Pointwise estimate for degenerate elliptic systems

Sebastian Schwarzacher
Dominic Breit, Andrea Cianchi & Lars Diening

22. September 2014

p-Laplacian system

Find solution $u : \Omega \to \mathbb{R}^N$ satisfying the following

system of partial differential equations

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -\operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Solution u is minimizer of $\mathcal{J}(w) := \int \frac{1}{p}|\nabla w|^p \, dx + \int \nabla w \cdot F \, dx.$

Theorem

For $F \in L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ exists solution $u \in W_0^{1,p}$ and $|\nabla u|^{p-2}\nabla u \in L^{p'}$.

Linear Calderón-Zygmond theory $p = 2$

The linear problem:
$-\Delta u := -\mathrm{div}(\nabla u) = -\mathrm{div}\, F.$
We have a representation formula:

$$\partial_i u(x) = \sum_j (\partial_i \partial_j K) * F_j(x).$$

with $K(x) = c|x|^{2-n}$. Pointwise estimate:
$|\nabla u(x)| \le c|\nabla^2 K * F(x)|.$

Theorem (Calderón-Zygmund, singular integrals)

If $F \in X$, then $\nabla u \in X : \|\nabla u\|_X \le c\,\|F\|_X.$

Examples: $X = L^q$ for $1 < q < \infty$, $X = \mathrm{BMO}$, $X = C^\beta$.

Non-linear Calderón-Zygmund theory

For $u \in W^{1,p}$ with

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\mathrm{div}(F)$$

we have:

Theorem (Non-linear Calderón-Zygmund theory)

If $F \in X$, then $|\nabla u|^{p-2}\nabla u \in X$.

Case: $X = L^s$ with $s \geq p'$ [Iwaniec '82; Kinnunen-Zhou '01].

Case: $X = \mathrm{BMO}, \mathrm{VMO}, C^\beta$ [DiBenedetto-Manfredi '93, Diening-Kaplický-Schwarzacher '12].

Non-linear Calderón-Zygmund theory

For $u \in W^{1,p}$ with

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\mathrm{div}(F)$$

we have:

Theorem (Non-linear Calderón-Zygmund theory)

If $F \in X$, then $|\nabla u|^{p-2}\nabla u \in X$.

Case: $X = L^s$ with $s \geq p'$ [Iwaniec '82; Kinnunen-Zhou '01].

Case: $X = \mathrm{BMO}, \mathrm{VMO}, C^\beta$ [DiBenedetto-Manfredi '93, Diening-Kaplický-Schwarzacher '12].

Question: Do we get pointwise estimates that inherit the above estimates?

Potential estimates

Let μ be a Radon measure.

$$\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = \mu, \text{ a measure } N = 1$$
$$|\nabla v(x)|^{p-1} \leq c\left(|\cdot|^{n-1} * |\mu|\right)(x)$$

- Kilpelainen-Maly ('92/'94): Nonlinear potential estimates;
- Duzaar-Mingione ('09, '10)
- Kuusi-Mingione ('13): linear potentials;
- Parabolic versions by Kuusi-Mingione ('14).

The Fefferman Stein operator:

$$\mathcal{M}_s^\sharp f(x) := \sup_{B \ni x} \left(⨍_B |f - (f)_B|^s dy \right)^{\frac{1}{s}},$$

- $f \in \mathrm{BMO} :\Leftrightarrow \|\mathcal{M}_s^\sharp f\|_\infty < \infty$.
- We have $L^\infty \subsetneq \mathrm{BMO} \subsetneq \cap_q L^q$;
- Weighted BMO_ω-spaces: $\mathcal{M}_\omega^\sharp f(x) := \sup_{B \ni x} \frac{1}{\omega(r_B)} ⨍_B |f - (f)_B| dy$,
- Campanato characterization $f \in C^\beta :\Leftrightarrow \|\mathcal{M}_\omega^\sharp f\|_\infty < \infty$ for $\omega(r) = r^\beta$.

Main result

Breit-Cianchi-Diening-Schwarzacher

$$\mathcal{M}^\sharp(|\nabla u|^{p-2}\nabla u)(x) \leq c\mathcal{M}^\sharp_{p'}(F)(x)$$

- Note that $\|\mathcal{M}^\sharp(f)\|_\infty \sim \|\mathcal{M}^\sharp_s(f)\|_\infty$;
- For $q > s$ we have $\|\mathcal{M}^\sharp_s(f)\|_q \sim \|f\|_q$;
- Also for weighted $\mathcal{M}^\sharp_\omega$;
- Local version available.

Consequences

Refinement of known local results:

- $F \in L^q \;\Rightarrow\; |\nabla u|^{p-2}\nabla u \in L^q$ for $q > p'$;
- $F \in \mathrm{BMO} \;\Rightarrow\; |\nabla u|^{p-2}\nabla u \in \mathrm{BMO}$;
- $F \in C^\beta$ in $x \;\Rightarrow\; |\nabla u|^{p-2}\nabla u \in C^\alpha$ in x.

By the continuity of $\mathcal{M}^\sharp$ more spaces including borderline

- Lorentz spaces;
- Orlicz spaces;
- Exponential estimates $F \in L^{\exp} \;\Rightarrow\; |\nabla u|^{p-2}\nabla u \in L^{\exp}$;

Pointwise estimates of measure valued right hand side for $N \geq 2$. If $\mu \in L^{p'}$, then

$$|\nabla u|^{p-1} \leq c\mathbb{W}_{\frac{1}{p'+1},p'+1}(|\mu|^{p'}).$$