On convergent series expansions of solutions of Navier's equation near singular points

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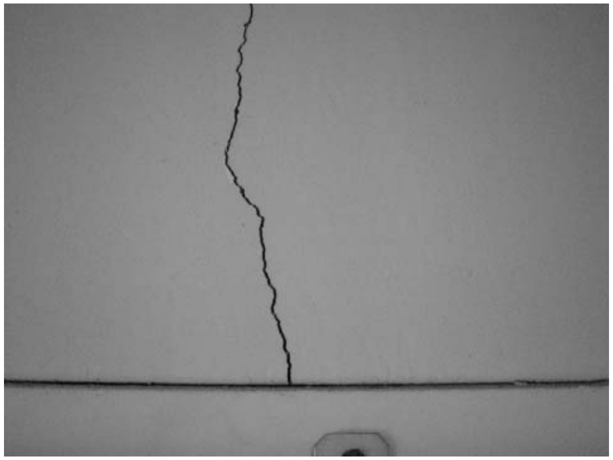
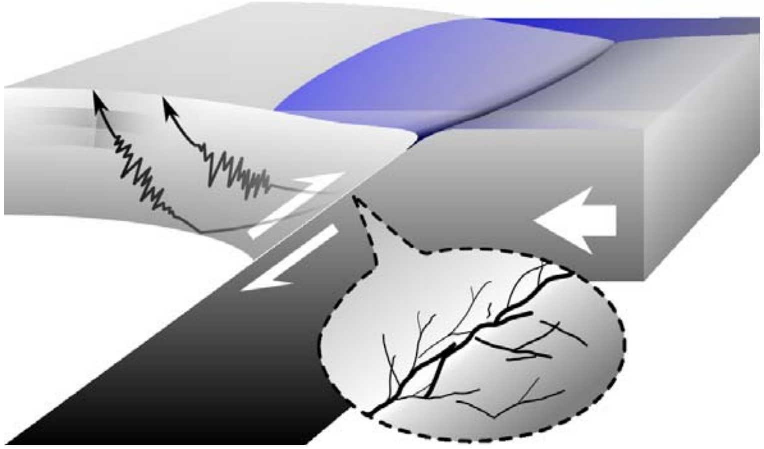
1. Introduction

- Motivation

⇒ Fracture Phenomena, Earthquakes





⇒ Difficulty in mathematical analysis

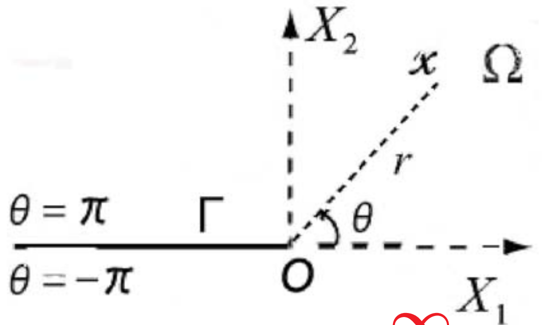
· The domain has singularities

· Solutions of the governing equation has singularity

⇒ It's very important

to analyze precise behavior of the solution at the singular points.

For example,



$$\begin{cases} Au = 0 \quad \text{in} \quad r \in (0, R),\ \theta \in (-\pi, \pi), \\ Tu = 0 \quad \text{on} \qquad\qquad \theta = \pm\pi. \end{cases}$$

$$\Rightarrow u(r,\theta) \stackrel{?}{=} \sum_{n=1}^{\infty} C_n r^{\alpha_n} \Phi_n(\theta)$$

C_1: Stress intensity Factor

α_n: Order of singularities

References

Costabel,M. & Dauge,M., Comm.PDE., 19(1994) 1677–1726

Grisvard, P., Elliptic problems in nonsmooth domains, 1985

Kondrat'ev, V. A. & Oleinik, O. A., Russian Math. Surveys, 38(1983), 3 – 76

Kozlov, V., Maz'ya, V. G. & Rossmann, J., Elliptic Boundary Value Problems in Domains With Point Singularities, 1997

Aims:

Governing Eq.: 2D linearized elasticity system

Singular point: a tip of a linear crack or inclusion

· to derive convergent series expansions of the solution near the tip

· to know how the order of singularities is determined

Maybe, it's depending on geometrical property of the crack, properties of the governing equation, boundary conditions on the crack, material properties, etc

Papers

[1] M. I. & H. I., Inverse Problems 23(2007) 589-607

[2] M. I. & H. I., Inverse Problems 24(2008) 025005

[3] H. I., V. K. & A. T., Appl. of Math. 56(2011) 69-97

[4] H. I., A. K., E. R. & A. T., ZAMM 92(2012) 716-730

Tools :

Lekhnitskii, S. G., Anisotropic Plates, 1968

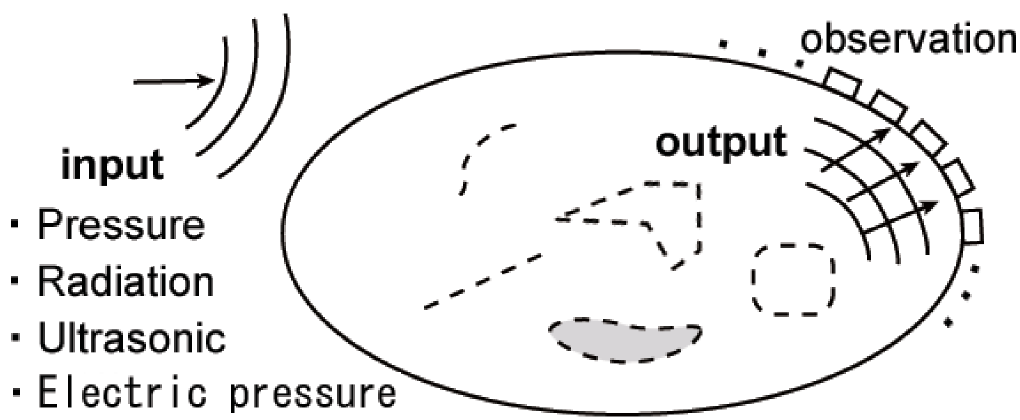
Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity, 1963

Rice, J. R., J. Appl. Mech., 55(1988), 98-103

Application: (Ref. [1] – [2] with Prof. Ikehata)

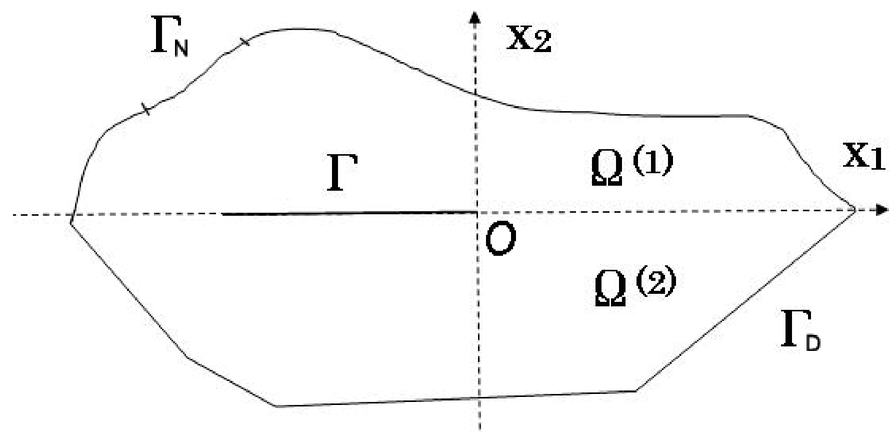
Inverse problems (Nondestructive evaluation)

Esp., reconstruction issue by means of Enclosure method



• **Preliminaries**

Domain



$\Omega \subset \mathbb{R}^2$: a bounded domain with Lipschitz boundary

2D linearized elasticity

$\Omega^{(1)} = \Omega \cap \{x_2 > 0\}$, $\Omega^{(2)} = \Omega \cap \{x_2 < 0\}$: Lipschitz

Γ': interface of $\Omega^{(k)}$, Γ: a line segment PO lies on Γ'

$\Gamma_N \subset \partial\Omega^{(1)} \setminus \Gamma'$, $\Gamma_N \cap \overline{\Gamma'} = \emptyset$, $\Gamma_D = \partial\Omega \setminus \Gamma_N$

The linearized elasticity equation (Navier's eq.):

For the displacement vector $u = (u_1, u_2)^{\mathrm{T}}$ in Ω

$$Au := \sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = 0 \quad i = 1, 2.$$

$C = (C_{ijkl})_{i,j,k,l=1,2}$: the elasticity tensor (constant)

- Anisotropic: 6 independent components

  C_{1111}, C_{2222}, C_{1212}, C_{1122}, C_{1112}, C_{1222}

- Isotropic: 2 independent components

$C_{1111} = C_{2222}$, $C_{1112} = C_{1222} = 0$, $2C_{1212} = C_{1111} - C_{1122}$

$\lambda := C_{1122}$, $\mu := C_{1212}$: *Lamé* constants in Ω satisfying $\mu > 0$ and $\lambda + \mu > 0$. And we define $\kappa := \frac{\lambda+3\mu}{\lambda+\mu}$.

$\sigma = (\sigma_{ij})_{i,j=1,2} = \sum_{k,l=1}^{2} C_{ijkl}\frac{\partial u_k}{\partial x_l}$: the stress tensor

$Tu := \sigma n$, $n = (n_1, n_2)^{\mathrm{T}}$:the unit outward normal to $\partial\Omega$

2. Case 1 : Ω is homogeneous, isotropic or anisotropic.

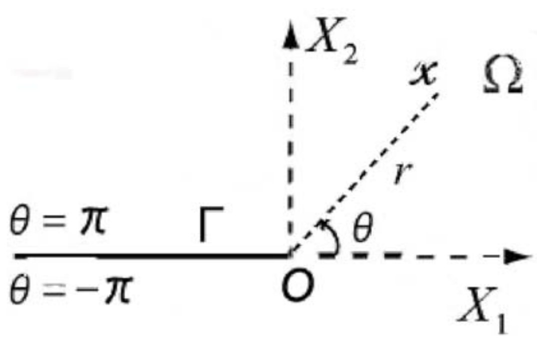
Γ is a linear crack (traction-free)

- Boundary value problem: for given $g \in L^2(\Gamma_N)$

$$(*)_1 \begin{cases} Au = 0 & \text{in} \quad \Omega \setminus \overline{\Gamma}, \\ Tu = 0 & \text{on} \quad \Gamma, \\ Tu = g & \text{on} \quad \Gamma_N, \\ u = 0 & \text{on} \quad \Gamma_D. \end{cases}$$

In the case of isotropic material

Ref.[1]: M. I. & H. I. 2007 Inverse Problems 23 589-607



$$\begin{cases} Au = 0 \ \text{in} \ r \in (0, R), \ \theta \in (-\pi, \pi), \\ Tu = 0 \ \text{on} \qquad \theta = \pm\pi. \end{cases}$$

How to derive :

Existence of the weak solution of $(*)_1$

Regularity results $\Downarrow$ Poincaré lemma

Airy's stress function $\triangle^2 U = 0$

$$\sigma_{11} = \frac{\partial^2 U}{\partial x_2^2},\ \sigma_{22} = \frac{\partial^2 U}{\partial x_1^2},\ \sigma_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2}$$

$\Downarrow$

Goursat-Kolosov-Muskhelishvili stress functions

$\exists \phi(z), \omega(z)$ s.t. $U = 2\mathrm{Re}\left\{\overline{z}\phi(z) + \int \omega(z)\, \mathrm{d}z\right\}$

$\Downarrow$

Analytic continuation

$\Downarrow$

Convergent series expansions near the crack tip

Proposition 1 $\exists A_n,\ B_n \in \mathbb{R},\ \rho \in \mathbb{R}^3$ s. t.

$$u(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{2\mu} r^{\frac{n}{2}} \left(A_n \varphi_n(\theta) - B_n \psi_n(\theta)\right) + F(x)\rho,$$

$$\begin{aligned}
\varphi_n(\theta) &= \begin{pmatrix} \kappa \cos \frac{n}{2}\theta - \frac{n}{2} \cos\left(\frac{n}{2} - 2\right)\theta + \left\{\frac{n}{2} + (-1)^n\right\} \cos \frac{n}{2}\theta \\ \kappa \sin \frac{n}{2}\theta + \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\theta - \left\{\frac{n}{2} + (-1)^n\right\} \sin \frac{n}{2}\theta \end{pmatrix} \\
\psi_n(\theta) &= \begin{pmatrix} \kappa \sin \frac{n}{2}\theta - \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\theta + \left\{\frac{n}{2} - (-1)^n\right\} \sin \frac{n}{2}\theta \\ -\kappa \cos \frac{n}{2}\theta - \frac{n}{2} \cos\left(\frac{n}{2} - 2\right)\theta + \left\{\frac{n}{2} - (-1)^n\right\} \cos \frac{n}{2}\theta \end{pmatrix}.
\end{aligned}$$

The series is convergent, absolutely in H^1 and uniformly in $r \le R' < R$ for each R'. For $n \ge 4$, A_n, B_n satisfy $\exists c > 0$ such that

$$|A_n| + |B_n| \le \frac{c}{\sqrt{n}} R^{-\frac{n}{2}} \| \nabla u \|_{L^2} .$$

In the case of anisotropic material

Ref.[2]: M. I. & H. I. 2008 Inverse Problems 24 025005

In this case we employ Lekhnitskii formalism instead of Goursat-Kolosov-Muskhelishvili stress functions.

⇒ Airy's stress function U satisfies $\boxed{\triangle_1 \triangle_2 U = 0}$,

$$\triangle_k = \mu_k \bar{\mu}_k \partial_{x_1}^2 - (\mu_k + \bar{\mu}_k)\partial_{x_1}\partial_{x_2} + \partial_{x_2}^2 \quad (k = 1, 2).$$

$\mu_k = \alpha_k + i\beta_k$, $\bar{\mu}_k = \alpha_k - i\beta_k$ for $\beta_k > 0$ is the root of the corresponding characteristic equation.

Note: In isotropic case, $\mu_1 = \mu_2 = i$.

In anisotropic case ($\mu_1 \neq \mu_2$) U can be given by two arbitrary analytic functions of the generalized complex variables $z_k = x_1 + \mu_k x_2$. In a case ($\mu_1 = \mu_2 = \alpha + i\beta$), Affine transformation $\hat{x}_1 = x_1 + \alpha x_2$, $\hat{x}_2 = \beta x_2$ ledas to $z = z_1 = z_2 = \hat{x}_1 + i\hat{x}_2$ (Isotropic case).

· <u>Anisotropic ($\mu_1 \neq \mu_2$)</u>

Proposition 2 $\exists A_n, B_n \in \mathbb{R}, \boldsymbol{k} \in \mathbb{R}^3$ s. t.

$$\boldsymbol{u}(r,\theta) = \sum_{n=0}^{\infty} r^{\frac{n}{2}} \mathrm{Re}\left[A_n \frac{i^{n-1}\boldsymbol{\psi}_{1,n}(\theta)}{\mu_1 - \mu_2} + B_n \frac{i^{n-1}\boldsymbol{\psi}_{2,n}(\theta)}{\mu_1 - \mu_2} \right] + \boldsymbol{F}(x)\boldsymbol{k},$$

where

$$\boldsymbol{\psi}_{1,n}(\theta) = \begin{pmatrix} \mu_1 p(\mu_2)(\cos\theta + \mu_2 \sin\theta)^{\frac{n}{2}} - \mu_2 p(\mu_1)(\cos\theta + \mu_1 \sin\theta)^{\frac{n}{2}} \\ \mu_1 q(\mu_2)(\cos\theta + \mu_2 \sin\theta)^{\frac{n}{2}} - \mu_2 q(\mu_1)(\cos\theta + \mu_1 \sin\theta)^{\frac{n}{2}} \end{pmatrix}$$

$$\boldsymbol{\psi}_{2,n}(\theta) = \begin{pmatrix} -p(\mu_1)(\cos\theta + \mu_1 \sin\theta)^{\frac{n}{2}} + p(\mu_2)(\cos\theta + \mu_2 \sin\theta)^{\frac{n}{2}} \\ -q(\mu_1)(\cos\theta + \mu_1 \sin\theta)^{\frac{n}{2}} + q(\mu_2)(\cos\theta + \mu_2 \sin\theta)^{\frac{n}{2}} \end{pmatrix}.$$

The series is convergent, absolutely in H^1 and uniformly in the neighbourhood of O. Constants $p(\mu_k)$ and $q(\mu_k)$ are explicitly given by material constants C_{ijkl}.

• Anisotropic ($\mu_1 = \mu_2 \equiv \mu_0 = \alpha + i\beta$)

Proposition 3 $\exists A_n,\ B_n \in \mathbb{R}\ (n = 0, 1, 2, \cdots)$ s. t.

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{\frac{n}{2}} \mathrm{Re}\left[\tilde{\psi}_{1,n}(\theta)\right] + B_n r^{\frac{n}{2}} \mathrm{Im}\left[\tilde{\psi}_{2,n}(\theta)\right] + F(x)\rho,$$

where

$$\tilde{\psi}_{1,n}(\theta) = \begin{pmatrix} p(\mu_0)\left\{\frac{n}{2}\Theta^{\frac{n}{2}-2} - \left(\frac{n}{2} + (-1)^n\right)\Theta^{\frac{n}{2}} + \overline{\Theta}^{\frac{n}{2}}\right\} + 4S_{11}\beta^2\Theta^{\frac{n}{2}} \\ q(\mu_0)\left\{\frac{n}{2}\Theta^{\frac{n}{2}-2} - \left(\frac{n}{2} + (-1)^n\right)\Theta^{\frac{n}{2}} + \overline{\Theta}^{\frac{n}{2}}\right\} + S_{22}\frac{4\beta^2}{|\mu_0|^2}\overline{\mu_0}\Theta^{\frac{n}{2}} \end{pmatrix}$$

$$\tilde{\psi}_{2,n}(\theta) = \begin{pmatrix} p(\mu_0)\left\{-\frac{n}{2}\Theta^{\frac{n}{2}-2} + \left(\frac{n}{2} - (-1)^n\right)\Theta^{\frac{n}{2}} + \overline{\Theta}^{\frac{n}{2}}\right\} - 4S_{11}\beta^2\Theta^{\frac{n}{2}} \\ q(\mu_0)\left\{-\frac{n}{2}\Theta^{\frac{n}{2}-2} + \left(\frac{n}{2} - (-1)^n\right)\Theta^{\frac{n}{2}} + \overline{\Theta}^{\frac{n}{2}}\right\} - S_{22}\frac{4\beta^2}{|\mu_0|^2}\overline{\mu_0}\Theta^{\frac{n}{2}} \end{pmatrix}$$

with $\Theta = \cos\theta + \mu_0 \sin\theta$. The series is convergent, absolutely in H^1 and uniformly in the neighbourhood of O.

3. Case 2 (a linear interfacial crack with friction)

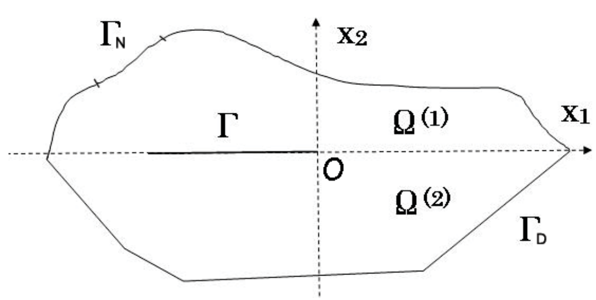
Ref.[3]: H. I., V. K. & T. A. 2011 Appl. of Math. 56 69-97

For given surface force $g \in L^2(\Gamma_N)$,

and a small constant friction coefficient $f \in (0,1)$,

find $u^{(1)} \in H^1(\Omega^{(1)})$ and $u^{(2)} \in H^1(\Omega^{(2)})$ satisfying

$$
(*)_2 \begin{cases}
A(\partial_x, \lambda^{(k)}, \mu^{(k)})u^{(k)} = 0 & \text{in } \Omega^{(k)} \ (k=1,2), \\
T(\partial_x, \lambda^{(1)}, \mu^{(1)})u^{(1)} = g & \text{on } \Gamma_N, \\
u^{(k)} = 0 & \text{on } \Gamma_D, \\
[u_1] = [u_2] = [\sigma_{12}] = [\sigma_{22}] = 0 & \text{on } \Gamma' \setminus \overline{\Gamma}, \\
\text{non-penetration } (*)_{np} & \text{on } \Gamma, \\
\text{Coulomb friction } (*)_f & \text{on } \Gamma.
\end{cases}
$$



Jump of u at Γ:

$[u] := u^{(1)} - u^{(2)}$ on Γ.

- Conditions on the crack

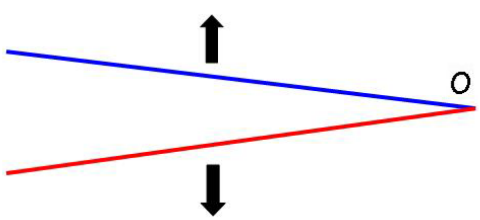
$$(*)_{np}: \ [\sigma_{22}] = 0, \ \sigma_{22}^{(k)} \leq 0, \ [u_2] \geq 0, \ \sigma_{22}^{(k)}[u_2] = 0,$$

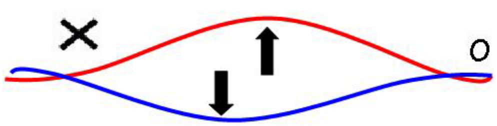
$$(*)_{f}: \ [\sigma_{12}] = 0, \ |\sigma_{12}^{(k)}| \leq -f\sigma_{22}^{(k)}, \ \sigma_{12}^{(k)}[u_1] + f\sigma_{22}^{(k)}|[u_1]| = 0.$$

Case A: open crack

$$[u_2] > 0$$

$$\sigma_{12}^{(k)} = \sigma_{22}^{(k)} = 0$$





Case B: Stick state

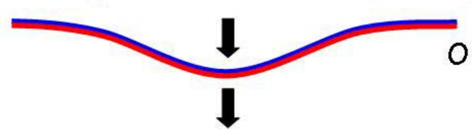
$$[u_2] = 0$$

$$[u_1] = 0$$

$$[\sigma_{22}] = [\sigma_{12}] = 0$$

$$\sigma_{22}^{(k)} \leq 0$$

$$|\sigma_{12}^{(k)}| \leq -f\sigma_{22}^{(k)}$$



Case C: slip state

$$[u_2] = 0$$

$$[u_1] \neq 0$$

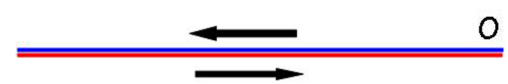
$$[\sigma_{22}] = [\sigma_{12}] = 0$$

$$\sigma_{22}^{(k)} \leq 0$$

$$\sigma_{12}^{(k)} \pm f\sigma_{22}^{(k)} = 0$$

"+": $[u_1] > 0$

"−": $[u_1] < 0.$



Existence of the solution of $(*)_2$

Notation

A bilinear form: $\mathcal{E}_D(u,v) := \int_D \sigma_{ij}(u)\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x \;\; i,j=1,2,$

$u=u^{(1)}$, $\sigma=\sigma^{(1)}$ in $\Omega^{(1)}$, $u=u^{(2)}$, $\sigma=\sigma^{(2)}$ in $\Omega^{(2)}$.

$\mathcal{R}=\{F(x)c=(c_1+c_0x_2, c_2-c_0x_1)\ \forall c=(c_1,c_2,c_0)\in\mathbb{R}^3\}$.

$\mathcal{K}=\{v\in H^1(\Omega\backslash\overline{\Gamma}) : \ v|_{\Gamma_D}=0,\ [v_2]\geq 0 \quad$ on $\Gamma\}$: convex

- The weak formulation

The quasi-variational inequality $(\ddagger)$:

Find $u\in\mathcal{K}$ satisfying for an arbitrary $v\in\mathcal{K}$

$$\mathcal{E}_{\Omega\backslash\overline{\Gamma}}(u,v-u)-\langle f\sigma_{22}, |[v_1]|-|[u_1]|\rangle_\Gamma \geq \int_{\Gamma_N} g\cdot(v-u)\mathrm{d}S_x.$$

$\langle\cdot,\cdot\rangle_\Gamma$: the duality pairing between $H_{00}^{-1/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$

Lions–Magenes sp.: $H^{1/2}_{00}(\Gamma) = \{s \in H^{1/2}_0(\Gamma) : \rho^{-1/2}s \in L^2(\Gamma)\}$

where $\rho \in C^{1,1}_0(\overline{\Gamma})$ s.t. $\rho > 0$, $\lim_{x \to O or P} \rho(x)/\mathrm{dist}(x, O or P) \neq 0$.

$$s \in H^{1/2}_{00}(\Gamma) \quad \Leftrightarrow \quad \bar{s} = \begin{cases} s & \text{on} \quad \Gamma \\ 0 & \text{on} \quad \Gamma' \setminus \overline{\Gamma} \end{cases} \in H^{1/2}(\Gamma').$$

- Difficulty

1. Cannot describe as a minimization problem

   $\times \Rightarrow$ standard variational method

2. Estimate of $\langle f\sigma_{22}, |[v_1]| - |[u_1]|\rangle_\Gamma$ without restriction on compact support of f

Our result

- **Preliminaries**

Korn inequality $\Rightarrow$

there exist $0 < \underline{C}_0 \leq \overline{C}_0 < \infty$ s.t. $\forall u \in H^1(\Omega \setminus \overline{\Gamma})$

$$\underline{C}_0 \|u\|^2_{H^1(\Omega\setminus\overline{\Gamma})} \leq \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(u,u) \leq \overline{C}_0 \|u\|^2_{H^1(\Omega\setminus\overline{\Gamma})}$$

Equivalent norm in $H^1(\Omega \setminus \overline{\Gamma})$ as $\|u\|^2_{1,\Omega\setminus\overline{\Gamma}} := \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(u,u)$

Continuity of the trace operator $\Rightarrow$

there exist C_1, C_2 s.t. $1 \leq C_1 C_2 < \infty$ and

$$\|[u]\|_{H^{1/2}_{00}(\Gamma)} \leq C_1 \|u\|_{1,\Omega\setminus\overline{\Gamma}} \quad \forall u \in H^1(\Omega \setminus \overline{\Gamma}),$$

$$\|Tu\|_{H^{-1/2}_{00}(\Gamma)} \leq C_2 \|u\|_{1,\Omega\setminus\overline{\Gamma}} \; \forall u \in H^1(\Omega \setminus \overline{\Gamma}) \text{ s.t. } Au = 0$$

$\underline{C}_0$, $\overline{C}_0$, C_1, C_2 depend on $\lambda^{(k)}$, $\mu^{(k)}$ for $k = 1, 2$, and on the geometry of Ω.

- Existence Theorem

Theorem If $f < \dfrac{1}{C_1 C_2} \leq 1$ holds, then there exists a solution $u \in \mathcal{K}$ of the quasi-variational inequality $(\ddagger)$.

Lemma 4 The solution $u \in \mathcal{K}$ of the quasi-variational inequality $(\ddagger)$ obeys the interior C^∞-regularity in $\Omega^{(1)}$ and $\Omega^{(2)}$. The boundary stress components σ_{i2}, $i = 1, 2$ are pointwise functions inside the crack Γ.

Remark 1 Uniqueness of the solution for $(*)_2$ remains an open problem.

Convergent expansions near the crack tip

Polar coordinate system:

$(x_1, x_2) = (r\cos\theta, r\sin\theta)$ with respect to the origin O

$$B_R := B_R(O), \quad B_R^{(1)} := B_R \cap \Omega^{(1)}, \quad B_R^{(2)} := B_R \cap \Omega^{(2)}$$

$R > R'$

Assumption:

Whole on the crack $B_R \cap \Gamma$ one of three cases; open crack, stick state, slip state, is imposed, that is, no switches among three cases on $B_R \cap \Gamma$.

How to derive

The interior & boundary regularity results of Lemma 4

$+$

Poincaré lemma

$\Downarrow$

Goursat-Kolosov-Muskhelishvili stress functions

$\phi^{(k)}(z),\ \omega^{(k)}(z) \in H^1(B_R^{(k)})$ $(k = 1, 2)$, $z = x_1 + ix_2$

$$2\mu^{(k)}(u_1^{(k)} + iu_2^{(k)}) \ = \ \kappa^{(k)}\phi^{(k)}(z) - \overline{\omega^{(k)}(z)} + (\overline{z} - z)\overline{\phi^{(k)\prime}(z)}$$

$\Downarrow$

Riemann–Hilbert problem: For any analytic $\Phi(z)$ in B_R

$$a\phi^{(1)'}(z) + b\phi^{(1)'}(\overline{z}) = \Phi(z) \quad \text{on} \quad B_R \cap \Gamma.$$

$\Downarrow$ local Taylor series expansions

Convergent expansions near the crack tip

Proposition 5 $\exists a_n, b_n \in \mathbb{C}$ satisfying some conditions and $c \in \mathbb{R}^3$ such that for $k = 1, 2$

$$u^{(k)}(r,\theta) = \sum_{n=0}^{\infty} r^{n+1} \left\{ \mathrm{Re}\left[r^{-\alpha} a_n\right] A_n^{(k)}(\epsilon,\theta) + \mathrm{Im}\left[r^{-\alpha} a_n\right] B_n^{(k)}(\epsilon,\theta) \right\}$$
$$+ \sum_{n=0}^{\infty} r^{n+1} \left\{ \mathrm{Re}\left[b_n\right] C_n^{(k)}(\epsilon,\theta) + \mathrm{Im}\left[b_n\right] D_n^{(k)}(\epsilon,\theta) \right\} + F(x)c$$

The series are convergent, absolutely in H^1 and generalized uniform for $k = 1, 2$, respectively.

- open crack case: $\alpha = \frac{1}{2} + i\epsilon$, where $\epsilon := \frac{1}{2\pi} \ln\left(\frac{1+\beta}{1-\beta}\right)$,

Dunders parameter $\beta := \frac{\mu^{(2)}(\kappa^{(1)}-1)-\mu^{(1)}(\kappa^{(2)}-1)}{\mu^{(2)}(\kappa^{(1)}+1)+\mu^{(1)}(\kappa^{(2)}+1)}$.

$$[u_2] > 0 \text{ on } \Gamma \Leftrightarrow \sum_{n=0}^{\infty} (-1)^n r^{\frac{1}{2}+n} \mathrm{Re}\left[a_n r^{-i\epsilon}\right] > 0$$

$\Rightarrow \mathrm{Re}[a_0] \geq 0$ corresponding to A.M.Khludnev results

- stick state case: $\alpha = 0 \Rightarrow$ real analytic

- slip state case:

$$\cot(\pi\alpha) = \mp f\beta$$

where "$-$" for $[u_1] > 0$ on Γ, "$+$" for $[u_1] < 0$ on Γ.

$\Rightarrow$ we can choose $0 < \alpha < 1/2$ satisfying inequality type conditions on Γ, if $\beta \neq 0$.

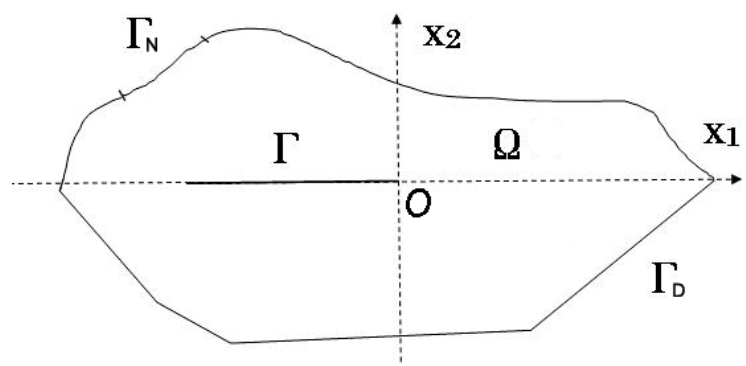
$\Rightarrow$ if $\beta = 0$, $\alpha = 1/2$.

Note:

- It enables us to have an a priori regularity of the solution near the crack tip; open crack case implies $u \notin H^{3/2}$, the solution is smooth in the stick state case, and for $\beta \neq 0$, $u \in H^{3/2}$ in the slip state case.

4. Case 3

- Motivation: Crack problem⇒Rigid inclusion problem
- Aim: To derive the convergent series expansion of solution of boundary value problem at a tip of rigid line inclusion Γ and the Irwin's formula



$\Omega \subset \mathbb{R}^2$: a bounded domain with Lipschitz boundary homogeneous isotropic linearized elasticity

- Applications:

Delamination phenomena (branched crack),
Inverse problems (nondestructive evaluation), etc.

Problem 1 (without delamination) Given $g \in L^2(\Gamma_N)$, find $u \in H^1(\Omega)$ and $\rho_0 \in \mathcal{R}(\Gamma)$ satisfying

$$(*)_3' \begin{cases} Au = 0 & \text{in} \quad \Omega \setminus \overline{\Gamma}, \\ Tu = g & \text{on} \quad \Gamma_N, \\ u = 0 & \text{on} \quad \Gamma_D, \\ u = \rho_0 & \text{on} \quad \Gamma, \\ \int_\Gamma [\sigma n]_\Gamma \cdot \rho \, \mathrm{d}S_x = 0 & \text{for} \quad \forall \rho \in \mathcal{R}(\Gamma). \end{cases}$$

Problem 2 (with delamination in a linear case) Given $g \in L^2(\Gamma_N)$, find $u \in H^1(\Omega \setminus \overline{\Gamma})$ and $\rho_0 \in \mathcal{R}(\Gamma)$ satisfying

$$(*)_3 \begin{cases} Au = 0 & \text{in} \quad \Omega \setminus \overline{\Gamma}, \\ Tu = g & \text{on} \quad \Gamma_N, \\ u = 0 & \text{on} \quad \Gamma_D, \\ Tu^+ = 0, \quad u^- = \rho_0 & \text{on} \quad \Gamma, \\ \int_\Gamma [\sigma n]_\Gamma \cdot \rho \, \mathrm{d}S_x = 0 & \text{for} \quad \forall \rho \in \mathcal{R}(\Gamma). \end{cases}$$

Weak formulation and Existence Theorem

$$
\begin{aligned}
\text{Set} \quad u \in H^1_{\Gamma_D}(\Omega) &:= \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = 0\}, \\
\Pi(u) &:= \frac{1}{2}\mathcal{E}_\Omega(u,u) - \int_{\Gamma_N} g \cdot u \, \mathrm{d}S_x, \\
\mathcal{E}_\Omega(u,v) &:= \int_\Omega \sigma_{ij} \frac{\partial}{\partial x_j} v_i \, \mathrm{d}x, \\
\mathcal{K} &:= \{v \in H^1_{\Gamma_D}(\Omega) \mid v|_\Gamma \in \mathcal{R}(\Gamma)\} \\
\mathcal{K}^- &:= \{v \in H^1_{\Gamma_D}(\Omega \setminus \overline{\Gamma}) \mid v^-|_\Gamma \in \mathcal{R}(\Gamma)\}.
\end{aligned}
$$

- **Weak formulation (minimization problem)**

Problem 1 or 2 $\Longrightarrow$ $\boxed{\Pi(u) = \inf_{v \in \mathcal{K} \text{ or } \mathcal{K}^-} \Pi(v)}$

$\Leftrightarrow$ find $u \in \mathcal{K}$ or $\mathcal{K}^-$ s.t. for any $w \in \mathcal{K}$ or $\mathcal{K}^-$

$$
\mathcal{E}_\Omega(u,w) - \int_{\Gamma_N} g \cdot w \, \mathrm{d}S_x = 0.
$$

Note that the condition $\int_\Gamma [\sigma n]_\Gamma \cdot \rho \, \mathrm{d}S_x = 0 \ \forall \rho \in \mathcal{R}(\Gamma)$ make sense as $\langle [\sigma n]_{\Gamma'}, v \rangle^{00}_{\Gamma'} = 0$, $\forall v \in \mathcal{K}^{(-)}$ with a duality pairing $\langle \cdot, \cdot \rangle^{00}_{\Gamma'}$ in the Lions–Magenes space $H^{1/2}_{00}(\Gamma')$, Γ': the interface of $\Omega^{(1)}$ and $\Omega^{(2)}$, i.e. extention of Γ.

Theorem

- For Problem 1 there exists a unique solution $u \in \mathcal{K}$ of the minimization problem.
- For Problem 2 there exists a unique solution $u \in \mathcal{K}^{-}$ of the minimization problem.

Lemma 6 The each solutions u obey the interior C^∞-regularity in $\Omega^{(1)}$ and $\Omega^{(2)}$. The boundary stress components σ_{i2}, $i = 1, 2$ are pointwise functions in Γ.

The convergent series expansions

- How to derive:

Existence of the weak solution of $(*)_3'$ and $(*)_3$

Regularity results Lemma 6 $\Downarrow$ Poincaré lemma

Goursat-Kolosov-Muskhelishvili stress functions

$\phi(z)$, $\omega(z) \in H^1(B_R)$, $z = x_1 + ix_2$

$$2\mu(u_1 + iu_2) \;=\; \tilde{\kappa}\phi(z) - \overline{\omega(z)} + (\bar{z} - z)\overline{\phi'(z)}$$

$\Downarrow$

Riemann–Hilbert problems: Given $\Phi(z), \Psi(z)$, on $B_R \cap \Gamma$

$$a\phi'(z) + b\phi'(\bar{z}) = \Phi(z) \quad \text{or} \quad \begin{cases} a\phi'(z) + b\omega'(\bar{z}) = \Phi(z) \\ c\phi'(\bar{z}) + d\omega'(z) = \Psi(z). \end{cases}$$

$\Downarrow$

Convergent expansions around O

• For Problem 1

Proposition 7 $\exists P_m, Q_m \in \mathbb{R}$, $\alpha, \alpha_1, \beta_1, \beta_2 \in \mathbb{R}$ s. t.

$$u(r,\theta) = \sum_{m=1}^{\infty} \frac{r^{\frac{m}{2}}}{2\mu} \left\{ P_m R_{1,m}(\theta) - Q_m S_{1,m}(\theta) \right\} + F_1,$$

$$R_{1,m}(\theta) = \begin{pmatrix} \kappa(1-(-1)^m)\cos\frac{m}{2}\theta + \frac{m}{2}\cos\frac{m}{2}\theta - \frac{m}{2}\cos\left(\frac{m}{2}-2\right)\theta \\ \kappa(1+(-1)^m)\sin\frac{m}{2}\theta - \frac{m}{2}\sin\frac{m}{2}\theta + \frac{m}{2}\sin\left(\frac{m}{2}-2\right)\theta \end{pmatrix}$$

$$S_{1,m}(\theta) = \begin{pmatrix} \kappa(1+(-1)^m)\sin\frac{m}{2}\theta + \frac{m}{2}\sin\frac{m}{2}\theta - \frac{m}{2}\sin\left(\frac{m}{2}-2\right)\theta \\ -\kappa(1-(-1)^m)\cos\frac{m}{2}\theta + \frac{m}{2}\cos\frac{m}{2}\theta - \frac{m}{2}\cos\left(\frac{m}{2}-2\right)\theta \end{pmatrix}$$

$F_1 = \begin{pmatrix} \alpha_1 r\sin\theta + \beta_1 \\ -\alpha r\cos\theta + \beta_2 \end{pmatrix}$. The series is convergent, absolutely in $H^1(B_R)$ and generalized uniform. For $m \geq 1$, P_m and Q_m satisfy $|P_m| + |Q_m| \leq c\frac{1}{\sqrt{m}} R^{-\frac{m}{2}} \|\nabla u\|_{L^2(B_R)}$.

- For Problem 2

Proposition 8 $\exists c_m \in \mathbb{C}$, α, β_1, $\beta_2 \in \mathbb{R}$ s. t.

$$u = \sum_{m=1}^{\infty} \frac{e^{\varepsilon\theta}}{2\mu} r^{\frac{2m-1}{4}} \left\{ \mathrm{Re}[c_m r^{-\varepsilon i}] R_{2,m}(\theta) - \mathrm{Im}[c_m r^{-\varepsilon i}] S_{2,m}(\theta) \right\} + F_2,$$

$$R_{2,m}(\theta) = \begin{pmatrix} \left(\kappa + \frac{2m-1}{4}\right) \cos \frac{2m-1}{4}\theta - \left((-1)^m \sqrt{\kappa} e^{-2\varepsilon\theta} - \varepsilon\right) \sin \frac{2m-1}{4}\theta - \frac{2m-1}{4} \cos \frac{2m-9}{4}\theta - \varepsilon \sin \frac{2m-9}{4}\theta \\ \left(\kappa - \frac{2m-1}{4}\right) \sin \frac{2m-1}{4}\theta - \left((-1)^m \sqrt{\kappa} e^{-2\varepsilon\theta} - \varepsilon\right) \cos \frac{2m-1}{4}\theta + \frac{2m-1}{4} \sin \frac{2m-9}{4}\theta - \varepsilon \cos \frac{2m-9}{4}\theta \end{pmatrix}$$

$$S_{2,m}(\theta) = \begin{pmatrix} \left(\kappa + \frac{2m-1}{4}\right) \sin \frac{2m-1}{4}\theta - \left((-1)^m \sqrt{\kappa} e^{-2\varepsilon\theta} + \varepsilon\right) \cos \frac{2m-1}{4}\theta - \frac{2m-1}{4} \sin \frac{2m-9}{4}\theta + \varepsilon \cos \frac{2m-9}{4}\theta \\ -\left(\kappa - \frac{2m-1}{4}\right) \cos \frac{2m-1}{4}\theta + \left((-1)^m \sqrt{\kappa} e^{-2\varepsilon\theta} + \varepsilon\right) \sin \frac{2m-1}{4}\theta - \frac{2m-1}{4} \cos \frac{2m-9}{4}\theta - \varepsilon \sin \frac{2m-9}{4}\theta \end{pmatrix}$$

$F_2 = \begin{pmatrix} \alpha r \sin\theta + \beta_1 \\ -\alpha r \cos\theta + \beta_2 \end{pmatrix}$ and $\varepsilon := \dfrac{\log \kappa}{4\pi}$. The series is convergent, absolutely in $H^1(B_R)$ and generalized uniform. For $m \geq 1$, it holds $|c_m| \leq c \dfrac{R^{-\frac{m}{2}+\frac{1}{4}}}{\sqrt{2m-1}} \|\nabla u\|_{L^2(B_R)}$.

Remark 2 [Irwin's formula]

Using smooth perturbation method, it holds that

$$\Pi'(u) = J$$
$$= -\int_S \frac{1}{2}\sigma_{ij}(u)n_1\frac{\partial}{\partial x_j}u_i - \sigma_{ij}(u)n_j u_{i,1} - \alpha\sigma_{2j}(u)n_j \,\mathrm{d}S_x$$

where S is any closed curve around the tip O of the rigid inclusion which does not contains another tip inside of S.

For Problem 1 $$J = \frac{\pi\kappa(\lambda+2\mu)}{2\mu(\lambda+\mu)}(P_1^2 + Q_1^2).$$

For Problem 2

$$J = -\frac{\pi(\lambda+2\mu)\sqrt{\kappa}}{4\mu(\lambda+\mu)}\left\{(3+16\varepsilon^2)\mathrm{Im}[c_1\overline{c_2}] - 8\varepsilon\mathrm{Re}[c_1\overline{c_2}]\right\}.$$

5.Summary

Case	singular point	material	boundary condition	order of singularity
1	a linear crack tip	homogeneous, anisotropic	traction-free	$\frac{n}{2}$
2	a tip of a linear interfacial crack	two dissimilar homogeneous and isotropic media	non-penetration, Coulomb's friction	$n + 1 - \alpha$ open crack: $\alpha = \frac{1}{2} + i\epsilon$, stick state: $\alpha = 0$, slip state: $\cot(\pi\alpha) = \mp f\beta$
3	a tip of a rigid line inclusion	homogeneous, isotropic	rigidity traction-free & rigidity	$n/2$ $\frac{2n-1}{4} - \varepsilon i$

6. Future works

- 3D case

- Nonlinear elasticity

- Elastodynamics (thermoelasticity)