furthermore $\boldsymbol{b}(\cdot,\cdot)$ *satisfies the continuous inf-sup condition on* $\boldsymbol{H}\times\boldsymbol{Q}$*, i.e. there exists* $\beta > 0$ *such that*

$$\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}\\ \boldsymbol{v}\neq 0}} \frac{\boldsymbol{b}(\boldsymbol{v},(q,\xi))}{\|\boldsymbol{v}\|_{\boldsymbol{H}}} \ge \beta\,\|(q,\xi)\|_{\boldsymbol{Q}} \quad \forall\,(q,\xi)\in\boldsymbol{Q}\,.$$

Theorem

Assume that $\boldsymbol{f} \in \boldsymbol{L}^{\infty}(\Omega)$ *and that* $\|\boldsymbol{f}\|_{\infty,\Omega} < \frac{1}{\tilde{C}\,\gamma}$*. Then, there exists a unique* $(\boldsymbol{u},(p,\lambda)) \in \boldsymbol{H} \times \boldsymbol{Q}$ *solution of our variational formulation* (3)*. Moreover, there holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}} + \left\{1 - \tilde{C}\,\gamma\,\|\boldsymbol{f}\|_{\infty,\Omega}\right\}\|(p,\lambda)\|_{\boldsymbol{Q}} \leq \tilde{C}\left\{\|\boldsymbol{F}\|_{\boldsymbol{H}'} + \|\boldsymbol{G}\|_{\boldsymbol{Q}'}\right\} \leq \tilde{C}\left\{\gamma\,|\Omega|\,\|\boldsymbol{f}\|_{\infty,\Omega} + \|g\|_{0;-1/2,\Gamma_N}\right\}. \tag{7}$$

Proof

Step 1: Introduce affine operator $\boldsymbol{T} : \boldsymbol{H} \times \boldsymbol{Q} \to \boldsymbol{H} \times \boldsymbol{Q}$ that, given $(\boldsymbol{w}, (r, \eta)) \in \boldsymbol{H} \times \boldsymbol{Q}$, defines

$$\boldsymbol{T}(\boldsymbol{w}, (r, \eta)) := (\bar{\boldsymbol{u}}, (\bar{p}, \bar{\lambda})) \in \boldsymbol{H} \times \boldsymbol{Q} \tag{8}$$

as the unique solution of (3) when p is replaced by r on the right hand side of $(3)_1$, that is

$$\begin{aligned} \boldsymbol{a}(\bar{\boldsymbol{u}}, \boldsymbol{v}) + \boldsymbol{b}(\boldsymbol{v}, (\bar{p}, \bar{\lambda})) &= \boldsymbol{F}_r(\boldsymbol{v}) + \boldsymbol{F}(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}, \\ \boldsymbol{b}(\bar{\boldsymbol{u}}, (q, \xi)) &= \boldsymbol{G}(q, \xi) \qquad \forall\, (q, \xi) \in \boldsymbol{Q}. \end{aligned} \tag{9}$$

given $r \in L^2(\Omega)$, the linear functionals $\boldsymbol{F}_r : \boldsymbol{H} \to \boldsymbol{R}$, $\boldsymbol{F} : \boldsymbol{H} \to \boldsymbol{R}$ and $\boldsymbol{G} : \boldsymbol{Q} \to \boldsymbol{R}$ defined by

$$\boldsymbol{F}_r(\boldsymbol{v}) := \gamma \int_\Omega r\, \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}, \tag{10}$$

$$\boldsymbol{F}(\boldsymbol{v}) := \gamma \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}, \tag{11}$$

and

$$\boldsymbol{G}(q, \xi) := \langle g, \xi \rangle_{\Gamma_N} \quad \forall\, (q, \xi) \in \boldsymbol{Q}, \tag{12}$$

The linear functionals satisfy

$$|\boldsymbol{F}_r(\boldsymbol{v})| \leq \gamma \, \|r\|_{0,\Omega} \, \|\boldsymbol{f}\|_{\infty,\Omega} \, \|\boldsymbol{v}\|_{0,\Omega} \quad \forall \, \boldsymbol{v} \in \boldsymbol{H} \tag{13}$$

$$|\boldsymbol{F}(\boldsymbol{v})| \leq \gamma \, |\Omega| \, \|\boldsymbol{f}\|_{\infty,\Omega} \, \|\boldsymbol{v}\|_{0,\Omega} \quad \forall \, \boldsymbol{v} \in \boldsymbol{H} \tag{14}$$

and

$$|\boldsymbol{G}(q, \xi)| \leq \|g\|_{0;-1/2,\Gamma_N} \, \|\xi\|_{0;1/2,\Gamma_N} \quad \forall \, (q, \xi) \in \boldsymbol{Q} \, , \tag{15}$$

which shows that $\boldsymbol{F}_r \in \boldsymbol{H}'$, $\boldsymbol{F} \in \boldsymbol{H}'$ and $\boldsymbol{G} \in \boldsymbol{Q}'$.

Step 2: Rewrite original problem (3) as: Find $(\boldsymbol{u}, (p, \lambda)) \in \boldsymbol{H} \times \boldsymbol{Q}$ such that

$$\boldsymbol{T}(\boldsymbol{u}, (p, \lambda)) = (\boldsymbol{u}, (p, \lambda)) \tag{16}$$

that is, to find a fixed point of $\boldsymbol{T}$.

Step 3: Apply Banach fixed point Theorem.

Variational formulation

Step 3: From the superposition principle:

$$\boldsymbol{T}(\boldsymbol{w},(r,\eta)) = (\boldsymbol{u}_0,(p_0,\lambda_0)) + \boldsymbol{S}(\boldsymbol{w},(r,\eta)) \quad \forall\,(\boldsymbol{w},(r,\eta)) \in \boldsymbol{H}\times\boldsymbol{Q}, \qquad (17)$$

where $(\boldsymbol{u}_0,(p_0,\lambda_0)) \in \boldsymbol{H}\times\boldsymbol{Q}$ is the unique solution of the auxiliary problem

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{u}_0,\boldsymbol{v}) + \boldsymbol{b}(\boldsymbol{v},(p_0,\lambda_0)) &= \boldsymbol{F}(\boldsymbol{v}) && \forall\,\boldsymbol{v}\in\boldsymbol{H}, \\ \boldsymbol{b}(\boldsymbol{u}_0,(q,\xi)) &= \boldsymbol{G}(q,\xi) && \forall\,(q,\xi)\in\boldsymbol{Q}, \end{aligned} \qquad (18)$$

and $\boldsymbol{S}:\boldsymbol{H}\times\boldsymbol{Q}\to\boldsymbol{H}\times\boldsymbol{Q}$ is the linear operator that, given $(\boldsymbol{w},(r,\eta))\in\boldsymbol{H}\times\boldsymbol{Q}$, defines

$$\boldsymbol{S}(\boldsymbol{w},(r,\eta)) := (\bar{\boldsymbol{u}},(\bar{p},\bar{\lambda})) \in \boldsymbol{H}\times\boldsymbol{Q} \qquad (19)$$

as the unique solution of (9) with $\boldsymbol{F}$ and $\boldsymbol{G}$ replaced by null functionals, that is

$$\begin{aligned} \boldsymbol{a}(\bar{\boldsymbol{u}},\boldsymbol{v}) + \boldsymbol{b}(\boldsymbol{v},(\bar{p},\bar{\lambda})) &= \boldsymbol{F}_r(\boldsymbol{v}) && \forall\,\boldsymbol{v}\in\boldsymbol{H}, \\ \boldsymbol{b}(\bar{\boldsymbol{u}},(q,\xi)) &= 0 && \forall\,(q,\xi)\in\boldsymbol{Q}. \end{aligned} \qquad (20)$$

Note here that (17) confirms the term affine given in advance to $\boldsymbol{T}$. Furthermore, the continuous dependence result for (18) and (20) establishes the existence of a same constant $\tilde{C} := \tilde{C}(\|\boldsymbol{a}\|,\alpha,\beta) > 0$ such that

$$\|(\boldsymbol{u}_0,(p_0,\lambda_0))\|_{\boldsymbol{H}\times\boldsymbol{Q}} \le \tilde{C}\left\{\|\boldsymbol{F}\|_{\boldsymbol{H}'} + \|\boldsymbol{G}\|_{\boldsymbol{Q}'}\right\}, \qquad (21)$$

and

$$\|\boldsymbol{S}(\boldsymbol{w},(r,\eta))\|_{\boldsymbol{H}\times\boldsymbol{Q}} \le \tilde{C}\|\boldsymbol{F}_r\|_{\boldsymbol{H}'}. \qquad (22)$$

Variational formulation

In particular, (22) and (13) yield

$$\|\boldsymbol{S}(\boldsymbol{w},(r,\eta))\|_{\boldsymbol{H}\times\boldsymbol{Q}} \le \tilde{C}\,\gamma\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|r\|_{0,\Omega} \quad \forall\,(\boldsymbol{w},(r,\eta)) \in \boldsymbol{H}\times\boldsymbol{Q}\,,$$

and hence, given $(\boldsymbol{w}_1,(r_1,\eta_1))$, $(\boldsymbol{w}_2,(r_2,\eta_2)) \in \boldsymbol{H}\times\boldsymbol{Q}$, we can use (17) to find that

$$\begin{aligned}\|\boldsymbol{T}(\boldsymbol{w}_1,(r_1,\eta_1)) - \boldsymbol{T}(\boldsymbol{w}_2,(r_2,\eta_2))\|_{\boldsymbol{H}\times\boldsymbol{Q}} &= \|\boldsymbol{S}\big((\boldsymbol{w}_1,(r_1,\eta_1)) - (\boldsymbol{w}_2,(r_2,\eta_2))\big)\|_{\boldsymbol{H}\times\boldsymbol{Q}} \\ &\le \tilde{C}\,\gamma\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|r_1 - r_2\|_{0,\Omega} \\ &\le \tilde{C}\,\gamma\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|(\boldsymbol{w}_1,(r_1,\eta_1)) - (\boldsymbol{w}_2,(r_2,\eta_2))\|_{\boldsymbol{H}\times\boldsymbol{Q}}\,,\end{aligned}$$

which shows that $\boldsymbol{T}$ is a contraction whenever

$$\|\boldsymbol{f}\|_{\infty,\Omega} < \frac{1}{\tilde{C}\,\gamma}\,.$$

Step 4:
Observe using (16) and (17), that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}} + \|(p,\lambda)\|_{\boldsymbol{Q}} = \|(\boldsymbol{u},(p,\lambda))\|_{\boldsymbol{H}\times\boldsymbol{Q}} = \|\boldsymbol{T}(\boldsymbol{u},(p,\lambda))\|_{\boldsymbol{H}\times\boldsymbol{Q}}$$
$$= \|(\boldsymbol{u}_0,(p_0,\lambda_0)) + \boldsymbol{S}(\boldsymbol{u},(p,\lambda))\|_{\boldsymbol{H}\times\boldsymbol{Q}},$$

which, thanks to the triangle inequality and the estimates (21) and (22), leads to

$$\|\boldsymbol{u}\|_{\boldsymbol{H}} + \|(p,\lambda)\|_{\boldsymbol{Q}} \leq \tilde{C}\left\{\|\boldsymbol{F}\|_{\boldsymbol{H}'} + \|\boldsymbol{G}\|_{\boldsymbol{Q}'} + \|\boldsymbol{F}_p\|_{\boldsymbol{H}'}\right\}. \tag{23}$$

In turn, it is clear from (13), (14), and (15), that

$$\|\boldsymbol{F}_p\|_{\boldsymbol{H}'} \leq \gamma\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|(p,\lambda)\|_{\boldsymbol{Q}}, \quad \|\boldsymbol{F}\|_{\boldsymbol{H}'} \leq \gamma\,|\Omega|\,\|\boldsymbol{f}\|_{\infty,\Omega}, \quad \text{and} \quad \|\boldsymbol{G}\|_{\boldsymbol{Q}'} \leq \|g\|_{0;-1/2,$$

which, together with (23), yield

$$\|\boldsymbol{u}\|_{\boldsymbol{H}} + \left\{1 - \tilde{C}\gamma\,\|\boldsymbol{f}\|_{\infty,\Omega}\right\}\|(p,\lambda)\|_{\boldsymbol{Q}} \leq \tilde{C}\left\{\|\boldsymbol{F}\|_{\boldsymbol{H}'} + \|\boldsymbol{G}\|_{\boldsymbol{Q}'}\right\}$$
$$\leq \tilde{C}\left\{\gamma\,|\Omega|\,\|\boldsymbol{f}\|_{\infty,\Omega} + \|g\|_{0;-1/2,\Gamma_N}\right\}.$$

Galerkin scheme. Main Results

Let $\boldsymbol{H}_h$ and $\boldsymbol{Q}_h$ be finite dimensional subspaces of $\boldsymbol{H}$ and $\boldsymbol{Q}$, respectively, and consider the Galerkin scheme: Find $(\boldsymbol{u}_h, (p_h, \lambda_h)) \in \boldsymbol{H}_h \times \boldsymbol{Q}_h$ such that

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \boldsymbol{b}(\boldsymbol{v}_h, (p_h, \lambda_h)) &= \gamma \int_\Omega p_h \boldsymbol{f} \cdot \boldsymbol{v}_h + \gamma \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{H}_h\,, \\ \boldsymbol{b}(\boldsymbol{u}_h, (q_h, \xi_h)) &= \langle g, \xi_h \rangle_{\Gamma_N} \qquad \forall\, (q_h, \xi_h) \in \boldsymbol{Q}_h. \end{aligned} \tag{24}$$

Then, we let $\boldsymbol{V}_h$ be the discrete kernel of $\boldsymbol{b}$, that is

$$\boldsymbol{V}_h := \Big\{ \boldsymbol{v}_h \in \boldsymbol{H}_h : \quad \boldsymbol{b}(\boldsymbol{v}_h, (q_h, \xi_h)) = 0 \quad \forall\, (q_h, \xi_h) \in \boldsymbol{Q}_h \Big\}\,, \tag{25}$$

and assume that there exists $\widehat{\alpha} > 0$, independent of h, such that

$$\boldsymbol{a}(\boldsymbol{v}_h, \boldsymbol{v}_h) \ge \widehat{\alpha}\, \|\boldsymbol{v}_h\|^2_{\boldsymbol{H}} \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h\,. \tag{26}$$

In addition, we also suppose that there exists $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{v}_h \in \boldsymbol{H}_h \\ \boldsymbol{v}_h \neq 0}} \frac{\boldsymbol{b}(\boldsymbol{v}_h, (q_h, \xi_h))}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}_h}} \ge \widehat{\beta}\, \|(q_h, \xi_h)\|_{\boldsymbol{Q}_h} \quad \forall\, (q_h, \xi_h) \in \boldsymbol{Q}_h\,. \tag{27}$$

Then, introducing corresponding discrete operators $\boldsymbol{T}_h : \boldsymbol{H}_h \times \boldsymbol{Q}_h \to \boldsymbol{H}_h \times \boldsymbol{Q}_h$ and $\boldsymbol{S}_h : \boldsymbol{H}_h \times \boldsymbol{Q}_h \to \boldsymbol{H}_h \times \boldsymbol{Q}_h$, and the particular discrete solutions $(\boldsymbol{u}_{h,0}, (p_{h,0}, \lambda_{h,0})) \in \boldsymbol{H}_h \times \boldsymbol{Q}_h$, analogously to the definitions given for the continuous case (cf. (8), (17), (19)), we find that (24) is equivalent to the fixed point equation: Find $(\boldsymbol{u}_h, (p_h, \lambda_h)) \in \boldsymbol{H}_h \times \boldsymbol{Q}_h$ such that

$$\boldsymbol{T}_h(\boldsymbol{u}_h, (p_h, \lambda_h)) = (\boldsymbol{u}_h, (p_h, \lambda_h)),$$

where

$$\boldsymbol{T}_h(\boldsymbol{w}_h, (r_h, \eta_h)) = (\boldsymbol{u}_{h,0}, (p_{h,0}, \lambda_{h,0})) + \boldsymbol{S}_h(\boldsymbol{w}_h, (r_h, \eta_h)) \qquad \forall\, (\boldsymbol{w}_h, (r_h, \eta_h)) \in \boldsymbol{H}_h \times \boldsymbol{Q}_h\,.$$

Theorem

Assume that $\boldsymbol{f} \in \boldsymbol{L}^\infty(\Omega)$ *and that* $\|\boldsymbol{f}\|_{\infty,\Omega} \le \frac{1}{2\,\widehat{C}\,\gamma}$, *where* $\widehat{C} := \widehat{C}(\|\boldsymbol{a}\|, \widehat{\alpha}, \widehat{\beta})$ *is the continuous dependence constant for the discrete problems defining* $(\boldsymbol{u}_{h,0}, (p_{h,0}, \lambda_{h,0}))$ *and the operator* $\boldsymbol{S}_h$. *Then, there exists a unique* $(\boldsymbol{u}_h, (p_h, \lambda_h)) \in \boldsymbol{H}_h \times \boldsymbol{Q}_h$ *solution of* (24), *and there holds*

$$\begin{aligned} 2\,\|\boldsymbol{u}_h\|_{\boldsymbol{H}} + \|(p_h, \lambda_h)\|_{\boldsymbol{Q}} &\le 2\,\widehat{C}\left\{\|\boldsymbol{F}|_{\boldsymbol{H}_h}\|_{\boldsymbol{H}_h'} + \|\boldsymbol{G}|_{\boldsymbol{Q}_h}\|_{\boldsymbol{Q}_h'}\right\} \\ &\le 2\,\widehat{C}\left\{\gamma\,|\Omega|\,\|\boldsymbol{f}\|_{\infty,\Omega} + \|g\|_{0;-1/2,\Gamma_N}\right\}. \end{aligned}$$

Galerkin scheme. Main Results

Theorem

Assume that $\boldsymbol{f} \in \boldsymbol{L}^\infty(\Omega)$ *and that*

$$\|\boldsymbol{f}\|_{\infty,\Omega} \,\le\, \frac{1}{2\,\gamma}\ \min\left\{\frac{1}{\tilde{C}}, \frac{1}{\widehat{C}}, \frac{\widehat{\alpha}\,\widehat{\beta}}{\widehat{\alpha} + \|\boldsymbol{a}\|}\right\},$$

where $\tilde{C} = \tilde{C}(\|\boldsymbol{a}\|, \alpha, \beta) > 0$ *and* $\widehat{C} = \widehat{C}(\|\boldsymbol{a}\|, \widehat{\alpha}, \widehat{\beta}) > 0$ *are the continuous dependence constants specified above. Then, the continuous and discrete problems* (3) *and* (24) *have unique solutions* $(\boldsymbol{u}, (p, \lambda)) \in \boldsymbol{H} \times \boldsymbol{Q}$ *and* $(\boldsymbol{u}_h, (p_h, \lambda_h)) \in \boldsymbol{H}_h \times \boldsymbol{Q}_h$*. Moreover, there exists a constant* $C > 0$*, depending on* $\|\boldsymbol{a}\|$, $\|\boldsymbol{b}\|$, $\widehat{\alpha}$, $\widehat{\beta}$, γ, *and* $\|\boldsymbol{f}\|_{\infty,\Omega}$, *such that*

$$\|(\boldsymbol{u}, (p, \lambda)) - (\boldsymbol{u}_h, (p_h, \lambda_h))\|_{\boldsymbol{H}\times\boldsymbol{Q}} \,\le\, C\,\mathrm{dist}\Big((\boldsymbol{u}, (p, \lambda)), \boldsymbol{H}_h \times \boldsymbol{Q}_h\Big)\,.$$

Galerkin scheme. Specific finite element subspaces

We let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of discretizations.
Given an integer $k \geq 0$, we define the finite element subspace $\boldsymbol{H}_h$ for the approximation of $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div};\Omega)$ as the global Raviart-Thomas space of order k, that is,

$$\boldsymbol{H}_h := \left\{ \boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div};\Omega) : \quad \boldsymbol{v}_h|_T \in \mathbf{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h \right\}. \tag{28}$$

The finite element subspace for the pressure p is given by the global space of piecewise polynomials of degree $\leq k$, that is

$$Q_h^p := \left\{ q_h \in L^2(\Omega) : \quad q_h|_T \in P_k(T) \quad \forall\, T \in \mathcal{T}_h \right\}. \tag{29}$$

To define the finite element subspace for λ, we first let $\Gamma_{N,h}$ be the partition of Γ_N inherited from the triangulation $\mathcal{T}_h$, and define the meshsize $h_N := \max\left\{|e| : \quad e \in \Gamma_{N,h}\right\}$. Note here that e denotes either edges of triangles (in $\boldsymbol{R}^2$) or faces of tetrahedra (in $\boldsymbol{R}^3$). Then, we let $\Gamma_{N,\tilde{h}}$ be another partition of Γ_N, independent of $\Gamma_{N,h}$, and define $\tilde{h}_N := \max\left\{|e| : \quad e \in \Gamma_{N,\tilde{h}}\right\}$. Then, for the same integer $k \geq 0$ employed in the definitions (28) and (29), we introduce

$$Q_{\tilde{h}}^{\lambda} := \left\{ \xi_{\tilde{h}} \in H_{00}^{1/2}(\Gamma_N) : \quad \xi_{\tilde{h}}|_e \in P_{k+1}(e) \quad \forall\, e \in \Gamma_{N,\tilde{h}} \right\}, \tag{30}$$

with $h_N \leq C_0\, \tilde{h}_N$ and then set

$$\boldsymbol{Q}_{h,\tilde{h}} := Q_h^p \times Q_{\tilde{h}}^{\lambda}. \tag{31}$$

Consider $\Omega \subset \boldsymbol{R}^2$ and $k = 0$. For each $T \in \mathcal{T}_h$ we define the a posteriori error indicator:

$$\begin{aligned}
\theta_T^2 := \; & \|\mathrm{div}\, \boldsymbol{u}_h\|_{0,T}^2 + h_T^2 \, \|\mathbf{r}(\boldsymbol{u}_h, p_h; \boldsymbol{f})\|_{0,T}^2 + h_T^2 \, \|\mathrm{curl}\{\mathbf{r}(\boldsymbol{u}_h, p_h; \boldsymbol{f})\}\|_{0,T}^2 \\
& + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Omega)} h_e \, \|[\mathbf{r}(\boldsymbol{u}_h, p_h; \boldsymbol{f}) \cdot \boldsymbol{s}]\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Gamma_N)} h_e \left\| \mathbf{r}(\boldsymbol{u}_h, p_h; \boldsymbol{f}) \cdot \boldsymbol{s} - \frac{d\lambda_h}{d\boldsymbol{s}} \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_N)} h_e \Big\{ \|\lambda_h + p_h\|_{0,e}^2 + \|g - \boldsymbol{u}_h \cdot \boldsymbol{\nu}\|_{0,e}^2 \Big\} \\
& + \sum_{e \in \mathcal{E}_h(T) \cap \mathcal{E}_h(\Gamma_D)} h_e \, \|\mathbf{r}(\boldsymbol{u}_h, p_h; \boldsymbol{f}) \cdot \boldsymbol{s}\|_{0,e}^2 \,,
\end{aligned} \tag{32}$$

where

$$\mathbf{r}(\boldsymbol{u}_h, p_h; \boldsymbol{f}) := \gamma \, \boldsymbol{f} + \gamma \, p_h \, \boldsymbol{f} - \alpha_0 \, \gamma \, \boldsymbol{u}_h$$

Remark

Note here that the inclusion of the expression $\|g - \boldsymbol{u}_h \cdot \boldsymbol{\nu}\|_{0,e}^2$ in the definition of θ_T^2 requires the Neumann datum to be smoother than $H_{00}^{-1/2}(\Gamma_N)$, namely $g \in L^2(\Gamma_N)$.

Theorem
Assume that $\mathbf{f} \in \mathbf{L}^\infty(\Omega)$, $g \in L^2(\Gamma_N)$, *and that they are piecewise polynomials on* $\mathcal{T}_h$ *and* $\Gamma_{N,h}$, *respectively, for each* $h > 0$. *Let* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3) *and* (24), *respectively. Then, there exist constants* $C_{\texttt{rel}} > 0$ *and* $C_{\texttt{eff}} > 0$, *independent of h, such that*

$$C_{\texttt{eff}}\, \boldsymbol{\theta} \;\leq\; \|(\mathbf{u}, (p, \lambda)) - (\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \;\leq\; C_{\texttt{rel}}\, \boldsymbol{\theta}\,. \qquad (33)$$

where the global a posteriori error estimator

$$\boldsymbol{\theta} := \left\{ \sum_{T\in\mathcal{T}_h} \theta_T^2 \right\}^{1/2} .$$

A posteriori error analysis. Preliminary results

- **Clément interpolator** Let $I_h : H^1(\Omega) \to X_h$ be the Clément interpolation operator, where

$$X_h := \left\{ \varphi_h \in C(\bar{\Omega}) : \quad \varphi_h|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h \right\}.$$

The local approximation properties of I_h are summarized in the following lemma.

Lemma

There exist $c_1,\, c_2 > 0$, independent of h, such that for all $\varphi \in H^1(\Omega)$ there holds

$$\|\varphi - I_h(\varphi)\|_{0,T} \,\leq\, c_1\; h_T\, \|\varphi\|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_h$$

and

$$\|\varphi - I_h(\varphi)\|_{0,e} \,\leq\, c_2\; h_e^{1/2}\, \|\varphi\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)\,,$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_h : \quad T' \cap T \neq \emptyset\}$ and $\Delta(e) := \cup\{T' \in \mathcal{T}_h : \quad T' \cap e \neq \emptyset\}$.

- **Helmholtz decomposition**

Lemma

For each $\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega)$ there exist $\boldsymbol{\zeta} \in \mathbf{H}^1(\Omega)$ and $\varphi \in H^1(\Omega)$, with $\int_\Omega \varphi = 0$, such that $\mathbf{v} = \boldsymbol{\zeta} + \mathbf{curl}\,\varphi$ in Ω and

$$\|\boldsymbol{\zeta}\|_{1,\Omega} + \|\varphi\|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{\mathrm{div},\Omega}\,, \qquad (34)$$

where C is a positive constant independent of $\mathbf{v}$.

A posteriori error analysis. Preliminary results

- **Raviart-Thomas interpolator** Let $\Pi_h : \boldsymbol{H}^1(\Omega) \to \boldsymbol{H}_h$ be the usual Raviart-Thomas interpolation operator, which is characterized by the identity

$$\int_e \Pi_h(\boldsymbol{w}) \cdot \boldsymbol{\nu} = \int_e \boldsymbol{w} \cdot \boldsymbol{\nu} \qquad \forall\, e \in \mathcal{E}_h\,, \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)\,. \tag{35}$$

It is easy to show, using (35), that

$$\operatorname{div}(\Pi_h(\boldsymbol{w})) \;=\; \mathcal{P}_h(\operatorname{div} \boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)\,, \tag{36}$$

where $\mathcal{P}_h$ is the $L^2(\Omega)$-orthogonal projector onto Q_h^p (cf. (29)).

Lemma
Π_h satisfies the following approximation properties

$$\|\boldsymbol{w} - \Pi_h(\boldsymbol{w})\|_{0,T} \;\leq\; C\, h_T\, \|\boldsymbol{w}\|_{1,T} \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)\,, \tag{37}$$

and

$$\|(\boldsymbol{w} - \Pi_h(\boldsymbol{w})) \cdot \boldsymbol{\nu}\|_{0,e} \;\leq\; C\, h_e^{1/2}\, \|\boldsymbol{w}\|_{1,T_e} \quad \forall\, e \in \mathcal{E}_h \cap \partial T_e, \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)\,, \tag{38}$$

where T_e in (38) is a triangle of $\mathcal{T}_h$ containing e on its boundary.

Numerical results

N stands for the number of degrees of freedom defining the finite element subspaces $\boldsymbol{H}_h$ and $\boldsymbol{Q}_h$ (equivalently, the number of unknowns of (24)), and the individual and global errors are denoted by:

$$\mathtt{e}(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{div},\Omega}, \qquad \mathtt{e}(p) := \|p - p_h\|_{0,\Omega}, \qquad \mathtt{e}(\lambda) := \|\lambda - \lambda_h\|_{0;1/2,\Gamma_N},$$

$$\text{and} \qquad \mathtt{e} := \left\{[\mathtt{e}(\boldsymbol{u})]^2 + [\mathtt{e}(p)]^2 + [\mathtt{e}(\lambda)]^2\right\}^{1/2},$$

Furthermore, we define the effectivity index

$$\mathtt{eff}(\boldsymbol{\theta}) := \mathtt{e}/\boldsymbol{\theta},$$

and we let $r(\boldsymbol{u})$, $r(p)$, $r(\lambda)$, and r be the experimental rates of convergence given by

$$r(\boldsymbol{u}) := \frac{\log(\mathtt{e}(\boldsymbol{u})/\mathtt{e}'(\boldsymbol{u}))}{\log(h/h')}, \; r(p) := \frac{\log(\mathtt{e}(p)/\mathtt{e}'(p))}{\log(h/h')}, \; r(\lambda) := \frac{\log(\mathtt{e}(\lambda)/\mathtt{e}'(\lambda))}{\log(h/h')}, \; r := \frac{\log\left(\mathtt{e}/\mathtt{e}'\right)}{\log(h/h')}$$

where h and h' denote two consecutive meshsizes with errors $\mathtt{e}$ and $\mathtt{e}'$, respectively. Postprocessing error associated to the inverse change of variables needed to recover the original pressure field P from p_h, and its associated rate as

$$\mathtt{e}(P) = \|P + \gamma^{-1}\log(p_h + 1)\|_{0,\Omega} \text{ and } r(P) := \frac{\log(\mathtt{e}(P)/\mathtt{e}'(P))}{\log(h/h')}.$$

Example 1

We consider the domain $\Omega := (0,1)^2$ with $\Gamma_D = (0,1) \times \{0\}$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$, and choose $\boldsymbol{f}$ and g so that the exact solutions of (1) and (2) are given by the smooth functions

$$U = \boldsymbol{u}(x_1, x_2) := \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2), \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad p(x_1, x_2) := x_1^2 + x_1 x_2,$$

$$\lambda(x_1, x_2) := -p|_{\Gamma_N}, \quad P(x_1, x_2) := -\gamma^{-1}\log(p + 1).$$

We set $\alpha_0 = 0.1$, $\gamma = 10$ and study the accuracy of the discretization using piecewise constant approximations for the pressure field, $\mathbf{RT}_0$ approximations for velocities, and piecewise linear approximations for the Lagrange multiplier.

Fixed point tolerance of $\epsilon_{fp} = 1e - 8$

Numerical results

Algorithm 2 Fixed point iteration

Set a tolerance ϵ_{fp} and define $\mathtt{res}(0) := 2\epsilon_{fp}$
Set $j = 0$ and choose an initial guess for the pressure p_h^0 satisfying $p_h^0|_{\Gamma_D} = 0$
for $j = 1, \ldots, j_{\max}$ **do**
  Solve the discrete problem

$$\boldsymbol{a}(\boldsymbol{u}_h^j, \boldsymbol{v}_h) + \boldsymbol{b}(\boldsymbol{v}_h, (p_h^j, \lambda_h^j)) = \gamma \int_\Omega (p_h^{j-1} + 1)\, \boldsymbol{f} \cdot \boldsymbol{v}_h \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{H}_h,$$
$$\boldsymbol{b}(\boldsymbol{u}_h^j, (q_h, \xi_h)) = \langle g, \xi_h \rangle_{\Gamma_N} \qquad \forall\, (q_h, \xi_h) \in \boldsymbol{Q}_h,$$

  Compute the pressure residual $\mathtt{res}(j) = \|p_h^j - p_h^{j-1}\|_{0,\Omega}$
  Update the pressure $p_h^{j-1} \leftarrow p_h^j$
  if $\mathtt{res}(j) < \epsilon_{fp}$ **or** $j \geq j_{\max}$ **then**
    break
  else
    continue
  end if
end for

N	h	$\mathtt{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}(\lambda)$	$r(\lambda)$	$\mathtt{e}(P)$	$r(P)$	$\mathtt{eff}(\boldsymbol{\theta})$	iter
Problem (3.1) solved iteratively via Algorithm 2											
11	1.414210	0.777153	–	0.454962	–	1.52585	–	–	–	0.238183	10
32	0.707107	0.465914	0.539419	0.232287	0.969840	0.759940	0.956320	0.024021	–	0.264271	9
104	0.353553	0.264449	0.817072	0.116629	0.993975	0.042501	2.247691	0.014861	0.692782	0.260534	10
368	0.176777	0.137101	0.947764	0.058315	0.999992	0.015887	1.419630	0.007794	0.931046	0.253384	11
1376	0.088388	0.069199	0.986393	0.029155	1.000112	0.008627	0.880807	0.003943	0.982878	0.250735	12
5312	0.044194	0.034682	0.996562	0.014577	1.000041	0.004781	0.851562	0.001977	0.995583	0.249815	11
20864	0.022097	0.017352	0.999138	0.007288	1.000011	0.002578	0.891039	0.000990	0.998306	0.249517	11
82688	0.011049	0.008677	0.999784	0.003644	1.000010	0.001352	0.930602	0.000496	0.997220	0.249431	12
329216	0.005524	0.004339	0.999946	0.001822	1.000000	0.000696	0.958398	0.000249	0.989965	0.249414	11
1313792	0.002762	0.002169	0.999987	0.000911	1.000000	0.000354	0.975827	0.000128	0.961434	0.249416	11
Linear non-symmetric problem (3.3)											
11	1.414210	0.677153	–	0.454962	–	1.525851	–	–	–	0.238183	1
32	0.707107	0.465914	0.539419	0.232287	0.969840	1.059940	0.525629	0.0240221	–	0.264271	1
104	0.353553	0.264449	0.817072	0.116629	0.993975	0.042501	4.640351	0.0148613	0.692794	0.260534	1
368	0.176777	0.137102	0.947764	0.058315	0.999992	0.015887	1.419632	0.0077944	0.931050	0.253384	1
1376	0.088388	0.069199	0.986393	0.029155	1.000112	0.008627	0.880807	0.0039436	0.982917	0.250732	1
5312	0.044194	0.034682	0.996562	0.014577	1.000041	0.004781	0.851562	0.0019776	0.995739	0.249815	1
20864	0.022097	0.017351	0.999138	0.007289	1.000010	0.002578	0.891039	0.0009895	0.998933	0.249517	1
82688	0.011049	0.008677	0.999784	0.003644	1.000000	0.001352	0.930603	0.0004948	0.999721	0.249431	1
329216	0.005524	0.004339	0.999946	0.001822	1.000000	0.000696	0.958398	0.0002474	0.999882	0.249414	1
1313792	0.002762	0.002169	0.999987	0.000911	1.000000	0.000353	0.975827	0.0001237	0.999771	0.249416	1

Table 6.1: Example 1: Experimental convergence for the mixed finite element approximation of the Darcy problem (2.2) and postprocessed pressure $P_h = \gamma^{-1}\log(p_h + 1)$ on a sequence of uniformly refined triangulations of $\Omega = (0,1)^2$, using a fixed point formulation with symmetric iterations (top) and a linear non-symmetric formulation (bottom). Here we have considered the parameters $\alpha_0 = 0.1$, $\gamma = 10$.

Numerical results

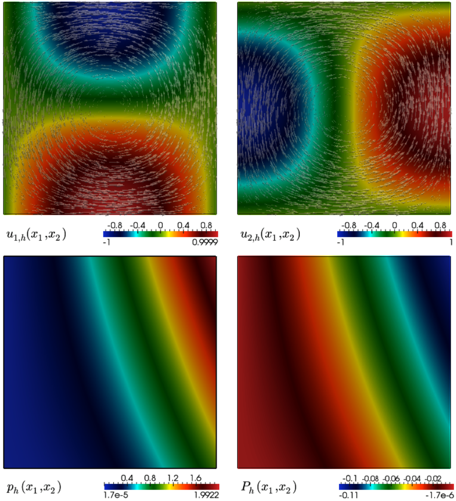

Figure 6.1: Example 1: Approximate velocity components (top), pressure distribution (bottom left), and postprocessed pressure (bottom right), computed using a mesh of 313041 vertices and 626080 elements.

Numerical results

Example 2
Nonconvex *pacman* domain $\Omega = \{(x_1, x_2) \in \boldsymbol{R}^2 : x_1^2 + x_2^2 \leq 1\} \setminus (0,1)^2$, with boundaries $\Gamma_N = (0,1) \times \{0\} \bigcup \{0\} \times (0,1)$ and $\Gamma_D = \partial\Omega \setminus \Gamma_D$, where the model problems (1),(2) admit the following exact solutions

$$U = \boldsymbol{u}(x_1, x_2) := \left((x_1 - c)^2 + (x_2 - c)^2\right)^{-1/2} \begin{pmatrix} c - x_2 \\ x_1 - c \end{pmatrix},$$

$$p(x_1, x_2) := \frac{1 - x_1^2 - x_2^2}{(x_1 - c)^2 + (x_2 - c)^2},$$

$$\lambda(x_1, x_2) := -p|_{\Gamma_N}, \quad P(x_1, x_2) := -\gamma^{-1} \log(p + 1),$$

with $c = 0.025$, satisfying the homogeneous boundary condition on Γ_D. Both pressure and velocity fields exhibit singularities close to the origin.

Numerical results

Algorithm 1 Mesh adaptation procedure

Set $i = 0$ and construct an initial mesh $\mathcal{T}_{h_0}$
for $i = 0, \dots, i_{\max}$ **do**
    Solve the discrete problem (4.1) on the current mesh $\mathcal{T}_{h_i}$ using Algorithm 2
    for $T \in \mathcal{T}_{h_i}$ **do**
        Compute the error indicator θ_T associated to T using (5.2)
        if $\theta_T < \epsilon$ **or** $i \geq i_{\max}$ **then**
            break
        else
            continue
        end if
        if $\theta_T \geq \frac{3}{5} \max\{\theta_L : L \in \mathcal{T}_{h_i}\}$ **then**
            Refine T according to the *blue-green* strategy
        end if
    end for
    Update the mesh $\mathcal{T}_{h_i} \leftarrow \mathcal{T}_{h_{i+1}}$
end for

N	h	$\mathtt{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathtt{e}(p)$	$r(p)$	$\mathtt{e}(\lambda)$	$r(\lambda)$	$\mathtt{e}(P)$	$r(P)$	$\mathtt{eff}(\boldsymbol{\theta})$	iter
Quasi-uniform refinement											
225	0.396463	11.43581	–	39.30172	–	5.905905	–	0.093927	–	0.035166	33
675	0.234657	10.09455	0.116705	28.75717	0.595641	3.798687	0.757472	0.069779	0.792872	0.018529	34
1625	0.153371	8.885424	0.244053	25.28118	0.302922	2.147508	0.661754	0.041202	0.838860	0.016287	32
3530	0.102059	6.453057	0.218888	18.87943	0.716865	1.412040	0.529988	0.029676	0.776951	0.013298	30
7960	0.068675	5.761473	0.400416	13.89542	0.773727	0.950806	0.873166	0.021216	0.820906	0.026307	27
19930	0.043191	3.869916	0.519673	9.780521	0.784410	0.524016	0.816079	0.014728	0.888218	0.009790	33
56730	0.025888	3.053998	0.727651	6.672843	0.822838	0.319769	0.729324	0.009356	0.822051	0.007904	35
178440	0.014914	1.622855	1.146492	4.555427	0.849742	0.203720	0.866195	0.006388	0.644759	0.068482	32
618660	0.008210	1.063797	0.698373	2.461297	0.796696	0.123352	0.711992	0.004243	0.477433	0.016053	34
Adaptive refinement											
861	0.210978	5.282240	–	38.54491	–	5.709012	–	0.070252	–	0.766485	13
1247	0.163169	3.171521	1.073922	19.82960	1.136872	4.367726	0.630054	0.045942	0.962860	0.768496	14
1931	0.135411	1.751252	1.015801	9.571121	1.033150	1.353830	0.899916	0.036824	1.011852	0.769635	15
3401	0.118863	0.902784	1.038221	4.444250	1.210593	0.869012	0.970478	0.021465	0.955669	0.779505	13
6843	0.096292	0.563009	1.052795	2.193142	1.320374	0.613204	0.982651	0.017271	0.940252	0.767345	14
17350	0.078102	0.167351	1.092846	1.290872	1.139381	0.391614	0.885738	0.012131	0.948918	0.750103	16
46492	0.062494	0.135441	0.930274	0.796067	0.980809	0.234962	0.954510	0.007226	0.958351	0.725237	13
127844	0.051637	0.084252	0.950035	0.481753	0.993065	0.172287	0.893301	0.003629	0.963105	0.731722	14
329880	0.042528	0.038337	0.989902	0.311465	0.982892	0.123530	0.986044	0.001995	0.953084	0.697151	13
783742	0.036014	0.010492	0.978930	0.134684	0.946813	0.101078	0.959217	0.000642	0.930247	0.756734	14

Table 6.2: Example 2: Experimental convergence for the mixed finite element approximation of the Darcy problem (2.2) and postprocessed pressure $P_h = \gamma^{-1}\log(p_h+1)$ on a sequence of quasi-uniformly (top) and adaptively (bottom) refined meshes of $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\} \setminus (0,1)^2$. Here we have considered the parameters $\alpha_0 = 0.1$, $\gamma = 10$.

Numerical results

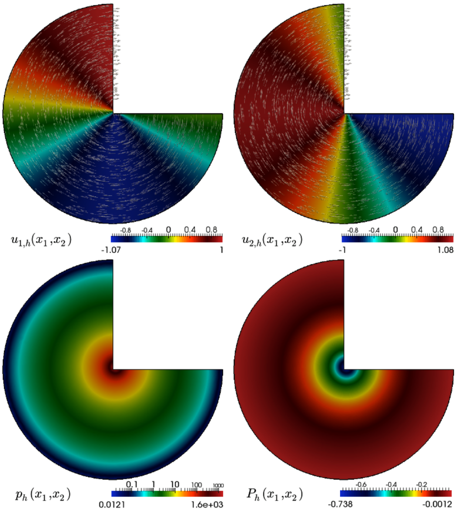

Figure 6.4: Example 2: Approximate velocity components (top), computed pressure distribution (bottom left) and postprocessed pressure (bottom right) obtained on an adapted mesh of 313041 vertices and 626080 elements.

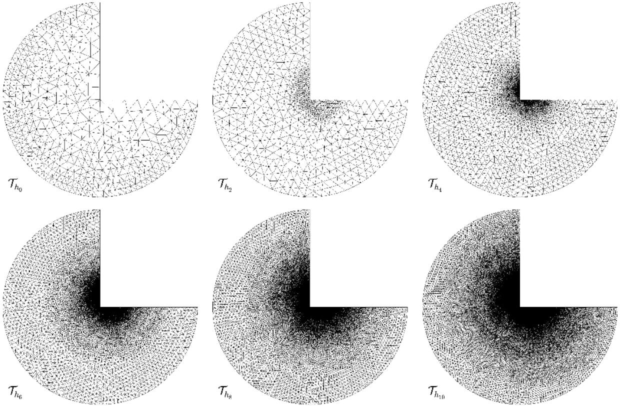

Figure 6.3: Example 2: Initial coarse mesh and adapted meshes after 2,4,6,8, and 10 iterations of Algorithm 1.

Numerical results

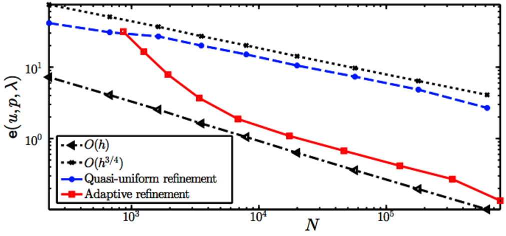

Figure 6.2: Example 2: Decay of the total error with respect to the number of degrees of freedom using a quasi-uniform and an adaptive refinement strategy (see individual errors in Table 6.2).

Example 3
$\Omega = (0,1)^3$ where the Dirichlet boundary is the bottom lid of the cube $\Gamma_D = (0,1) \times (0,1) \times \{0\}$, and the remaining faces constitute the Neumann boundary Γ_N. We construct $\boldsymbol{f}, g$ so that the exact solutions of the original and auxiliary Darcy problems (2), (2) are given by

$$U = \boldsymbol{u}(x_1, x_2, x_3) := \begin{pmatrix} \cos(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) \\ \sin(2\pi x_1)\cos(2\pi x_2)\sin(2\pi x_3) \\ -2\sin(2\pi x_1)\sin(2\pi x_2)\cos(2\pi x_3) \end{pmatrix},$$

$$p(x_1, x_2, x_3) := \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) + x_1 x_2 x_3,$$

$$P(x_1, x_2, x_3) := -\gamma^{-1}\log(p+1),$$

$$\lambda(x_1, x_2, x_3) := -p|_{\Gamma_N}.$$

As in the preceding tests, we choose the model parameters $\alpha_0 = 0.1$, $\gamma = 10$. Using as a base an initial tetrahedral mesh of 8 vertices and 18 elements, we perform eight successive refinements and we compute experimental errors in different norms.
Now we relax the fixed point tolerance to $\epsilon_{fp} = 1e-6$.

Numerical results

N	h	$\mathsf{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathsf{e}(p)$	$r(p)$	$\mathsf{e}(\lambda)$	$r(\lambda)$	$\mathsf{e}(P)$	$r(P)$	iter
32	1.414211	2.043353	–	0.495995	–	1.747510	–	0.124762	–	16
195	0.707107	1.066271	0.968334	0.273696	0.857752	0.809612	1.152151	0.073987	0.794381	15
2616	0.282843	0.519971	0.987055	0.155025	0.620363	0.426881	0.820118	0.029708	0.995827	15
19931	0.141421	0.246593	1.076301	0.079143	0.969966	0.259207	0.879427	0.019232	0.627301	13
96000	0.083189	0.145919	0.988807	0.046887	0.986583	0.168196	1.086805	0.011871	0.909225	12
340107	0.054392	0.095618	0.994823	0.030734	0.994049	0.082207	1.063027	0.007858	0.971003	11
974840	0.038222	0.067251	0.997451	0.021619	0.997142	0.041281	1.053195	0.005544	0.988516	10
2397651	0.028284	0.034231	0.999417	0.015553	0.998497	0.021958	0.920208	0.003473	0.983695	14
3256096	0.018757	0.018630	0.988616	0.007387	0.998462	0.011346	0.983085	0.001824	0.886841	14

Table 6.3: Example 3: Convergence results for the mixed finite element approximation of the Darcy problem (2.2) and postprocessed pressure $P_h = \gamma^{-1}\log(p_h + 1)$ on a sequence of uniformly refined triangulations of $\Omega = (0,1)^3$. Here we have considered the parameters $\alpha_0 = 0.1$, $\gamma = 10$.

Numerical results

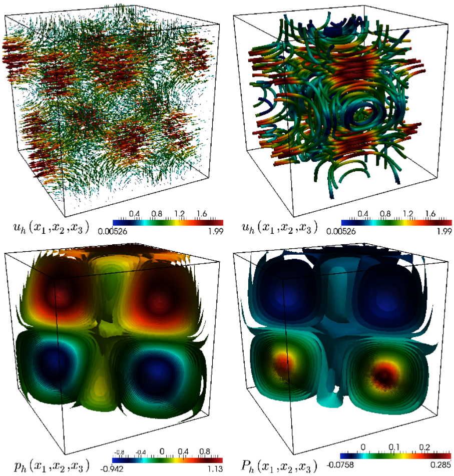

Figure 6.5: Example 2: Approximate velocity vectors and streamlines (top), computed pressure distribution (bottom left) and postprocessed pressure (bottom right) obtained on a uniform mesh of 286906 vertices and 573412 tetrahedral elements.

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