Splitting in time schemes: from fluids to phase-field models

Francisco Guillen-Gonzalez,
guillen@us.es
Depto EDAN and IMUS, Universidad de Sevilla, Spain.
Collaborators:
J.V.Gutierrez-Santacreu (USE, Spain),
M.V.Redondo-Neble (UCA, Spain)
G.Tierra (Praha, Czech R.),
R.Cabrales (Bio-Bio, Chile)

MORE, 21-26 September 2014, Liblice.

Model

- Differential system for $(\boldsymbol{u}, p)$ (velocity, pressure)

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot(2\nu \boldsymbol{D}\boldsymbol{u}) + \nabla p = 0 & (\boldsymbol{u}\text{-system}) \\ \nabla\cdot\boldsymbol{u} = 0 & (p\text{-equation}) \end{cases} \tag{1}$$

  with $\boldsymbol{D}\boldsymbol{u} = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t)/2$ and $\nu > 0$ the viscosity coefficient.

- BCs:

$$\boldsymbol{u}|_{\partial\Omega} = 0$$

- ICs:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \quad \text{in } \Omega$$

Energy's law

$$\frac{d}{dt}E_{kin}(\boldsymbol{u}) + \int_\Omega 2\nu|\boldsymbol{D}\boldsymbol{u}|^2 = 0, \tag{2}$$

One-step time scheme

Time-discrete schemes: Given a (uniform) partition of time interval $[0, T]$: $\{0 = t_0, t_1, \cdots, t_N = T\}$, with time step $k = T/N$, to design how to construct $(\boldsymbol{u}^n, p^n) \sim (\boldsymbol{u}(t_n), p(t_n))$

Step n: Given $\boldsymbol{u}^{n-1}$, to compute $(\boldsymbol{u}^n, p^n)$.

Notation: $\boldsymbol{u}^0 := \boldsymbol{u}^{n-1}$ (data) and $\boldsymbol{u} = \boldsymbol{u}^n$ (unknown). Idem for the pressure.

No-incremental pressure scheme (Chorin-Temam)

Reformulated as splitting (or segregated) scheme:

Sub-step 0: Given $\boldsymbol{u}^0$, find p^0 solving

$$-k\,\Delta p^0 + \varepsilon p^0 + \nabla\cdot\boldsymbol{u}^0 = 0 \quad \text{in } \Omega, \quad \partial_n p^0|_{\partial\Omega} = 0$$

Sub-step 1: Given $\boldsymbol{u}^0, p^0$, find $\boldsymbol{u}$ solving

$$\frac{1}{k}(\boldsymbol{u}-\boldsymbol{u}^0) + C(\boldsymbol{u}^0,\boldsymbol{u}) - \nabla\cdot(2\nu\boldsymbol{D}\boldsymbol{u}) + \nabla p^0 = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = 0.$$

Reformulation of convective terms: $C(\boldsymbol{u},\boldsymbol{v}) := (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \frac{1}{2}(\nabla\cdot\boldsymbol{u})\boldsymbol{v}$

Then

$$\int_\Omega C(\boldsymbol{u},\boldsymbol{v})\cdot\boldsymbol{v} = 0 \quad \forall\,\boldsymbol{u},\boldsymbol{v} \quad \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0.$$

Unconditional energy-stability

Idea: To get the energy-stability of the scheme, it is necessary to maintain the energy law (2) at discrete level.

Reformulation of No-incremental Algorithm, by using the "auxiliary velocity":

$$\boldsymbol{u}_\star^0 := \boldsymbol{u}^0 - k\,\nabla p^0$$

Sub-step 0:

$$\nabla \cdot \boldsymbol{u}_\star^0 + \varepsilon p^0 = 0, \quad i.e. \quad (\boldsymbol{u}_\star^0, \nabla \overline{p}) = \varepsilon (p^0, \overline{p}) \quad \forall\, \overline{p}$$

Sub-step 1: (without pressure!!)

$$\frac{1}{k}(\boldsymbol{u} - \boldsymbol{u}_\star^0) + C(\boldsymbol{u}^0, \boldsymbol{u}) - \nabla \cdot (\nu \boldsymbol{D}\boldsymbol{u}) = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = 0.$$

Stability estimates

Notation: $\|\cdot\| = \|\cdot\|_{L^2}$ and $(f,g) = \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x}$

Sub-step 0: Multiplying by $\boldsymbol{u}^0_\star$:

$$\frac{1}{k}\left(\frac{1}{2}\|\boldsymbol{u}^0_\star\|^2 - \frac{1}{2}\|\boldsymbol{u}^0\|^2\right) + \mathrm{ND0} + \mathrm{ND}_{pen} = 0,$$

where

$$\mathrm{ND0} = \frac{k}{2}\|\frac{\boldsymbol{u}^0_\star - \boldsymbol{u}^0}{k}\|^2 = \frac{1}{2}\|\sqrt{k}\nabla p^0\|^2, \quad \mathrm{ND}_{pen} = \varepsilon\,\|p^0\|^2$$

Sub-step 1: Multiplying by $\boldsymbol{u}$:

$$\frac{1}{k}\left(\frac{1}{2}\|\boldsymbol{u}\|^2 - \frac{1}{2}\|\boldsymbol{u}^0_\star\|^2\right) + \mathrm{ND1} + \int_\Omega 2\nu|\boldsymbol{Du}|^2 = 0$$

where

$$\mathrm{ND1} = \frac{k}{2}\|\frac{\boldsymbol{u} - \boldsymbol{u}^0_\star}{k}\|^2$$

Stability estimates

Adding (Notation: discrete time-derivative $\delta_t a_n = (a_n - a_{n-1})/k$)):

Theorem (Unconditional energy-stability)

$$\delta_t E_{kin}(\boldsymbol{u}^n) + \int_\Omega 2\nu |\boldsymbol{D}\boldsymbol{u}^n|^2 + \mathrm{ND0} + \mathrm{ND}_{pen} + \mathrm{ND1} = 0,$$

RK: No-incremental Algorithm is a "fully dissipative" scheme.

Estimates: "Integrating in time" (adding in n and multiplying by k):

- Velocity : $(\boldsymbol{u}^n)_n$ in $l^\infty(L^2) \cap l^2(H^1)$,
- Pressure, via numerical dissipation terms:

$$\mathrm{ND0} \quad \Rightarrow \quad (\sqrt{k}\nabla p^n)_n \quad \text{bounded. in } l^2(L^2)$$

$$\mathrm{ND}_{pen} \quad \Rightarrow \quad (\sqrt{\varepsilon} p^n)_n \quad \text{bounded. in } l^2(L^2)$$

- Pressure, via "inf-sup" LBB's condition ....

Sub-optimal error estimates [J.Shen, J.L.Guermond, ...]

- Velocity:
$$(u(t_n) - \boldsymbol{u}^n)_n \sim O(k) \text{ in } l^\infty(L^2) \cap l^2(H^1),$$
- Pressure:
$$(p(t_n) - p^n)_n \sim O(k^{1/2}) \text{ in } l^2(L^2)$$

Drawbacks:

-
$$\nabla \cdot \boldsymbol{u} = k\,\Delta p - \varepsilon p \sim O(k + \varepsilon)$$
- artificial BC for pressure
$$\partial_n p|_{\partial\Omega} = 0$$

Incremental pressure scheme (Van-Kan)

Sub-step 0: Given $\boldsymbol{u}^0$ and p^{-1}, find $\delta_t p^0$ solving

$$-k^2\,\Delta\delta_t p^0 + \varepsilon\,\delta_t p^0 + \nabla\cdot\boldsymbol{u}^0 = 0 \quad \text{in } \Omega, \quad \partial_n \delta_t p^0|_{\partial\Omega} = 0,$$

and

$$p^0 = p^{-1} + k\,\delta_t p^0$$

Sub-step 1: Find $\boldsymbol{u}$ solving

$$\frac{1}{k}(\boldsymbol{u}-\boldsymbol{u}^0)+C(\boldsymbol{u}^0,\boldsymbol{u})-\nabla\cdot(2\nu\boldsymbol{D}\boldsymbol{u})+\nabla(p^0 + k\,\delta_t p^0) = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = 0.$$

Unconditional energy-stability

Reformulation of Incremental Algorithm by using

$$\boldsymbol{u}^0_\star := \boldsymbol{u}^0 - k^2\,\nabla\delta_t p^0$$

Sub-step 0:

$$\nabla\cdot\boldsymbol{u}^0_\star + \varepsilon\,\delta_t p^0 = 0, \quad i.e. \quad (\boldsymbol{u}^0_\star, \nabla\overline{p}) = \varepsilon(\delta_t p^0, \overline{p}) \quad \forall\,\overline{p}$$

Sub-step 1: (without part of pressure!!)

$$\frac{1}{k}(\boldsymbol{u} - \boldsymbol{u}^0_\star) + C(\boldsymbol{u}^0, \boldsymbol{u}) - \nabla\cdot(2\nu\boldsymbol{D}\boldsymbol{u}) + \nabla p^0 = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = 0.$$

Stability estimates

Sub-step 0: Multiplying by $\boldsymbol{u}_\star^0$:

$$\frac{1}{k}\left(\frac{1}{2}\|\boldsymbol{u}_\star^0\|^2 - \frac{1}{2}\|\boldsymbol{u}^0\|^2\right) + \mathrm{ND0} + \mathrm{ND}_{pen} = 0,$$

where

$$\mathrm{ND0} = \frac{k}{2}\|\frac{\boldsymbol{u}_\star^0 - \boldsymbol{u}^0}{k}\|^2 = \frac{k}{2}\|k\,\nabla\delta_t p^0\|^2, \quad \mathrm{ND}_{pen} = \varepsilon\,\|\delta_t p^0\|^2$$

Sub-step 1: Multiplying by $\boldsymbol{u}$:

$$\frac{1}{k}\left(\frac{1}{2}\|\boldsymbol{u}\|^2 - \|\boldsymbol{u}_\star^0\|^2\right) + \mathrm{ND1} + \int_\Omega 2\nu|\boldsymbol{D}\boldsymbol{u}|^2 + (\nabla p^0, \boldsymbol{u}) = 0$$

where

$$\mathrm{ND1} = \frac{k}{2}\left\|\frac{\boldsymbol{u} - \boldsymbol{u}_\star^0}{k}\right\|^2$$

Adding:

$$\frac{1}{k}\left(\frac{1}{2}\|\boldsymbol{u}\|^2-\frac{1}{2}\|\boldsymbol{u}^0\|^2\right)+\int_\Omega 2\nu|\boldsymbol{Du}|^2+(\nabla p^0,\boldsymbol{u})+\mathrm{ND0}+\mathrm{ND}_{pen}+\mathrm{ND1}=0,$$

Lemma (Splitting error)

$$(\nabla p^0,\boldsymbol{u})=\delta_t\left(\frac{1}{2}\|k\,\nabla p\|^2+\frac{\varepsilon}{2}\|p\|^2\right)-\frac{k}{2}\,\|k\,\delta_t\nabla p\|^2-\frac{\varepsilon\,k}{2}\|\delta_t p\|^2$$

RK: Scheme introduces a "numerical source"

$$\mathrm{NS}:=-\frac{k}{2}\,\|k\,\delta_t\nabla p\|^2-\frac{\varepsilon\,k}{2}\|\delta_t p\|^2$$

and a "modified energy"

$$E_{mod}(p):=\frac{1}{2}\|k\,\nabla p\|^2+\frac{\varepsilon}{2}\|p\|^2$$

Developping ND1:

Lemma

$$\mathrm{ND}_{pen} + \mathrm{ND1} + \mathrm{NS} = \frac{k}{2}\|\delta_t \boldsymbol{u}_\star\|^2 + \frac{\varepsilon\, k}{2}\|\delta_t p - \delta_t p^0\|^2 + \frac{\varepsilon\, k}{2}\|\delta_t p^0\|^2$$

RK: The numerical source is controlled by the numerical dissipation.

Theorem (Unconditional energy-stability with modified energy)

$$\delta_t E_{kin}(\boldsymbol{u}) + \int_\Omega 2\nu|\boldsymbol{D}\boldsymbol{u}|^2$$

$$+\delta_t E_{mod}(p) + \mathrm{ND0} + \frac{k}{2}\|\delta_t \boldsymbol{u}_\star\|^2 + \frac{\varepsilon\, k}{2}\|\delta_t p - \delta_t p^0\|^2 + \frac{\varepsilon\, k}{2}\|\delta_t p^0\|^2 = 0.$$

RK: Incremental Algorithm is a dissipative scheme with a modified energy.

A priori Estimates

- Velocity:

$$(\boldsymbol{u}^n)_n \quad \text{bounded in } l^\infty(L^2) \cap l^2(H^1),$$

- Pressure, via the modified energy:

$$(k\,\nabla p^n)_n \quad \text{and} \quad (\sqrt{\varepsilon}\,p^n)_n \quad \text{bounded in } l^\infty(L^2)$$

Optimal accuracy [G-G & Redondo-Neble'14]

- Velocity:

$$(\boldsymbol{u}(t_n) - \boldsymbol{u}^n)_n \quad \text{and} \quad (\partial_t \boldsymbol{u}(t_n) - \delta_t \boldsymbol{u}^n)_n \sim O(k) \text{ in } l^\infty(L^2) \cap l^2(H^1),$$

- Pressure:

$$(p(t_n) - p^n)_n \sim O(k) \text{ in } l^\infty(L^2)$$

Two immiscible incompressible fluids (Abels-Garcke-Grun's model)

Model

Mixture of two immiscible and incompressible fluids with different (and constant) densities $\rho_1 > \rho_2 > 0$.

- Differential system (NS fluids + CH phase):

$$\left\{\begin{array}{rcl} \phi_t + \nabla\cdot(\phi\,\boldsymbol{u}) - \nabla\cdot(m(\phi)\nabla\mu) &=& 0,\\ -\lambda\varepsilon\Delta\phi + \dfrac{\lambda}{\varepsilon}f(\phi) &=& \mu,\\ \rho(\phi)\partial_t\boldsymbol{u} + \Big((\rho(\phi)\boldsymbol{u} - \rho_{dif}\,m(\phi)\nabla\mu)\cdot\nabla\Big)\boldsymbol{u} & & \\ \quad -\nabla\cdot(2\eta(\phi)\boldsymbol{D}\boldsymbol{u}) + \nabla p + \phi\nabla\mu &=& 0,\\ \nabla\cdot\boldsymbol{u} &=& 0, \end{array}\right. \tag{3}$$

- $\boldsymbol{u}$ velocity, p potential function (depending on pressure and phase)
- ϕ is an order parameter ($\phi \equiv -1$ in fluid 1 and $\phi \equiv 1$ in fluid 2)
- μ chemical potential
- ρ density (linear wrt. the phase): $\rho(\phi) = \frac{\rho_1+\rho_2}{2} + \frac{\rho_1-\rho_2}{2}\phi := \rho_{med} + \rho_{dif}\phi$

Moreover,

- $\eta(\phi) > 0$ viscosity of the mixture,
- $\lambda > 0$ the surface tension,
- $\varepsilon > 0$ a (small) parameter related to the thickness of the interface between the two fluids.
- $m(\phi) > 0$ is the mobility function
- $F(\phi)$ is the polynomial Ginzburg-Landau double well potential with minima in ± 1 (representing the pure phases):

$$F(\phi) = \frac{1}{4}(\phi^2 - 1)^2, \quad f(\phi) = F'(\phi) = (\phi^2 - 1)\phi.$$

Energy's law [Abels-Garcke-Grun'12]

Conservation of phase

$$\frac{d}{dt}\int_\Omega \phi(t,x) = 0. \tag{4}$$

Energy's law

Multiplying by $(\mu, \partial_t\phi, \boldsymbol{u}, p)$ in (3), and testing the ρ-equation by $|\boldsymbol{u}|^2/2$, the following (dissipative) energy law holds:

$$\frac{d}{dt}E(\boldsymbol{u}(t),\phi(t)) + 2\int_\Omega \eta(\phi)|\boldsymbol{D}\boldsymbol{u}|^2 + \int_\Omega m(\phi)|\nabla\mu|^2 = 0, \tag{5}$$

where the free energy is

$$E(\boldsymbol{u},\phi) = E_{kin}(\boldsymbol{u}) + \lambda\, E_{mix}(\phi) := \int_\Omega \rho(\phi)\frac{|\boldsymbol{u}|^2}{2} + \lambda\left(\frac{\varepsilon}{2}\int_\Omega |\nabla\phi|^2 + \frac{1}{\varepsilon}\int_\Omega F(\phi)\right),$$

Splitting in time scheme [G-G & Tierra, to appear in JCM]

Step 1: Compute $[\phi^n]$ a constant truncation of ϕ^n outside of $-1, 1$, and consider the modified density

$$\rho([\phi^n]) = \rho_{med} + \rho_{dif}[\phi^n] \tag{6}$$

Step 2: Find (ϕ^{n+1}, w^{n+1}) satisfying

$$\begin{cases} \delta_t \phi^{n+1} + \nabla \cdot ([\phi^n]\, \boldsymbol{u}_\star^n) - \nabla \cdot (m([\phi^n]) \nabla \mu^{n+1}) = 0, \quad \text{in } \Omega, \\ -\lambda\varepsilon \Delta \phi^{n+1} + \dfrac{\lambda}{\varepsilon} f_k(\phi^{n+1}, \phi^n) - \mu^{n+1} = 0, \quad \text{in } \Omega, \\ \dfrac{\partial \phi^{n+1}}{\partial n}\Big|_{\partial\Omega} = 0, \quad \dfrac{\partial w^{n+1}}{\partial n}\Big|_{\partial\Omega} = 0 \quad \text{in } (0, T). \end{cases} \tag{7}$$

where

$$\boldsymbol{u}_\star^n = \boldsymbol{u}^n - k \frac{[\phi^n]}{\rho([\phi^n])} \nabla \mu^{n+1}, \quad \text{in } \Omega, \tag{8}$$

and $f_k(\phi^{n+1}, \phi^n)$ denotes an approximation of $f(\phi(t_{n+1}))$.

Step 3: Find $(\boldsymbol{u}^{n+1}, p^{n+1})$ satisfying

$$\begin{cases} \rho([\phi^n])\delta_t \boldsymbol{u}^{n+1} + \dfrac{1}{2}\boldsymbol{u}^{n+1}\delta_t\rho([\phi^{n+1}]) - \nabla\cdot\Big(2\eta([\phi^n])\boldsymbol{D}\boldsymbol{u}^{n+1}\Big) \\ \quad + C\Big(\rho(\phi^n)\boldsymbol{u}^n - \rho_{dif}\, m([\phi^n])\nabla\mu^{n+1}, \boldsymbol{u}^{n+1}\Big) + \nabla p^{n+1} = -[\phi^n]\nabla\mu^{n+1}, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0, \\ \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0. \end{cases} \tag{9}$$

RK: To arrive at (9) we have used the following "residual expression"

$$\frac{1}{2}\Big(\delta_t\rho([\phi^{n+1}]) + div\big(\rho(\phi^n)\boldsymbol{u}^n - \rho_{dif}\, m(\phi^n)\nabla\mu^n\big)\Big)\boldsymbol{u}^{n+1}.$$

This idea has been used in [G-G & Gutierrez-Santacreu'08] to define a stable approximation of the density-dependent Navier-Stokes problem.

Unconditional energy-stability

Conservative property

$$\int_\Omega \phi^{n+1} dx = \int_\Omega \phi^n dx.$$

Theorem (Uncond. stability)

It holds

$$\begin{aligned} &\delta_t E(\boldsymbol{u}^{n+1}, \phi^{n+1}) + \int_\Omega 2\eta([\phi^n])|\boldsymbol{D}\boldsymbol{u}^{n+1}|^2 + \int_\Omega m([\phi^n])|\nabla \mu^{n+1}|^2 \\ &+ ND^{n+1}_{philic} + ND^{n+1}_{phobic} = 0, \end{aligned} \tag{10}$$

where the philic numerical dissipation is

$$ND^{n+1}_{philic} = k\frac{\lambda\varepsilon}{2}\|\delta_t \nabla \phi^{n+1}\|^2_{L^2} + \frac{k}{2}\int_\Omega \rho([\phi^n])\left(\left|\frac{\boldsymbol{u}^n - \boldsymbol{u}^n_\star}{k}\right|^2 + \left|\frac{\boldsymbol{u}^n_\star - \boldsymbol{u}^{n+1}}{k}\right|^2\right) dx$$

and the phobic numerical dissipation (or source) is

$$ND^{n+1}_{phobic} = \frac{\lambda}{\varepsilon}\left(\int_\Omega f_k(\phi^{n+1}, \phi^n)\delta_t \phi^{n+1} - \delta_t \int_\Omega F(\phi^{n+1})\right).$$

Nematic liquid crystals (Lin's model)

Model

- Energy:

$$E(\boldsymbol{u},\boldsymbol{d}) := E_{kin}(\boldsymbol{u}) + \lambda\, E_{nem}(\boldsymbol{d}) := \frac{1}{2}|\boldsymbol{u}|^2 + \lambda\left(\frac{1}{2}|\nabla \boldsymbol{d}|^2 + \frac{1}{\varepsilon^2}\int_\Omega F(\boldsymbol{d})\right)$$

  with $\lambda > 0$ an elastic constant and $F(\boldsymbol{d})$ a $C^2(\mathrm{R})$ truncate (for $|\boldsymbol{d}| > 1$) Ginzburg-Landau potential.

- Differential system (NS fluids + AC nematic): By using $\boldsymbol{w} = \dfrac{\delta E(\boldsymbol{u},\boldsymbol{d})}{\delta \boldsymbol{d}}$:

$$\left\{\begin{array}{rcl} \partial_t \boldsymbol{d} + (\nabla \boldsymbol{d})\boldsymbol{u} + \gamma\, \boldsymbol{w} &=& \boldsymbol{0} \\ -\lambda\,\Delta \boldsymbol{d} + \dfrac{\lambda}{\varepsilon^2}\boldsymbol{f}(\boldsymbol{d}) - \boldsymbol{w} &=& \boldsymbol{0} \\ \partial_t \boldsymbol{u} + C(\boldsymbol{u},\boldsymbol{u}) - \nu\Delta \boldsymbol{u} + \nabla p - (\nabla \boldsymbol{d})^t \boldsymbol{w} &=& \boldsymbol{0} \\ \nabla \cdot \boldsymbol{u} &=& 0 \end{array}\right. \tag{11}$$

Energy's law

$$\frac{d}{dt}E(\boldsymbol{u}(t),\boldsymbol{d}(t)) + \nu\int_\Omega |\nabla \boldsymbol{u}|^2 + \gamma\int_\Omega |\boldsymbol{w}|^2 = 0. \tag{12}$$

Linear stable approximation of the potential terms ([Shen])

The Taylor expansion of second order of $F(\boldsymbol{d})$ gives

$$\begin{aligned} F(\boldsymbol{d}^{n+1}) - F(\boldsymbol{d}^n) &= \nabla_{\boldsymbol{d}} F(\boldsymbol{d}^n) \cdot (\boldsymbol{d}^{n+1} - \boldsymbol{d}^n) \\ &+ \frac{1}{2}(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n)^T H_{\boldsymbol{d}} F(\boldsymbol{d}^{n+\theta})(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n), \end{aligned} \tag{13}$$

where $H_{\boldsymbol{d}} F(\boldsymbol{d})$ is the Hessian matrix of $F(\boldsymbol{d})$ and $\boldsymbol{d}^{n+\theta} = \theta \boldsymbol{d}^{n+1} + (1-\theta)\boldsymbol{d}^n$ for some $\theta \in (0,1)$. Since $F(\boldsymbol{d})$ is essentially quadratic then the euclidean norm of the hessian

$$\|H_{\boldsymbol{d}} F(\boldsymbol{d})\| \leq H_F, \quad \forall \boldsymbol{d}.$$

In particular,

$$\frac{1}{2}(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n)^T H_{\boldsymbol{d}} F(\boldsymbol{d}^{n+\theta})(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n) \leq \frac{H_F}{2}|\boldsymbol{d}^{n+1} - \boldsymbol{d}^n|^2 \tag{14}$$

Consequently, if one considers the approximation of the potential term $\boldsymbol{f}(\boldsymbol{d}(t_{n+1}))$ as

$$\boldsymbol{f}(\boldsymbol{d}^n) + \frac{H_F}{2}(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n)$$

(i.e. an explicit approximation of the potential term plus a large enough first-order linear dissipation term, from (13) and (14), one has

$$\left(\boldsymbol{f}(\boldsymbol{d}^n) + \frac{H_F}{2}(\boldsymbol{d}^{n+1} - \boldsymbol{d}^n)\right) \cdot (\boldsymbol{d}^{n+1} - \boldsymbol{d}^n) \geq F(\boldsymbol{d}^{n+1}) - F(\boldsymbol{d}^n). \tag{15}$$

This inequality will play an essential role for the energy-stability of schemes.

Description of the scheme

Step1 Find $(\boldsymbol{d}^{n+1}, \boldsymbol{w}^{n+1})$:

$$\left\{ \begin{array}{rcl} \delta_t \boldsymbol{d}^{n+1} + (\nabla \boldsymbol{d}^n)\widetilde{\boldsymbol{u}}^{n+1} + \gamma\, \boldsymbol{w}^{n+1} & = & 0, \\ -\lambda\, \Delta \boldsymbol{d}^{n+1} + \dfrac{\lambda}{\varepsilon^2} \boldsymbol{f}(\boldsymbol{d}^n) + \dfrac{\lambda}{\varepsilon^2} \dfrac{H_F}{2} (\boldsymbol{d}^{n+1} - \boldsymbol{d}^n) - \boldsymbol{w}^{n+1} & = & 0, \end{array} \right. \tag{16}$$

where

$$\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{u}^n + k\, (\nabla \boldsymbol{d}^n)^T \boldsymbol{w}^{n+1}. \tag{17}$$

RK: This problem can be decoupled at algebraic level.

Step2 Find $p^{n+1} \in P_h$:

$$k(\nabla p^{n+1}, \nabla \bar{p}) + j(p^{n+1}, \bar{p}) = -(\nabla \cdot \widetilde{\boldsymbol{u}}^{n+1}, \bar{p}), \quad \forall\, \bar{p} \in P_h \tag{18}$$

with the stabilization term (letting equal FE spaces for velocity and pressure)

$$j(p^{n+1}, \bar{p}) = \frac{S}{\nu}(p^{n+1} - \Pi_0(p^{n+1}), \bar{p} - \Pi_0(\bar{p})), \tag{19}$$

where $S > 0$ is an algorithmic constant and Π_0 is the L^2-orthogonal projection operator onto the piecewise constant finite-element space Y_h.

Step3 Find $\boldsymbol{u}^{n+1}$ s.t. $\boldsymbol{u}^{n+1}|_{\partial\Omega} = \boldsymbol{0}$ and satisfying

$$\delta_t \boldsymbol{u}^{n+1} + C(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}) - \nu\,\Delta \boldsymbol{u}^{n+1} + \nabla p^{n+1} - (\nabla \boldsymbol{d}^n)^T \boldsymbol{w}^{n+1} = 0. \qquad (20)$$

Reformulation. Defining

$$\widehat{\boldsymbol{u}}^{n+1} = \widetilde{\boldsymbol{u}}^{n+1} - k\,\nabla p^{n+1} \qquad (21)$$

then Step 3 is rewritten without pressure and elastic terms:

$$\frac{\boldsymbol{u}^{n+1} - \widehat{\boldsymbol{u}}^{n+1}}{k} + C(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}) - \nu\,\Delta \boldsymbol{u}^{n+1} + \nabla p^{n+1} - (\nabla \boldsymbol{d}^n)^T \boldsymbol{w}^{n+1} = 0.$$

Unconditional energy-stability

Theorem (Unconditional stability)

It holds:

$$\begin{aligned}&\delta_t E(\boldsymbol{u}^{n+1}, \boldsymbol{d}^{n+1}) + \nu \|\nabla \boldsymbol{u}^{n+1}\|^2 + \gamma \|\boldsymbol{w}^{n+1}\|^2 \\ &\quad + k\,\frac{\lambda}{2}\|\nabla \delta_t \boldsymbol{d}^{n+1}\|^2 + j(p^{n+1}, p^{n+1}) \\ &\quad + \frac{k}{2}\Big(\|\frac{\boldsymbol{u}^{n+1} - \widehat{\boldsymbol{u}}^{n+1}}{k}\|^2 + \|\frac{\widehat{\boldsymbol{u}}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{k}\|^2 + \|\frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^{n}}{k}\|^2\Big) \leq 0\end{aligned} \tag{22}$$

Conclusions and Future work.

Conclusions

1. Comparation between No-Incremental and Incremental splitting schemes in fluids
2. Application of No-incremental ideas to Two Phase-Field models, obtaining energy-stable first-order (fully) discrete schemes.

Future work

1. Application of Incremental ideas to Two Phase-Field models,
2. Convergence and accuracy of splitting in time schemes
3. To design second order splitting stable schemes
4. Large time convergence of splitting schemes in complex fluids

THANK YOU FOR YOUR ATTENTION