Accuracy of Momentum Balance and Surface Forces for Finite Element Computations in Elasticity

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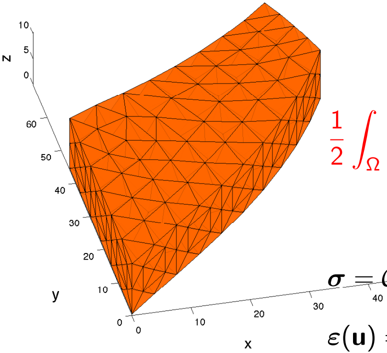
Overview

Incompressible Linear Elasticity: Variational Formulations

First-Order System Formulation of Linear Elasticity

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$
Stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{d \times d}$



$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}$$

$$\frac{1}{2} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) \, dx - \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx - \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{u} \, dx \longrightarrow \min$$

$$\boldsymbol{\sigma} = \mathcal{C} \, \boldsymbol{\varepsilon}(\mathbf{u}) := 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \, (\mathrm{tr} \, \boldsymbol{\varepsilon}(\mathbf{u})) \, \mathbf{I}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{A} \boldsymbol{\sigma} := \frac{1}{2\mu} \left(\boldsymbol{\sigma} - \frac{\lambda}{\lambda d + 2\mu} (\mathrm{tr} \, \boldsymbol{\sigma}) \, \mathbf{I} \right)$$

Incompressibility: $\lambda \to \infty$

Incompressible Linear Elasticity: Variational Formulations

First-Order System Formulation of Linear Elasticity

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$
Stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{d \times d}$

$$\varepsilon(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}$$

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \text{ in } \Omega \\ \mathcal{A}\boldsymbol{\sigma} - \varepsilon(\mathbf{u}) &= \mathbf{0} \text{ in } \Omega \\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} \text{ on } \Gamma_N \end{aligned}$$

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu} \left(\boldsymbol{\sigma} - \frac{\lambda}{\lambda d + 2\mu} (\operatorname{tr} \boldsymbol{\sigma}) \, \mathbf{I} \right)$$

$$\mathcal{A} = \begin{cases} \mathcal{C}^{-1} & \text{, if } \lambda < \infty \\ \frac{1}{2\mu} \operatorname{dev} & \text{, if } \lambda = \infty \end{cases}$$

$\mathbf{u} \in H^1_{\Gamma_D}(\Omega)^d$
$\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\operatorname{div}, \Omega)^d$ ($\boldsymbol{\sigma}^N \in H(\operatorname{div}, \Omega)^d$ s.t. $\boldsymbol{\sigma}^N \cdot \mathbf{n} = \mathbf{t}$ on Γ_N)

Stress tensor $\boldsymbol{\sigma}$ is of particular interest on its own:
- large stresses components cause inelastic behavior or damage
- $\boldsymbol{\sigma} \cdot \mathbf{n}$ represents surface forces caused by the deformation

Incompressible Linear Elasticity: Variational Formulations

Displacement-Pressure (Galerkin) Formulation $(\mathbf{u}^g, p^g)$

Insert new variable p into material (2nd) eqn:

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{\lambda d + 2\mu}(\operatorname{tr}\boldsymbol{\sigma})\,\mathbf{I}\right) = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - p\,\mathbf{I}\right) = \varepsilon(\mathbf{u})$$

and combine this with momentum balance (1st) equation:

Determine $\mathbf{u}^g \in H^1_{\Gamma_D}(\Omega)^d$, $p^g \in L^2(\Omega)$ such that

$$\begin{aligned} 2\mu\,(\varepsilon(\mathbf{u}^g), \varepsilon(\mathbf{v}))_{L^2(\Omega)} + (p^g, \operatorname{div}\mathbf{v})_{L^2(\Omega)} &= (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} + \langle \mathbf{t}, \mathbf{v}\rangle_{L^2(\Gamma_N)} \\ (\operatorname{div}\mathbf{u}^g, q)_{L^2(\Omega)} &= \frac{1}{\lambda}(p^g, q)_{L^2(\Omega)} \end{aligned}$$

holds for all $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)^d$, $q \in L^2(\Omega)$

$$\boldsymbol{\sigma}^g = 2\mu\,\varepsilon(\mathbf{u}^g) + p^g\,\mathbf{I} \in L^2(\Omega)^{d\times d}$$

Incompressible Linear Elasticity: Variational Formulations

Hellinger-Reissner (Mixed) Saddle-Point Formulation $(\boldsymbol{\sigma}^m, \mathbf{u}^m)$

Determine $\boldsymbol{\sigma}^m \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\mathrm{div}, \Omega)^d$, $\mathbf{u}^m \in L^2(\Omega)^d$ and $\gamma^m \in L^2(\Omega)^{d \times d, \mathrm{skew}}$ such that

$$
\begin{aligned}
(\mathcal{A}\boldsymbol{\sigma}^m, \boldsymbol{\tau})_{L^2(\Omega)} + (\mathbf{u}^m, \mathrm{div}\, \boldsymbol{\tau})_{L^2(\Omega)} + (\gamma^m, \mathrm{skew}\, \boldsymbol{\tau})_{L^2(\Omega)} &= 0 \\
(\mathrm{div}\, \boldsymbol{\sigma}^m + \mathbf{f}, \mathbf{v})_{L^2(\Omega)} &= 0 \\
(\mathrm{skew}\, \boldsymbol{\sigma}^m, \boldsymbol{\eta})_{L^2(\Omega)} &= 0
\end{aligned}
$$

holds for all $\boldsymbol{\tau} \in H_{\Gamma_N}(\mathrm{div}, \Omega)^d$, $\mathbf{v} \in L^2(\Omega)^d$ and $\boldsymbol{\eta} \in L^2(\Omega)^{d \times d, \mathrm{skew}}$

$$
\mathrm{skew}\boldsymbol{\tau} = \frac{1}{2}(\boldsymbol{\tau} - \boldsymbol{\tau}^T)
$$

$$
L^2(\Omega)^{d \times d, \mathrm{skew}} = \{\boldsymbol{\tau} \in L^2(\Omega)^{d \times d} : \boldsymbol{\tau} + \boldsymbol{\tau}^T = \mathbf{0}\}
$$

Incompressible Linear Elasticity: Variational Formulations

Finite element spaces for the Hellinger-Reissner formulation

Arnold/Brezzi/Douglas (1984) (PEERS),
Brezzi/Douglas/Marini (1986)
Stenberg (1988)

. . . ,

Arnold/Winther (2002): AW_k

Lonsing/Verfürth (2004): $PEERS_k$ and $BDMS_k$

. . . ,

Arnold/Falk/Winther (2007)
Boffi/Brezzi/Fortin (2009): $RT_k^d \times P_{k,\mathrm{disc}}^d \times P_{k,\mathrm{cont}}^{d\times d,\mathrm{skew}}$, $k \geq 1$
Pechstein/Schöberl (2011)

Incompressible Linear Elasticity: Variational Formulations

First-Order System Least Squares Formulation $(\boldsymbol{\sigma}^{ls}, \mathbf{u}^{ls})$

Determine $\boldsymbol{\sigma}^{ls} \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\mathrm{div}, \Omega)^d$ and $\mathbf{u}^{ls} \in H^1_{\Gamma_D}(\Omega)^d$ such that

$$\mathcal{F}(\mathbf{u}, \boldsymbol{\sigma}) := \|\mathrm{div}\, \boldsymbol{\sigma} + \mathbf{f}\|^2_{L^2(\Omega)} + \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega)}$$

is minimized

Equivalently: $\boldsymbol{\sigma}^{ls} \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\mathrm{div}, \Omega)^d$ and $\mathbf{u}^{ls} \in H^1_{\Gamma_D}(\Omega)^d$ s.t.

$$\begin{aligned}(\mathrm{div}\, \boldsymbol{\sigma}^{ls} + \mathbf{f}, \mathrm{div}\, \boldsymbol{\tau})_{L^2(\Omega)} + (\mathcal{A}\boldsymbol{\sigma}^{ls} - \boldsymbol{\varepsilon}(\mathbf{u}^{ls}), \mathcal{A}\boldsymbol{\tau})_{L^2(\Omega)} &= 0 \\ (\mathcal{A}\boldsymbol{\sigma}^{ls} - \boldsymbol{\varepsilon}(\mathbf{u}^{ls}), \boldsymbol{\varepsilon}(\mathbf{v}))_{L^2(\Omega)} &= 0\end{aligned}$$

holds for all $\boldsymbol{\tau} \in H_{\Gamma_N}(\mathrm{div}, \Omega)^d$ and $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)^d$

From now on: $(\ ,\ )$ instead of $(\ ,\ )_{L^2(\Omega)}$, $\|\ \ \|$ instead of $\|\ \ \|_{L^2(\Omega)}$

Incompressible Linear Elasticity: Variational Formulations

Approximation Properties for First-Order System Least Squares

Coercivity of the first-order system least squares bilinear form

$$\mathcal{B}(\mathbf{u},\boldsymbol{\sigma};\mathbf{v},\boldsymbol{\tau}) = (\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau}) + (\mathcal{A}\boldsymbol{\sigma} - \varepsilon(\mathbf{u}), \mathcal{A}\boldsymbol{\tau} - \varepsilon(\mathbf{v}))$$

in $H^1_{\Gamma_D}(\Omega)^d \times H_{\Gamma_N}(\operatorname{div},\Omega)^d$ with respect to

$$|||(\mathbf{v},\boldsymbol{\tau})||| = \left(\|\varepsilon(\mathbf{v})\|^2 + \|\operatorname{div}\boldsymbol{\tau}\|^2 + \|\boldsymbol{\tau}\|^2\right)^{1/2}$$

holds uniformly for $\lambda \to \infty$

Cai/St., SIAM J. Numer. Anal., 2004
based on ideas from Cai/Lazarov/Manteuffel/McCormick, 1994

$\Longrightarrow$ Optimal order convergence:

$$|||(\mathbf{u}-\mathbf{u}_h^{ls},\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^{ls})||| \eqsim \inf_{\mathbf{v}_h,\boldsymbol{\tau}_h} |||(\mathbf{u}-\mathbf{v}_h,\boldsymbol{\sigma}-\boldsymbol{\tau}_h)|||$$

for subspaces $\mathbf{V}_h \subset H^1_{\Gamma_D}(\Omega)^d$, $\boldsymbol{\Sigma}_h \subset H_{\Gamma_N}(\operatorname{div},\Omega)^d$

Incompressible Linear Elasticity: Variational Formulations

Finite Element Spaces and Approximation Properties

In comparison, for the displacement-pressure formulation:

$$|||(\mathbf{u} - \mathbf{u}_h^g, \mathbf{0})||| = \|\varepsilon(\mathbf{u} - \mathbf{u}_h^g)\| \eqsim \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\varepsilon(\mathbf{u} - \mathbf{v}_h)\|$$

(if an inf-sup stable Stokes finite element pair is used)

And, for the Hellinger-Reissner (mixed) formulation:

$$\begin{aligned} |||(\mathbf{0}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)||| &= \left(\|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)\|^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|\right)^{1/2} \\ &\eqsim \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h} \left(\|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|^2 + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|\right)^{1/2} \end{aligned}$$

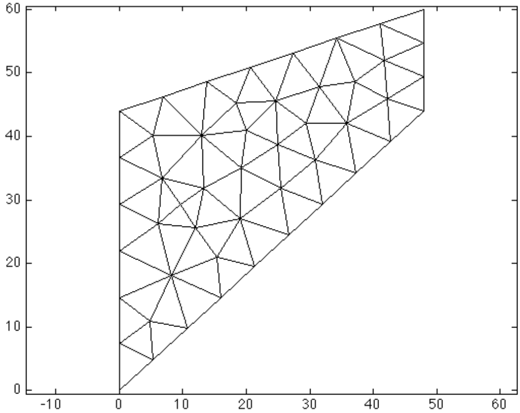
(if Σ_h is part of an inf-sup stable finite element combination)

$$\Longrightarrow \quad |||(\mathbf{u} - \mathbf{u}_h^{ls}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls})||| \eqsim |||(\mathbf{u} - \mathbf{u}_h^g, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)|||$$

Incompressible Linear Elasticity: Stress Approximation

Cook's Membrane

Finite element spaces based on a triangulation $\mathcal{T}_h$
$\mathbf{V}_h$: H^1-conforming $\mathcal{P}_2$ elements
$\mathbf{\Sigma}_h$: $H(\mathrm{div})$-conforming $\mathcal{RT}_1$ elements



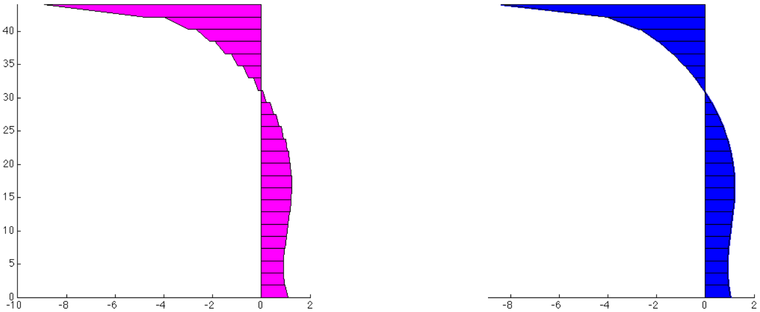
small-strain
(linear elasticity)
incompressible case
$\lambda = \infty$ $(\nu = 0.5)$

Boundary conditions:
$\mathbf{u} = \mathbf{0}$ at left
$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$ at top/bottom
$\boldsymbol{\sigma} \cdot \mathbf{n} = (0, 1)$ at right,

Incompressible Linear Elasticity: Stress Approximation

Cook's Membrane

Why is accurate momentum conservation important?
Normal traction $\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})$ on left boundary Γ

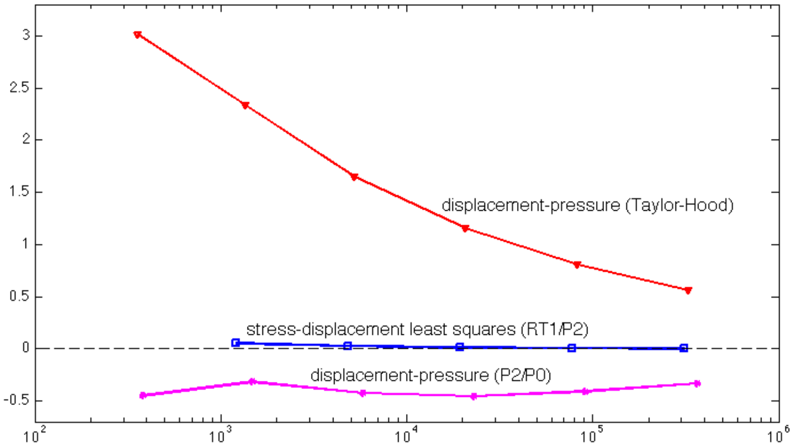


displacement-pressure (P2/P0) stress-displacement (RT1/P2)

Incompressible Linear Elasticity: Stress Approximation

Cook's Membrane

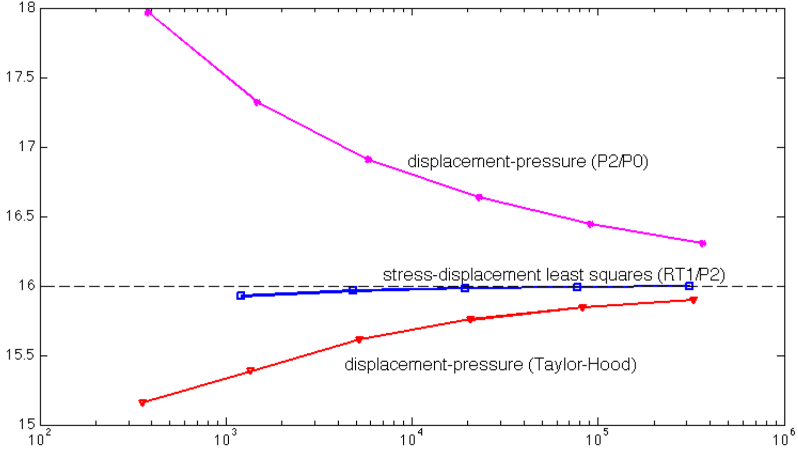
Resultant normal traction $\int_\Gamma \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})\, ds = 0$ on left boundary Γ



Incompressible Linear Elasticity: Stress Approximation

Cook's Membrane

Resultant tangential traction $\int_\Gamma \mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})\, ds = 16$ on Γ



Incompressible Linear Elasticity: Stress Approximation

Stress Approximations in $H(\mathrm{div}, \Omega)$

$\mathbf{e}_i$: constant unit test functions

$$
\begin{aligned}
|\langle \boldsymbol{\sigma}_h \cdot \mathbf{n} - \boldsymbol{\sigma} \cdot \mathbf{n}, \mathbf{e}_i \rangle_{0,\partial\Omega}| &= |(\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \mathbf{e}_i)| = |(\mathrm{div}\,\boldsymbol{\sigma}_h + \mathbf{f}, \mathbf{e}_i)| \\
&\leq |\Omega|^{1/2} \|\mathrm{div}\,\boldsymbol{\sigma}_h + \mathbf{f}\| \\
&\leq |\Omega|^{1/2} \left(\|\mathrm{div}\,\boldsymbol{\sigma}_h + \mathcal{P}_h \mathbf{f}\| + \|\mathbf{f} - \mathcal{P}_h \mathbf{f}\| \right)
\end{aligned}
$$

$\mathcal{P}_h$: $L^2(\Omega)$-orthogonal projection into $\mathbf{Z}_h$ (discont. $\mathcal{P}_1$-elements)

Hellinger-Reissner formulation using elements with $\mathrm{div}\,\boldsymbol{\Sigma}_h \subset \mathbf{Z}_h$
e.g. $\mathcal{RT}_1^d \times \mathcal{P}_{1,disc}^d \times \mathcal{P}_{1,cont}^{d\times d,\mathrm{skew}}$ (Boffi/Brezzi/Fortin, 2009):

$$
|\langle \boldsymbol{\sigma}_h^m \cdot \mathbf{n} - \boldsymbol{\sigma} \cdot \mathbf{n}, \mathbf{e}_i \rangle_{0,\partial\Omega}| \leq |\Omega|^{1/2} \|\mathbf{f} - \mathcal{P}_h \mathbf{f}\|
$$

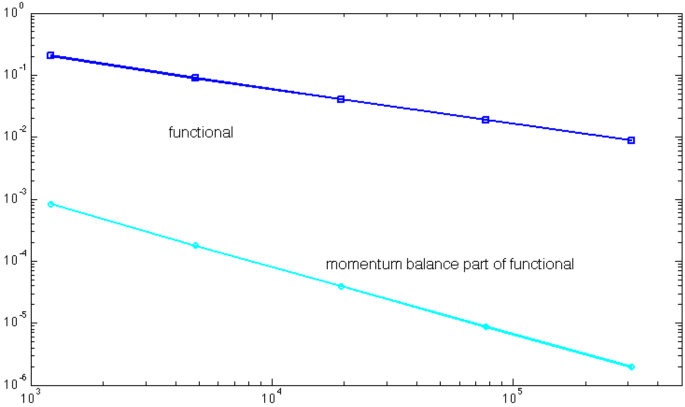
In any case:
$$
\|\boldsymbol{\sigma} \cdot \mathbf{n} - \boldsymbol{\sigma}_h \cdot \mathbf{n}\|_{-1/2,\partial\Omega} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div},\Omega}
$$

for any $H(\mathrm{div}, \Omega)$-conforming stress approximation

Accuracy of Momentum Balance

Computational Results

Observation for First-Order System Least Squares:
Momentum balance accuracy of higher order



$\mathcal{F}(\boldsymbol{\sigma}_h^{ls}, \mathbf{u}_h^{ls}) = \|\mathrm{div}\, \boldsymbol{\sigma}_h^{ls}\|_{L^2(\Omega)}^2 + \|\mathcal{A}\boldsymbol{\sigma}_h^{ls} - \boldsymbol{\varepsilon}(\mathbf{u}_h^{ls})\|_{L^2(\Omega)}^2$ vs. $\|\mathrm{div}\, \boldsymbol{\sigma}_h^{ls}\|_{L^2(\Omega)}^2$

Accuracy of Momentum Balance

A Duality Argument for Improved Momentum Balance

Regularity assumption (R_α):
$\boldsymbol{\sigma} \in H^\alpha(\Omega)^{d\times d}$, $\mathbf{u} \in H^{1+\alpha}(\Omega)^d$ for any linear elasticity problem with $(\boldsymbol{\sigma}, \mathbf{u}) \in H_{\Gamma_N}(\mathrm{div}, \Omega)^d \times H^1_{\Gamma_D}(\Omega)^d$.

This implies: $\mathcal{F}(\boldsymbol{\sigma}_h^{ls}, \mathbf{u}_h^{ls})^{1/2} \lesssim h^\alpha \|\boldsymbol{\sigma}\|_{\alpha,\Omega}$

Theorem: Let (R_α) be fulfilled for some $\alpha > 0$ and consider the first-order system least squares approximation $(\boldsymbol{\sigma}_h^{ls}, \mathbf{u}_h^{ls})$ with RT_k/conforming P_{k+1} spaces. Then, if $k \geq \alpha - 1$,

$$\|\mathrm{div}\,(\boldsymbol{\sigma}_h^{ls} - \boldsymbol{\sigma})\| \lesssim h^\alpha \mathcal{F}(\boldsymbol{\sigma}_h^{ls}, \mathbf{u}_h^{ls})^{1/2} + \inf_{\mathbf{z}_h \in \mathbf{Z}_h} \|\mathbf{f} - \mathbf{z}_h\| \, .$$

Proof: (based on an idea from Brandts/Chen/Yang, 2006)

Step 1: $\|\mathrm{div}\,(\boldsymbol{\sigma}_h^{ls} - \boldsymbol{\sigma})\| = \|\mathrm{div}\,\boldsymbol{\sigma}_h^{ls} + \mathbf{f}\| = \|\mathrm{div}\,\boldsymbol{\sigma}_h^{ls} + \mathbf{f}_h\| + \inf_{\mathbf{z}_h \in \mathbf{Z}_h} \|\mathbf{f} - \mathbf{z}_h\|$

with

$$\|\mathrm{div}\,\boldsymbol{\sigma}_h^{ls} + \mathbf{f}_h\| = \sup_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h} \frac{(\mathrm{div}\,\boldsymbol{\sigma}_h^{ls} + \mathbf{f}_h, \mathrm{div}\,\boldsymbol{\tau}_h)}{\|\mathrm{div}\,\boldsymbol{\tau}_h\|} = \sup_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h} \frac{(\mathrm{div}\,(\boldsymbol{\sigma}_h^{ls} - \boldsymbol{\sigma}), \mathrm{div}\,\boldsymbol{\tau}_h)}{\|\mathrm{div}\,\boldsymbol{\tau}_h\|}$$

Accuracy of Momentum Balance

A Duality Argument for Improved Momentum Balance

Step 2: For any $\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h$, define auxiliary boundary value problem:
$\boldsymbol{\Xi} \in H_{\Gamma_N}(\mathrm{div}, \Omega)^d$ and $\boldsymbol{\eta} \in H^1_{\Gamma_D}(\Omega)^d$ such that

$$\mathrm{div}\, \boldsymbol{\Xi} = \mathrm{div}\, \boldsymbol{\tau}_h \,,$$
$$\mathcal{A}\boldsymbol{\Xi} - \boldsymbol{\varepsilon}(\boldsymbol{\eta}) = \mathbf{0}$$

(R_α) implies $\boldsymbol{\Xi} \in H^\alpha(\Omega)^{d\times d}$ and $\boldsymbol{\eta} \in H^{1+\alpha}(\Omega)^d$.

Step 3: $\boldsymbol{\Xi}_h^m \in \boldsymbol{\Sigma}_h$, $\boldsymbol{\eta}_h^{ls} \in \mathbf{V}_h$

$$\begin{aligned}
&(\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}), \mathrm{div}\,\boldsymbol{\tau}_h) \\
&\quad = (\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}), \mathrm{div}\,\boldsymbol{\Xi}) \\
&\quad = (\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}), \mathrm{div}\,\boldsymbol{\Xi}) + (\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}) - \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h^{ls}), \mathcal{A}\boldsymbol{\Xi} - \boldsymbol{\varepsilon}(\boldsymbol{\eta})) \\
&\quad = (\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}), \mathrm{div}\,(\boldsymbol{\Xi} - \boldsymbol{\Xi}_h^m)) \\
&\qquad + (\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}) - \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h^{ls}), \mathcal{A}(\boldsymbol{\Xi} - \boldsymbol{\Xi}_h^m) - \boldsymbol{\varepsilon}(\boldsymbol{\eta} - \boldsymbol{\eta}_h^{ls})) \\
&\quad = (\mathcal{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls}) - \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h^{ls}), \mathcal{A}(\boldsymbol{\Xi} - \boldsymbol{\Xi}_h^m) - \boldsymbol{\varepsilon}(\boldsymbol{\eta} - \boldsymbol{\eta}_h^{ls}))
\end{aligned}$$

□

Stress-Displacement Formulation for Hyperelastic Materials

Hyperelastic Material Models

Deformation gradient $\mathbf{F}(\mathbf{u}) = \mathbf{I} + \nabla\mathbf{u}$

Left Cauchy-Green strain tensor $\mathbf{B}(\mathbf{u}) = \mathbf{F}(\mathbf{u})\mathbf{F}(\mathbf{u})^T$

Stored energy function $\psi : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$

$$\int_\Omega \psi(\mathbf{B}(\mathbf{u}))\,dx - \int_\Omega \mathbf{f}\cdot\mathbf{u}\,dx - \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{u}\,dx \longrightarrow \min$$

among all admissible $\mathbf{u} : \Omega \to \mathbb{R}^3$

1st Piola-Kirchhoff stress tensor $\mathbf{P} = \partial_{\mathbf{F}}\psi(\mathbf{B}(\mathbf{u}))$

Optimality condition:

$$\int_\Omega \partial_{\mathbf{F}}\psi(\mathbf{B}(\mathbf{u})) : \nabla\mathbf{v}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx + \int_{\Gamma_N} \mathbf{t}\cdot\mathbf{v}\,do$$

leads to first-order system:
Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\mathrm{div}\,\mathbf{P} + \mathbf{f} = \mathbf{0}$$

$$\mathbf{P}\mathbf{F}(\mathbf{u})^T - \partial_{\mathbf{F}}\psi(\mathbf{B}(\mathbf{u}))\mathbf{F}(\mathbf{u})^T = \mathbf{0}$$

Stress-Displacement Formulation for Hyperelastic Materials

First-Order System Formulation

Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{P} + \mathbf{f} &= \mathbf{0} \\ \mathbf{P}\mathbf{F}(\mathbf{u})^T - \mathcal{G}_{NH}(\mathbf{B}(\mathbf{u})) &= \mathbf{0} \end{aligned}$$

How to generalize the linear strain-stress relation $\mathcal{A} = \mathcal{C}^{-1}$?

Inverting the stress-strain relation $\boldsymbol{\Sigma} = \mathcal{G}_{NH}(\mathbf{B})$: $\mathbf{B} = \mathcal{A}_{NH}(\boldsymbol{\Sigma})$ leads to a nonlinear algebraic equation at each quadrature point

Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{P} + \mathbf{f} &= 0 \\ \mathcal{A}_{NH}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u}) &= \mathbf{0} \end{aligned}$$

where $\mathcal{A}_{NH} = \mathcal{G}_{NH}^{-1}$ for $\lambda < \infty$, $\mathcal{A}_{NH}$ also well-defined for $\lambda = \infty$

Stress-Displacement Formulation for Hyperelastic Materials

Inverting the Stress-Strain Relation

Determine $\mathbf{u} \in W^{1,4}_{\Gamma_D}(\Omega)^3$, $\mathbf{P} \in W^4_{\Gamma_N}(\mathrm{div}, \Omega)^3$ such that

$$\mathcal{F}(\mathbf{P}, \mathbf{u}) = \|\mathrm{div}\, \mathbf{P} + \mathbf{f}\|^2_{L^2(\Omega)} + \|\mathcal{A}_{NH}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u})\|^2_{L^2(\Omega)}$$

is minimized.

B. Müller/St./Schwarz/Schröder, SIAM J. Sci. Comput., to appear

Based on this formulation, accurate approximation of stresses and surface forces again achieved based on higher-order momentum balance accuracy

Conclusions

- First-order system least squares methods in solid mechanics provide simultaneous approximation of displacements and stresses
- Produces accurate results for local evaluations of stresses and traction forces important in connection to damage simulations
- Theoretical explanation by improved momentum balance accuracy due to closeness to Hellinger-Reissner mixed approach
- Generalization to hyperelastic materials based on inverting the stress-strain relation (see talk by Benjamin Müller on Tuesday)