# Regular Lagrangian flow and size-structure equations

Piotr Gwiazda

University of Warsaw, Institute of Applied Mathematics and Mechanics

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# Motivation

$\psi(t,x,r)$ is the function representing density of polymer molecules of length $r$ at time $t$ at $x$.

$$\partial_t \psi(t,x,r) + \mathrm{div}_x (u(t,x)\psi(t,x,r))$$
$$= \partial_r(\tau(r)\psi(t,x,r)) - \beta(r)\psi(t,x,r) + 2\int_r^\infty \beta(\tilde{r})\kappa(r,\tilde{r})\psi(t,x,\tilde{r})d\tilde{r}$$

- $\tau > 0$ is the polimerization rate,
- $\beta(r)$ - the rate of fragmentation, $\beta(r,\cdot)$ can depend also on macroscopic quantities, namely on the velocity of the solvent and the shear rate,
- $\kappa(r,\tilde{r})$ - the fragmentation kernel - represents the proportion of individuals of size $r$ born from a given dividing individual of size $\tilde{r}$

# Transport equation - formulation

Consider the Cauchy problem for the following system

$$\partial_t \psi(t,x) + u(t,x) \cdot \nabla_x \psi(t,x) = 0,$$

$$\psi(0,x) = \bar{\psi}(x).$$

Where $u(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity of the transported quantity $\psi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and $\bar{\psi} : \mathbb{R}^d \to \mathbb{R}$ is given.

# DiPerna and Lions renormalization scheme.

- DiPerna, R. J. and Lions, P.-L., Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. (1989).

We say that $\psi$ is a renormalized solution to the transport equation if it satisfies

$$\partial_t \beta(\psi) + u \cdot \nabla_x \beta(\psi) = 0,$$

for any $\beta \in C^1(\mathbb{R};\mathbb{R})$ (with proper growth condition).

Existence, uniqueness and stability (compactness) for rough coefficients.

# DiPerna and Lions renormalization scheme.

Note that if

- $u^n \to u$ in $L^1$,
- $u^n$ is bounded in $W^{1,1}$ (it is enough $BV_{loc}$ or even BD)
- $\mathrm{div} u^n = 0$ (it is enough $\in L^\infty$ or even BMO)

Then

$$\psi^n \to \psi \quad \text{in } L^p$$

# DiPerna and Lions renormalization scheme.

Indeed,

$$\partial_t \beta(\psi^n) + u^n \nabla_x \beta(\psi^n) = 0$$

is equivalent to

$$\partial_t \beta(\psi^n) + \mathrm{div}(u^n \beta(\psi^n)) = 0.$$

Let $n \to \infty$, $\beta(u^n) \rightharpoonup \beta$. Then

$$\partial_t \beta + \mathrm{div}(u\beta) = 0.$$

Choose now $\beta(\psi^n) = (\psi^n)^p$. By renormalized property we have

$$\partial_t(\psi^p) + \mathrm{div}(u\psi^p) = 0$$

and by unique solvability we conclude that

$$\beta = \psi^p$$

and hence from $\psi^n \rightharpoonup \psi$ in $L^p$ and $\|\psi^n\|_{L^p} \to \|\psi\|_{L^p}$ we have $\psi^n \to \psi$ in $L^p$.

# Integral operator

$$\partial_t \psi(t,x) + u(t,x)\cdot\nabla_x \psi(t,x) = \int \gamma(x,y)\psi(t,y)dy,$$

$$\psi(0,x) = \bar{\psi}(x).$$

The equation for a renormalized quantity is not a linear equation on $\beta(\psi)$ only!

$$\partial_t \beta(\psi(t,x)) + u(t,x)\cdot\nabla_x \beta(\psi(t,x)) = \beta'(\psi(t,x)) \int \gamma(x,y)\psi(t,y)dy,$$

$$\beta(\psi(0,x)) = \beta(\bar{\psi}(x)).$$

# Lagrangian description

- The typically used description in fluid dynamics is the Eulerian description.
- In our considerations the Lagrangian description becomes of significant interest
  If $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded smooth vector field, the flow of $\psi$ is a smooth map $X : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\frac{dX}{dt}(t, x) = u(t, X(t, x)), \quad X(0, x) = x.$$

# Lagrangian description

## Regular Lagrangian flows

When $u$ is merely integrable one defines the so-called *regular Lagrangian flows*, meaning that for a.a. $x \in \mathbb{R}^n$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of $\dot{\gamma}(t) = u(t,\gamma(t))$ for $t \in [0,T]$ with $\gamma(t) = x$. Moreover, there exists a constant $L$ independent of $t$ such that

$$\mathcal{L}^n(X(t,\cdot)^{-1}(A)) \leq L\,\mathcal{L}^n(A)$$

for every Borel set $A \subset \mathbb{R}^n$,

Crippa, Gianluca; De Lellis, Camillo Estimates and regularity results for the DiPerna-Lions flow. J. Reine Angew. Math. 616 (2008), 15–46

Consider

$$\partial_t \psi + u \cdot \nabla_x \psi = \int \gamma \psi dy$$

on the characteristics

$$(\partial_t \psi + u \nabla_x \psi)(t, X_u(t,x)) = \int \gamma(X_u(t,x), y)\psi(t,y)\, \mathcal{L}^n(dy)$$

and introduce the quantity

$$\tilde{\psi}(t,x) := \psi(t, X_u(t,x)).$$

Notice that

$$\partial_t \tilde{\psi}(t,x) = (\partial_t \psi + u \nabla_x \psi)(t, X_u(t,x))$$

Recall that by $\mathcal{L}^n$ we will mean the $n-$dimensional Lebesgue measure and we introduce a measure $\mu_t$ as follows

$$\mu_t(A) := \mathcal{L}^n(X_u^{-1}(t, A)) \quad \text{for every Borel set } A.$$

proceeding with a change of variables leads to the following problem

$$\partial_t \tilde{\psi}(t, x) = \int \gamma(X_u(t, x), X_u(t, y))\tilde{\psi}(t, y) D_{\mathcal{L}^n}\mu_t \, \mathcal{L}^n(dy)$$

where by $D_{\mathcal{L}^n}\mu_t$ we mean the density (Radon-Nikodym derivative) of the measure $\mu_t$ with respect to the Lebesgue measure $\mathcal{L}^n$. In case when $\mathrm{div} u = 0$ then $D_{\mathcal{L}^n}\mu_t = 1$ a.e. Otherwise, the change of variables is not a measure preserving map. For simplicity consider the case $\mathrm{div} u = 0$.

We assume that $\gamma \in L^1(0, T; L^1_y(L^\infty_x)))$
Consider a sequence $(u_k)$ such that

$$u_k \to u \text{ in } L^p([0, T]; W^{1,p}).$$

Then we define $X_{u_k}$ as a regular Lagrangian flow for $u_k$ and $\psi_k$ as a solution to equation with $u_k$ instead of $u$. Then

$$\tilde{\psi}_k(t, x) := \psi_k(t, X_{u_k}(t, x)).$$

We will claim that

$$X_{u_k} \to X_u \text{ in } \mathcal{C}([0, T]; L^1_{loc}) \tag{1}$$

moreover, by the reversibility of the flow follows that

$$X^{-1}_{u_k} \to X^{-1}_u \text{ in } \mathcal{C}([0, T]; L^1_{loc}).$$

Our target is to show that

$$\tilde{\psi}_k \to \tilde{\psi} \text{ in } \mathcal{C}([0, T]; L^1_{loc})$$

and consequently

$$\psi_k \to \psi \text{ in } \mathcal{C}([0, T]; L^1_{loc}).$$

Let us rewrite

$$\partial_t \tilde{\psi}(t,x) = \int \gamma(t, X_u(t,x), X_u(t,y))\tilde{\psi}(t,y)dy$$

as a linear ordinary differential equation in a Banach space as follows

$$\frac{d\tilde{\psi}(t)}{dt} = A^k(t)[\tilde{\psi}] \tag{2}$$

with an operator $A^k : L^1 \to L^1$

$$A^k(t)[\tilde{\psi}] := \int \gamma(t, X_{u_k}(t,x), X_{u_k}(t,y))\tilde{\psi}(y)\ \mu_t(dy)$$

As $X_u(0,x) = x$ implies that $\psi_0 = \tilde{\psi}_0$, then the solution to (2) is given by

$$\tilde{\psi}(t) = \exp\left(\int_0^t A^k(s)\ ds\right)[\psi_0]$$

where

$$\exp\left(\int_0^t A^k(s)\ ds\right)[\psi_0] = \sum_{i=1}^{\infty} \frac{\left(\int_0^t A^k(s)\ ds\right)^i}{i!}[\psi_0].$$

For simplicity we start with a case when the function $\gamma$ has a product form, namely there exist functions $\gamma_1, \gamma_2$

$$\gamma(X_u(t,x), X_u(t,y)) = \gamma_1(X_u(t,x)) \otimes \gamma_2(X_u(t,y))$$

with $\gamma_1 \in L^p$ and $\gamma_2 \in L^{p'}$. If $\psi_0 \in L^p$, $p > 1$ then the norm of the operator $\int_0^t A^k(s)\, ds$ equals to $t\|\gamma_1\|_{L^p} \cdot \|\gamma_2\|_{L^{p'}}$.

In order to apply the Lusin's theorem we observe that from the information that $u \in L^1(0, T; L^\infty)$ it follows that for a.a. $x$

$$\sup_{t \in [0,T]} |X_u(t,x)| \leq |x| + \|u\|_{L^1(0,T;L^\infty)}.$$

and we denote by $D$ the set of finite measure containing the values of $X_u$. Hence for a fixed $\epsilon > 0$ there exist continuous functions $\overline{\gamma}_i$, $i = 1, 2$ such that
$\mathcal{L}^n\{w \in D : \gamma_i(t,w) \neq \overline{\gamma}_i(t,w), i = 1, 2\} < \epsilon$. From

$$X_{u_k} \to X_u \text{ in } \mathcal{C}([0, T]; L^1_{loc})$$

it follows almost everywhere convergence of the sequence $X_{u_k}$ and hence we may conclude that

$$\overline{\gamma}_i(\cdot, X_{u_k}) \to \overline{\gamma}_i(\cdot, X_u) \text{ a.e. in } [0, T] \times D.$$