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On PDE analysis for flows of quasi-incompressible fluids

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Setting

Given $(\varrho^1, \mathbf{v}^1)$, $(\varrho^2, \mathbf{v}^2)$. Let

$$\varrho := \varrho^1 + \varrho^2, \quad c := \frac{\varrho^1}{\varrho}, \quad \mathbf{v} := \frac{1}{\varrho}(\varrho^1\mathbf{v}^1 + \varrho^2\mathbf{v}^2), \quad \mathbf{j} := \varrho^1(\mathbf{v}^1 - \mathbf{v}).$$

There arises:

$$\begin{aligned} \dot{\varrho} &= -\varrho \operatorname{div}\mathbf{v}, \\ \varrho\dot{c} &= -\operatorname{div}\mathbf{j}, \\ \varrho\dot{\mathbf{v}} &= \operatorname{div}\mathbf{T} + \varrho\mathbf{b}, \\ \varrho(e + \frac{1}{2}|\mathbf{v}|^2)^{\cdot} &= \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}_E) + \varrho\mathbf{b}\cdot\mathbf{v}, \end{aligned} \tag{1}$$

where

$$\dot{z} := \partial_t z + \mathbf{v}\cdot\nabla_x z.$$

Modelling–compatible constitutive theory

The modelling approach is inspired by [4] Rajagopal and Srinivasa in 2004.

Constitutive assumption:

$$e = e(\eta, \varrho, c), \quad \eta \text{ is the entropy.}$$

Quasi-incompressibility assumption:

$$\varrho = \varrho(c).$$

If the volume additivity holds, i.e. $1 = \varphi^1 + \varphi^2$ and $\varphi^\alpha \rho_m^\alpha = \varrho^\alpha$ (ϱ_m^α are the true densities), then

$$\varrho(c) = \frac{\varrho_m^1 \varrho_m^2}{(1-c)\varrho_m^1 + c\varrho_m^2}.$$

Modelling–quasi-incompressible fluids

For quasi-incompressible fluids:

$$\operatorname{div}\mathbf{v} = R(c)\operatorname{div}\mathbf{j}, \quad R(c) := \frac{\varrho'(c)}{\varrho^2(c)}.$$

With volume additivity:

$$R(c) = r_* := \frac{\varrho_m^1 - \varrho_m^2}{\varrho_m^1 \varrho_m^2}.$$

If $\varrho_m^1 = \varrho_m^2$, $\operatorname{div}\mathbf{v} = 0$, the fluids become incompressible.

Modelling–quasi-incompressible fluids

Now

$$e = e(\eta, \varrho(c), c) = \tilde{e}(\eta, c).$$

Define

$$\theta := \frac{\partial \tilde{e}}{\partial \eta}, \quad \mu := \frac{\partial \tilde{e}}{\partial c}.$$

Then

$$\varrho\dot{\eta} + \operatorname{div}\left(\frac{\mathbf{q}_\eta}{\theta}\right) = \frac{1}{\theta}\left(\mathbf{T}^d : \mathbf{D}^d - \mathbf{j}\cdot\nabla_x(\mu + mR(c)) - \frac{\mathbf{q}_\eta\cdot\nabla_x\theta}{\theta}\right), \tag{2}$$

where $m := \operatorname{tr}\mathbf{T}/3, \mathbf{q}_\eta := \mathbf{q}_E - \big(\mu + mR(c)\big)\mathbf{j}$ and

$$\mathbf{D} := \frac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^t\big), \quad \mathbf{T}^d := \mathbf{T} - \frac{1}{3}\operatorname{tr}\mathbf{T}, \quad \mathbf{D}^d := \mathbf{D} - \frac{1}{3}\operatorname{tr}\mathbf{D}.$$

Modelling–Boundary conditions

We assume

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T).$$

If $\mathbf{b} = 0$, the energy $\mathcal{E}(t)$ defined as

$$\mathcal{E}(t) := \int_\Omega \varrho\left(e + \frac{1}{2}|\mathbf{v}|^2\right) dx$$

is conserved if

$$\mathbf{q}_E \cdot \mathbf{n} = \mathbf{T}\mathbf{v} \cdot \mathbf{n}.$$

Cauchy tensor $\mathbf{T}$ is symmetric,

$$\mathbf{T}\mathbf{v} \cdot \mathbf{n} = \left(\mathbf{T}\mathbf{n}\right) \cdot \mathbf{v} = \left(\mathbf{T}\mathbf{n}\right)_\tau \cdot \mathbf{v}_\tau.$$

Modelling-entropy

Let

$$S(t) := \int_{\Omega} \varrho \eta dx.$$

Then

$$\frac{dS(t)}{dt} = \left(\frac{1}{\theta}\mathbf{T}^d, \mathbf{D}^d\right)_{\Omega} - \left(\frac{1}{\theta}\mathbf{j}, \nabla_x(\mu + mR(c))\right)_{\Omega} - \left(\frac{1}{\theta}\mathbf{q}_\eta, \frac{\nabla_x \theta}{\theta}\right)_{\Omega}$$
$$- \left(\frac{1}{\theta}(\mathbf{Tn})_\tau, \mathbf{v}_\tau\right)_{\partial\Omega} + \left(\frac{1}{\theta}(\mu + mR(c)), \mathbf{j}\cdot\mathbf{n}\right)_{\partial\Omega}$$

Requiring that $dS/dt \geq 0$ and that the terms forcing the duality are linearly related leads finally to

Modelling-conclusion

$$
\begin{aligned}
&\mathbf{T}^d = 2\nu \mathbf{D}^d, && \nu \ge 0, \quad \text{in } \Omega,\\
&\mathbf{j} = -\beta \nabla_x \big(\mu + mR(c)\big), && \beta \ge 0, \quad \text{in } \Omega,\\
&\mathbf{q}_E = -\kappa \frac{\nabla\theta}{\theta} - \frac{\beta}{2}\nabla_x \big(\mu + mR(c)\big)^2, && \kappa \ge 0, \quad \text{in } \Omega,\\
&\mathbf{v}_\tau = -\gamma (\mathbf{Tn})_\tau, && \gamma \ge 0, \quad \text{on } \partial\Omega,\\
&\mathbf{j}\cdot\mathbf{n} = \delta\big(\mu + mR(c)\big) && \delta \ge 0, \quad \text{on } \partial\Omega.
\end{aligned}
\tag{3}
$$

We obtain

$$
\begin{aligned}
\mathrm{div}\mathbf{v} &= -R(c)\mathrm{div}\left(\beta\nabla_x\big(\mu + mR(c)\big)\right),\\
\varrho(c)\dot{c} &= \mathrm{div}\left(\beta\nabla_x\big(\mu + mR(c)\big)\right),\\
\varrho(c)\dot{\mathbf{v}} &= \nabla_x m + \mathrm{div}(2\nu\mathbf{D}^d),\\
\varrho(c)\dot{e} &= m\mathrm{div}\mathbf{v} + 2\nu|\mathbf{D}^d|^2 + \mathrm{div}(\kappa\nabla_x \ln\theta) + \frac{1}{2}\mathrm{div}\Big(\beta\nabla_x\big(\mu + mR(c)\big)^2\Big).
\end{aligned}
$$

A special case

For the case where

$$R(c) = r_* = \frac{\varrho_m^1 - \varrho_m^2}{\varrho_m^1 \varrho_m^2}, \quad \beta = \kappa = \nu = 1, \quad \gamma = \delta = 0, \quad \theta = 1,$$

we have

$$\begin{aligned}
&\partial_t \varrho + \mathrm{div}(\varrho \mathbf{v}) = 0, \\
&\partial_t(\varrho \mathbf{v}) + \mathrm{div}(\varrho \mathbf{v} \otimes \mathbf{v}) + \frac{1}{r_*} \nabla_x q(\varrho) = 2 \mathrm{div} \mathbf{D}^d - \frac{1}{r_*^2} \nabla_x \Delta_x^{-1} \mathrm{div} \mathbf{v}, \\
&\mathbf{v} = 0 \quad \text{on } \partial\Omega \times (0, T).
\end{aligned} \tag{4}$$

It is suggested to consider strictly increasing q such that

$$\lim_{\varrho \to \frac{1}{1+r_*}+} q(\varrho) = -\infty, \quad \lim_{\varrho \to 1-} q(\varrho) = +\infty.$$

Question

1, Existence of weak solution?

2, Asymptotic behavior as $r_* \to 0$? From quasi-incompressible to incompressible?

Existence of weak solution

Suppose for some $\beta_0 > 5/2$:

$$\liminf_{\varrho \to \frac{1}{1+r_*}+} |q(\varrho)(\varrho - \frac{1}{1+r_*})^{\beta_0}| > 0, \quad \liminf_{\varrho \to 1-} |q(\varrho)(1-\varrho)^{\beta_0}| > 0.$$

Initial data:

$$\mathbf{v}_0 \in L^2(\Omega;\mathbb{R}^3), \quad \frac{1}{1+r_*} < \varrho_0 < 1, \quad \int_\Omega Q(\varrho_0)dx < \infty.$$

Pressure potential Q:

$$Q(\varrho) := \varrho \int_{\varrho_*}^{\varrho} \frac{q(z)}{z^2} dz,$$

where ϱ_* is the only zero point of $q(\cdot)$.

Existence of weak solution

Then there exists a global weak solution satisfying:

$$\frac{1}{1+r_*} \leq \varrho \leq 1, \quad q(\varrho) \in L^1, \quad \mathbf{v} \in C_w(L^2) \cap L^2(W_0^{1,2})$$

and

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \frac{1}{r_*} Q(\varrho) \right)(\tau, \cdot)\, dx + \int_0^\tau \int_\Omega 2|\mathbf{D}^d(\mathbf{v})|^2 + \frac{1}{r_*^2} |\nabla_x \Delta_x^{-1} \mathrm{div} \mathbf{v}|^2\, d$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{v}_0|^2 + \frac{1}{r_*} Q(\varrho_0) \right) dx.$$

Proof–Approximate solutions

We employ idea by Feireisl and Zhang [2].

Regularized pressure for $\alpha > 0$ small and $\gamma > 3/2$ large:

$$q_\alpha(\varrho) := \begin{cases} q(\frac{1}{1+r_*} + \alpha), & \varrho \leq \frac{1}{1+r_*} + \alpha, \\ q(\varrho), & \frac{1}{1+r_*} + \alpha \leq \varrho \leq 1 - \alpha, \\ q(1-\alpha) + (\varrho - 2)_+^\gamma, & \varrho \geq 1 - \alpha. \end{cases}$$

By replacing the pressure term q by q_α in (4), we obtain an approximate system.

By the theory developed by Lions [3] and Feireisl, Novotný and Petzeltová [1], we have global existence of finite energy weak solution $[\varrho_\alpha, \mathbf{v}_\alpha]$ to the approximate system:

Proof–Passing the limit

$$\partial_t \varrho_\alpha + \mathrm{div}(\varrho_\alpha \mathbf{v}_\alpha) = 0,$$

$$\partial_t(\varrho_\alpha \mathbf{v}_\alpha) + \mathrm{div}(\varrho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) + \frac{1}{r_*}\nabla_x q_\alpha(\varrho_\alpha)$$
$$= 2\mathrm{div}\mathbf{D}^d(\mathbf{v}_\alpha) - \frac{1}{r_*^2}\nabla_x \Delta_x^{-1}\mathrm{div}\mathbf{v}_\alpha.$$

Weak convergence

$$\varrho_\alpha \to \varrho \text{ weakly(*) in } L^\infty(0, T; L^\gamma(\Omega; \mathbb{R}^3)),$$
$$\mathbf{v}_\alpha \to \mathbf{v} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)).$$

For the nonlinear terms in the approximate system (at least in the sense of distribution):

$$\varrho_\alpha \mathbf{v}_\alpha \to \varrho \mathbf{v}, \quad \varrho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha \to \varrho \mathbf{v} \otimes \mathbf{v}.$$

Proof–Uniform L^1 estimate of the pressure

The difficulty is to show

$$q_\alpha(\varrho_\alpha) \to q(\varrho) \text{ weakly in } L^1?$$

Introduce

$$\varphi = \psi(t)\mathcal{B}(\varrho_\alpha - \langle\varrho_\alpha\rangle), \quad \langle\varrho_\alpha\rangle := \frac{1}{|\Omega|}\int_\Omega \varrho_\alpha \, dx,$$

with $\psi \in C_c^\infty(0,T)$ and $\mathcal{B}$ a bounded linear operator from $\{g \in L^p(\Omega), \quad \langle g\rangle = 0\}$ to $W_0^{1,p}(\Omega;\mathbb{R}^3)$ for $1 < p < \infty$ such that

$$\mathrm{div}\mathcal{B}(g) = g, \quad \mathcal{B}(g)|_{\partial\Omega} = 0.$$

Taking φ as a test function implies

$$\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \text{ bounded in } L^1((0,T)\times\Omega).$$

Proof–Equi-integrability of the pressure

Introduce

$$\varphi = \psi(t)\mathcal{B}(\eta_\alpha(\varrho_\alpha) - \langle \eta_\alpha(\varrho_\alpha)\rangle),$$

where $\psi \in C_c^\infty(0, T)$ and

$$\eta_\alpha(s) = \begin{cases} \log(s - \frac{1}{1+r*}) - \log(1-s), & \frac{1}{1+r_*} + \alpha \le s \le 1-\alpha, \\ -\log(\alpha), & s \ge 1-\alpha, \\ \log(\alpha), & s \le \frac{1}{1+r_*} - \alpha. \end{cases}$$

Taking φ as a test function implies

$$\{q_\alpha(\varrho_\alpha)\eta_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^1((0,T)\times\Omega).$$

This gives the equi-integrability of the pressure. Then

$$q_\alpha(\varrho_\alpha) \to \overline{q(\varrho)} \text{ weakly in } L^1((0,T)\times\Omega)).$$

Proof–Strong convergence of density

By employing the arguments of Lions, we can obtain

$$\int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \leq 0.$$

This imples

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho),$$

and furthermore

$$\varrho_\alpha \to \varrho \ a.e \text{ in } (0, T) \times \Omega.$$

Then

$$\overline{q(\varrho)} = q(\varrho).$$

Convergence for $r_* \to 0$

Let $[\varrho_{r_*}, \mathbf{v}_{r_*}]$ be the weak solution. It can be shown

$$\sup_{t \in (0,T)} \|\varrho_{r_*}(t,\cdot) - 1\|_{L^\infty(\Omega)} \leq \varepsilon$$

and

$$\mathbf{v}_{r_*} \to \mathbf{w} \text{ in } L^2((0,T) \times \Omega; R^3),$$

where $\mathbf{w}$ is a *weak* solution to the incompressible Navier-Stokes system

$$\mathrm{div}\mathbf{w} = 0,$$

$$\partial_t \mathbf{w} + \nabla \mathbf{w} \cdot \mathbf{w} + \nabla P = \Delta \mathbf{w}, \ \ \mathbf{w}|_{\partial\Omega} = 0.$$

Collaborators

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Thank you for your attention!