Renormalization Defects for Continuity Equations

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Incompressible Transport

Consider the Cauchy problem for the transport/continuity equations

$$
\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho v) &= 0 \\
\operatorname{div}(v) &= 0 \\
v \cdot \nu &= 0 \text{ on } \partial\Omega \\
\rho(\cdot, 0) &= \rho_0.
\end{aligned}
$$

Here, Ω is $\mathbb{R}^d$, $\mathbb{T}^d$, or a bounded (smooth) domain. Usually the velocity $v : \Omega \times [0, T] \to \mathbb{R}^d$ is considered given and the scalar density $\rho : \Omega \times [0, T] \to \mathbb{R}$ is sought for.

Associated ODE

If v and ρ_0 are smooth, we can solve the ODE

$$\partial_t X(x,t) = v(X(x,t),t)$$
$$X(x,0) = x$$

and thus obtain a unique bounded solution of the continuity equation given by

$$\rho(x,t) = \rho_0(X^{-1}(x,t)).$$

Uniqueness

To show uniqueness another way, multiply the equation by 2ρ and observe that by the chain rule and the divergence-free condition on v,

$$0 = 2\rho\partial_t\rho + 2\rho\,\mathrm{div}(\rho v) = \partial_t\left(\rho^2\right) + \mathrm{div}\left(\rho^2 v\right).$$

Integration in x then yields

$$\frac{d}{dt}\int_\Omega \rho^2 dx = 0,$$

so if $\rho_0 \equiv 0$ then $\rho \equiv 0$ for all times. Uniqueness for general ρ_0 follows by linearity.

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Weak Solutions

For $1 \leq p \leq \infty$ let q be the conjugate exponent and let $v \in L^1(0, T; L^q(\Omega))$ be weakly divergence-free.

Definition

The density $\rho \in L^\infty(0, T; L^p(\Omega))$ is a weak solution to the Cauchy problem for the continuity equation $\partial_t \rho + \operatorname{div}(\rho v) = 0$ if

$$\int_0^T \int_\Omega (\partial_t \phi + v \cdot \nabla \phi) \rho dxdt + \int_\Omega \rho_0(x) \phi(x, 0) dx = 0$$

for every $\phi \in C_c^1(\Omega \times [0, T))$.

Renormalized Solutions

For a weak solution, we can't apply the chain rule to obtain uniqueness. Instead one *postulates* the chain rule:

Definition (DiPerna–Lions '88)

A bounded weak solution ρ with initial data ρ_0 is called renormalized if

$$\begin{aligned} \partial_t \beta(\rho) + \operatorname{div}(\beta(\rho) v) &= 0 \\ \beta(\rho(\cdot, 0)) &= \beta \circ \rho_0 \end{aligned}$$

in the sense of distributions for every $\beta \in C^1(\mathbb{R}; \mathbb{R})$.

DiPerna–Lions Theory

Theorem (DiPerna–Lions '89)

Suppose $v \in L^1\left(0, T; W^{1,1}(\Omega)\right)$, $\mathrm{div}(v) = 0$, *and* $\rho_0 \in L^\infty(\Omega)$.

- *There exists a unique renormalized solution.*
- *Every bounded weak solution is renormalized.*
- *If* v *is time-independent, then there exists a flow* $X(x,t)$ *such that for a.e.* $x \in \Omega$, $X(x,\cdot) \in C^1$ *and*

$$\partial_t X(x,t) = v(X(x,t))$$
$$X(x,0) = x.$$

Moreover, $\rho(x,t) = \rho_0(X^{-1}(x,t))$ *gives the unique renormalized solution.*

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Remarks

- The argument for the renormalization of any bounded weak solution relies on a regularization of ρ using a mollifier η_ϵ and a commutator estimate for
$$(\mathrm{div}(\rho v)) * \eta_\epsilon - \mathrm{div}\left((\rho * \eta_\epsilon) v\right).$$

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Counterexamples

- DiPerna–Lions '89: There exists $v \in W^{s,1}_{loc}(\mathbb{R}^2)$ for all $s < 1$ such that for every $\rho_0 \not\equiv 0$, there are two renormalized solutions. (If $\rho_0 \equiv 0$, then zero is the only renormalized solution).
- Aizenman '78, Depauw '03, Colombini–Luo–Rauch '03: There exists $v \in L^1_{loc}\left(0, T; BV_{loc}(\mathbb{R}^2)\right)$ and a (non-renormalized) nontrivial solution $\rho \in L^\infty$ with $\rho_0 \equiv 0$.
- Alberti–Bianchini–Crippa '14: There exists $v \in C^{0,\alpha}(\mathbb{R}^2)$ for all $\alpha < 1$ and a nontrivial solution $\rho \in L^\infty$ with $\rho_0 \equiv 0$.

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Renormalization Defects

Let $\rho \in L^\infty$ be a weak solution to a continuity equation with velocity $v \in L^1(\Omega \times \mathbb{R})$, where $\operatorname{div}(v) = 0$. Let also $\beta \in C^1(\mathbb{R}; \mathbb{R})$. A distribution $f \in \mathcal{D}'(\Omega \times \mathbb{R}; \mathbb{R})$ is called renormalization defect of ρ (with respect to v and β) if

$$\partial_t \beta(\rho) + \operatorname{div}\left(\beta(\rho) v\right) = f.$$

For example, in Depauw's counterexample we have (for $\beta = |\cdot|^2$)

$$f = dx \otimes \delta_0(dt)$$

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Generalization of Depauw

Let $\beta \in C^1(\mathbb{R};\mathbb{R})$ be even and bijective from $\mathbb{R}^+$ onto $\mathbb{R}^+$.

Theorem 1 (Crippa–Gusev–Spirito–W. '14)

Let $f \in L^1(\mathbb{R})$ and $d \geq 2$. Then there exist $v \in L^\infty(\Omega \times \mathbb{R};\mathbb{R}^d)$, $\rho \in L^\infty(\Omega \times \mathbb{R})$ such that

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) &= 0 \\ \operatorname{div}(v) &= 0 \\ v \cdot \nu &= 0 \quad \textit{on } \partial\Omega \\ \partial_t \beta(\rho) + \operatorname{div}\left(\beta(\rho) v\right) &= f(t). \end{aligned}$$

Discussion

- For $f = \delta_0$ this yields the same defect as Depauw's example.
- This gives (to our knowledge) the first example of a renormalization defect that is absolutely continuous in t ("diffuse renormalization defect").
- The drawback is that the defect is only allowed to depend on t.

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Stationary Continuity Equation

Let now $\beta : \mathbb{R} \to \mathbb{R}$ be strongly convex.

Theorem 2 (Crippa–Gusev–Spirito–W. '14)

Let $d = 3$ and $f \in L^p(\Omega)$ for some $p > 3$. Then there exist $v \in L^\infty(\Omega)$, $\rho \in L^\infty(\Omega)$ such that

$$\begin{aligned} \operatorname{div}(\rho v) &= 0 \\ \operatorname{div}(v) &= 0 \\ v \cdot \nu &= 0 \quad \text{on } \partial\Omega \\ \operatorname{div}\left(\beta(\rho) v\right) &= f. \end{aligned}$$

Discussion

- This result is impossible for $d = 2$ (Bianchini–Gusev '14).
- In both theorems, one may take ρ to be strictly positive and bounded away from zero: Simply add a constant if necessary.
- Both results are achieved using convex integration techniques.

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Open Problems

- Can we do the nonstationary case for renormalization defects which depend on t and x?
- Is it possible to construct solutions with higher regularity in ρ and/or v (Hölder)?
- Is there a way to restore uniqueness? A candidate is the viscosity limit: Solve

$$\partial_t \rho + \mathrm{div}(\rho v) = \epsilon \Delta \rho$$

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