# Regularity issues in systems describing flows of incompressible fluids

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# Navier-Stokes-"Fourier"-"quantity"-like equations

$$\operatorname{div} \mathbf{v} = 0, \quad (1)$$

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p, \quad (2)$$

$$k_{,t} + \operatorname{div}(k\mathbf{v}) - \operatorname{div}(\mu(k)\nabla k) + \varepsilon(k) = \mathbf{S} \cdot \mathbf{D}(\mathbf{v}), \quad (3)$$

$$\text{other balance laws} = 0. \quad (4)$$

where

- $\mathbf{v}$ . . . velocity field
- $p$ . . . pressure
- $\mathbf{D}(\mathbf{v})$ . . . symmetric part of velocity gradient
- $\mathbf{S}$ constitutively determined part of the Cauchy stress
- $\nu$ . . . viscosity
- $k$ . . . a QUANTITY

# Regularity for systems with critical growth

- critical growth due to $\mathbf{S} \cdot \mathbf{D}(\mathbf{v})$
- regularity for general elliptic/parabolic systems with terms having critical growth does not hold
- single equation OK
- system requires
  - very special structure of $\mathbf{S}$ - here it can be coupled with $k, \dots,$ typically depends only on $\mathbf{D}(\mathbf{v})$ and not on $\nabla \mathbf{v}$, incompressibility constrain & pressure
  - very special form of the right hand sides - here nice in momentum equation, horrible in the other ones
- the presence of the convective term

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# Regularity for non-coupled problem

Consider

$$\mathbf{S} \sim (1 + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v}).$$

Results in 3D:

- spatially periodic problem
  - existence of weak solution for $r > \frac{6}{5}$
  - existence of strong solution for $r > \frac{11}{5}$
  - uniqueness for $r > \frac{5}{2}$, in case of better data for $r > \frac{11}{5}$
  - classical solution only in 2D
- general boundary conditions (slip, no-slip, no-stick,....)
  - existence of weak solution for $r > \frac{6}{5}$
  - existence of strong solution for mysterious $r > \frac{11}{5} + \varepsilon$
  - uniqueness for $r > \frac{5}{2}$, in case of better data $r > \frac{12}{5}$
  - classical solution only in 2D

# Time regularity - uniqueness for $r > \frac{11}{5}$

**Theorem (B, Etwein, Kaplický, Pražák)**

*Let $r > \frac{11}{5}$ and assume any arbitrary relevant boundary conditions. Then any weak solution satisfies*

$$\mathbf{v}_t \in L^{\infty}_{loc}(0,T;L^2) \cap L^2_{loc}(0,T;W^{1,2}). \tag{5}$$

*In particular the solution is unique in the sense of trajectories. Moreover, if the initial data are "smooth" then (5) holds without "loc" and the weak solution is unique.*

# Coupling and "equivalent" reformulation of the problem

Introduce a "new" quantity

$$\boxed{E := \frac{1}{2}|\mathbf{v}|^2 + k.}$$

Multiply (2) by $\mathbf{v}$ and add the result to (3) to get

$$E_{,t} + \operatorname{div}(\mathbf{v}(E+p)) - \operatorname{div}(\mu(k)\nabla k) - \operatorname{div}(\mathbf{S}\mathbf{v}) + \varepsilon(k) = 0. \tag{6}$$

- the bad term $\mathbf{S} \cdot \mathbf{D}(\mathbf{v})$ is not in (6)
- the pressure $p$ is needed! in weak formulation
- the presence of $\sim |\mathbf{v}|^3$ in the equation

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# Navier-Stokes-Fourier

Replace $k$ by $e$ and call it internal energy. Consider $\mathbf{S}$ of the form

$$\boxed{\mathbf{S} \sim \nu(e)(1+\gamma(e)+|\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}}\mathbf{D}(\mathbf{v}),} \qquad \boxed{\varepsilon(e) \equiv 0.}$$

**Theorem (B, Málek, Rajagopal)**

*Let $r > \frac{9}{5}$ and $\nu$ and $\gamma$ be not so bad (bounded from above and below and continuous). Then there exists a suitable weak solution for slip bc. Suitable means*

$$e_t + \operatorname{div}(e\mathbf{v}) - \operatorname{div}(\mu(e)\nabla e) \geq \mathbf{S}\cdot\mathbf{D}(\mathbf{v}).$$

**Theorem**

*Let $r > \frac{6}{5}$. Then there exists Eduard's weak solution, i.e., solution solving balance of linear momentum,*

$$\frac{d}{dt}\int_\Omega E = 0,$$

*and for any smooth nondecreasing nonnegative (on $\mathbb{R}^+$) concave bounded function $f$*

$$f(e)_t + \operatorname{div}(f(e)\mathbf{v}) - \operatorname{div}(\mu(e)f'(e)\nabla e) \geq f'(e)\mathbf{S}\cdot\mathbf{D}(\mathbf{v}) - \mu(e)f''(e)|\nabla e|^2$$

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# N-S-F reduction and regularity I

- spatially periodic setting, constant heat conductivity
- linear model, i.e., $r = 2$, i.e., $\mathbf{S} = \nu(e)\mathbf{D}(\mathbf{v})$
- neglect inertia

The system is still with critical growth + dependence only on symmetric gradient + presence of the pressure

**Theorem (B, Kaplický, Málek)**

*Let $\nu$ be smooth function bounded from above and below. Assume that $e(0,x) \geq e_{\min} > 0$. Assume in addition that*

$$\frac{|\nu'(e)|}{\nu(e)} \leq \frac{1}{16(e - e_{\min})}$$

*Then there exists a strong solution.*

**Even in 2D we are not able to handle the convective term.**

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# N-S-F reduction and regularity II

- spatially periodic setting, constant heat conductivity
- nonlinear model, but $\nu(e) = 1$, i.e., $\mathbf{S} = (1 + \gamma(e) + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v})$
- keep inertia

The system is still with critical growth but of much better structure

**Theorem (B, Málek, Shilkin)**

*Let $\gamma$ be smooth nonnegative bounded function. Assume in addition that $r \in [\frac{11}{5}, 4]$ and*

$$|\gamma'(e)| \le Ce^{-\alpha} \qquad \text{with } \alpha > \frac{1}{2}.$$

*Then there exists a strong solution.*
*In 2D it is enough to assume $r \in (1, 4]$. Moreover for $r > \frac{4}{3}$ we have a classical solution.*

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# Turbulent kinetic energy models

- introduce the equation for $E$
- assume the following growth conditions

$$C_1(1+k)^{\alpha} \le \nu(k) \le C_2(1+k)^{\alpha},$$
$$C_1(1+k)^{\beta} \le \mu(k) \le C_2(1+k)^{\beta},$$
$$C_1 k^{1+\gamma} \le \varepsilon(k) \le C_2 k^{1+\gamma},$$

with some

$$\boxed{\alpha, \beta, \gamma \in [0, \infty)}$$

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# Existence of weak solution

**Theorem (B, Lewandowski, Málek)**

*Let* $\alpha, \beta, \gamma$ *satisfy*

$$\gamma < \beta + \frac{2}{3}, \qquad \alpha < \frac{2\beta}{5} + \frac{2}{3}.$$

*Then there exists a weak solution for slip bc.*

# Relevance to regularity of Navier-Stokes equation

Criterium for regularity of 3D Navier-Sokes:

Theorem

*There exists* $\varepsilon > 0$ *such that if* $\mathbf{v}$ *is a suitable weak solution to N-S in* $(-1,0) \times B_1(0)$ *satisfying*

$$\boxed{\int_{-1}^{0}\int_{B_1(0)} \nu_0 |\mathbf{D}(\mathbf{v})|^2 \le \varepsilon}$$

*then*

$$\boxed{|\mathbf{v}| \le C \;\; in\; (-\frac{1}{2}, 0] \times B_{\frac{1}{2}}(0)}$$

# Conjecture for Turbulent kinetic energy model

**Conjecture (B, Lewandowski, Málek)**

*There exists* $\varepsilon > 0$ *such that if* $\mathbf{v}$ *is a solution to N-S-F like system in* $(-1,0) \times B_1(0)$ *satisfying*

$$\boxed{\int_{-1}^{0}\int_{B_1(0)} \nu(k)|\mathbf{D}(\mathbf{v})|^2 \le \varepsilon}$$

*then*

$$\boxed{|\mathbf{v}|^2 + 2k \le C \ \ in \ (-\frac{1}{2}, 0) \times B_{\frac{1}{2}}(0)}$$

# Conjecture implies regularity for turbulent kinetic energy models

Assume that

$$\boxed{\nu(k) = \nu_0 k^\alpha, \qquad \mu(k) = \mu_0 k^\alpha}$$

with $\alpha \in \mathbb{R}_+$.

Then we can introduce a scaling similar BUT different to Navier-Stokes, i.e., if $(\mathbf{v}, k)$ solves the problem in

$$(-\lambda^{2(1-\alpha)}, 0) \times B_{\lambda^{1-2\alpha}}(0)$$

for some $\lambda > 0$. Then

$$\boxed{\mathbf{v}_\lambda(t,x) := \lambda \mathbf{v}(\lambda^{2(1-\alpha)}t, \lambda^{1-2\alpha}x), \qquad k_\lambda(t,x) := \lambda^2 k(\lambda^{2(1-\alpha)}t, \lambda^{1-2\alpha}x)}$$

solves the same problem in $(-1,0) \times B_1(0)$.

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# Conjecture implies regularity for some turbulent kinetic energy models

**Theorem (B, Málek, Lewandowski)**

*Let Conjecture be true. Then for $\alpha \in [\frac{1}{6}, \frac{1}{2}]$ the solution is bounded.*

Apply the Conjecture to $(\mathbf{v}_\lambda, k_\lambda)$:

$$\int_{-1}^{0} \int_{B_1(0)} \nu(k_\lambda) |\mathbf{D}(\mathbf{v}_\lambda)|^2 \, dx \, dt$$
$$= \lambda^{6\alpha - 1} \int_{-\lambda^2(1-\alpha)}^{0} \int_{B_{\lambda^{1-2\alpha}}(0)} k^\alpha |\mathbf{D}(\mathbf{v})|^2 \, dx \, dt \overset{\lambda \to 0}{\to} 0.$$

Moreover, some estimate on the dimension of possible singularities.

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# Kolmogorov model of turbulence

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nu_0 \operatorname{div} \left( \frac{k}{\omega} \mathbf{D}(\mathbf{v}) \right) &= -\nabla p, \\ \omega_{,t} + \operatorname{div}(\omega \mathbf{v}) - \kappa_1 \operatorname{div} \left( \frac{k}{\omega} \nabla \omega \right) &= -\kappa_2 \omega^2, \\ k_{,t} + \operatorname{div}(k \mathbf{v}) - \kappa_3 \operatorname{div} \left( \frac{k}{\omega} \nabla k \right) &= -k\omega + \kappa_4 \frac{k}{\omega} |\mathbf{D}(\mathbf{v})|^2. \end{aligned}$$

- $\mathbf{v}$ is the statistical mean velocity of the fluid
- $k$ denotes the turbulent kinetic energy
- $\omega$ is related to the length scale $\ell$ by the relation $\omega := C\sqrt{k}/\ell$
- $\nu_0, \kappa_1, \ldots, \kappa_4$ are assumed to be given positive constants

# Existence of solution

**Theorem (B, Málek)**

*There exists a weak solution.*

*Moreover, for any $a, b > 0$ and any weak solution $(\mathbf{v}, k, \omega)$ in $(-b, 0) \times B_a(0)$, the scaled quantities*

$$\mathbf{v}_{a,b}(x,t) := \frac{b}{a}\mathbf{v}(ax, bt),$$
$$k_{a,b}(x,t) := \frac{b^2}{a^2} b(ax, bt),$$
$$\omega_{a,b}(x,t) := b\omega(ax, bt)$$

*solves the same problem in $(-1, 0) \times B_1(0)$*

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