Ways to measure compactness

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Outline

- Different approximation widths, their geometrical meaning
- Function spaces: Sobolev, Besov, Triebel-Lizorkin type
- Discretization techniques: Wavelets and other -lets
- Finite-dimensional spaces
- Methods and Results
- Remarks / Conclusion

Compactness (of sets or operators) is a central notion in both analysis and numerics

- Convergence of (sub)sequences
- Finite coverings
- Spectral properties
- Further properties of functions and operators
- Many generalizations

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How to quantify compactness?

. . . to get speed of convergence, size of coverings, . . .

s-numbers

X, Y Banach spaces, $\quad T \in \mathcal{L}(X, Y)$

$(S_1) \qquad ||T|| = s_1(T) \geq s_2(T) \geq s_3(T) \geq \cdots \geq 0$

$(S_2) \qquad s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$

$(S_3) \qquad s_{m+n-1}(ST) \leq s_m(S)s_n(T)$

Generalization to (quasi-) Banach spaces

Approximation numbers

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Image of every element is approximated from an $n-1$-dimensional subspace, in a linear way.

The smaller $a_n(T)$ is, the better approximation is possible.

Typically (depends on X and Y), $a_n(T) \to 0$ is equivalent to compactness of T

Other (=non-linear) ways

Gelfand numbers:

$$c_n(T) = \inf\{||TJ_M^X|| : \operatorname{codim}(M) < n\}$$

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Approximation of Tx with $x \in M$ by zero element - lower bound for non-linear approximation based on information from $n-1$ linear measurements

Other (=non-linear) ways

Kolmogorov numbers:

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Image of every element is approximated from an $n-1$-dimensional subspace, in a non-linear way.

Easily,

$$c_n(T) \leq a_n(T), \qquad d_n(T) \leq a_n(T)$$

Yet another ways...

Entropy numbers $e_n(T)$

$$e_n(T) = \inf\Big\{\varepsilon > 0 : T(B_X) \subset \bigcup_{i=1}^{2^{n-1}} (y_i + \varepsilon B_Y) \quad \text{for some} \quad y_i \in Y\Big\}$$

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Mean Gaussian widths, Bernstein widths, and many others

Finite-dimensional spaces

- Widths of $id : \ell_p^m \to \ell_q^m$ were studied in connection with geometry of Banach spaces (since 1950's)
- Contributions of Stechkin, Kashin, Gluskin, Carl, Pietsch, Kühn, Pajor, Tikhomirov, Talagrand, and many others
- Some constructions are deterministic, some are random
- For (almost) all widths, the behavior is known
- Results available also for diagonal and other special operators
- Some tricky open problems left still open...

Function spaces

$\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain

$$L_p(\Omega) \;\ldots\; \text{Lebesgue spaces}$$

$$W_p^s(\Omega) \;\ldots\; \text{Sobolev spaces}$$

more general:
Besov spaces $B^s_{p,q}(\Omega)$ and
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$$id : W^{s_1}_{p_1}(\Omega) \to W^{s_2}_{p_2}(\Omega)$$

is compact if, and only if,

$$s_1 - s_2 > \left(\frac{d}{p_1} - \frac{d}{p_2}\right)_+$$

Discretization - transfer to sequence spaces

Wavelet decomposition

$$f \in W_p^s(\Omega) : f = \sum_{\nu=0}^{\infty} \sum_{m} \lambda_{\nu m} \psi_{\nu m}$$

$$\lambda = \{\lambda_{\nu m}\}_{\nu m} \in w_p^s$$

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$$S : f \to \lambda, \quad S^{-1} : \lambda \to f$$

$$\begin{array}{ccc} W_{p_1}^{s_1}(\Omega) & \xrightarrow{\;id\;} & W_{p_2}^{s_2}(\Omega) \\ S \downarrow & & \uparrow S^{-1} \\ w_{p_1}^{s_1} & \xrightarrow[id']{} & w_{p_2}^{s_2} \end{array}$$

$$s_n(id) \approx s_n(id')$$

Example

$$a_n(id : B^{s_1}_{1,q_1}(\Omega) \to B^{s_2}_{\infty,q_2}(\Omega))$$

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$$a_n(id : \ell_{q_1}(2^{j(s_1-d)}\ell_1^{2^{jd}}) \rightarrow \ell_{q_2}(2^{js_2}\ell_\infty^{2^{jd}}))$$

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$$id_j : \ell_1^{2^{jd}} \to \ell_\infty^{2^{jd}} \qquad id = \sum_{j=0}^{\infty} id_j$$

Results: Approximation numbers

$$id : W^{s_1}_{p_1}(\Omega) \to W^{s_2}_{p_2}(\Omega)$$

$$s = s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+$$

$$a_n \approx n^{-\frac{s}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \quad \text{if}$$

- $1 \le p_1 \le p_2 \le 2$
- $2 \le p_1 \le p_2 \le \infty$
- $1 \le p_2 \le p_1 \le \infty$

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$$a_n \asymp n^{-\frac{s}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \quad \text{if} \quad \begin{array}{l} \bullet\ 1 \le p_1 \le p_2 \le 2 \\ \bullet\ 2 \le p_1 \le p_2 \le \infty \\ \bullet\ 1 \le p_2 \le p_1 \le \infty \end{array}$$

If $1 \le p_1 < 2 < p_2 \le \infty$

$$a_n \asymp \begin{cases} n^{-\frac{s}{d}-\frac{1}{2}+\max\left(1-\frac{1}{p_2},\frac{1}{p_1}\right)} & \frac{s}{d} > \max\left(1-\frac{1}{p_2},\frac{1}{p_1}\right) \\ n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{p_2}\right)\frac{\min(p_1',p_2)}{2}} & \frac{s}{d} < \max\left(1-\frac{1}{p_2},\frac{1}{p_1}\right) \end{cases}$$

Further results

- Nowadays classical machinery
- Many forerunners - back to 1960's, with spline interpolation, trigonometric polynomial, etc.
- Similar tables for other widths, Besov and Triebel-Lizorkin spaces, anisotropic spaces, spaces of dominating mixed smoothness, ...
- Especially - estimates of sampling numbers give bounds on recovery of functions from limited number of function values (here the reduction to sequence spaces is not so direct)
- Discretization of functions and isomorpisms to sequence spaces allow us to use other techniques, i.e. random projections, Compressed Sensing, or Matrix Completion

Conclusion

- Estimates from above can usually be transformed into algorithms for approximation (based on wavelets)
- Estimates from below give a benchmark for performance of other algorithms
- Probabilistic vs. deterministic constructions

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Thank you for your attention!