Existence of global weak solutions to compressible isentropic finitely extensible nonlinear bead-spring chain models for dilute polymers

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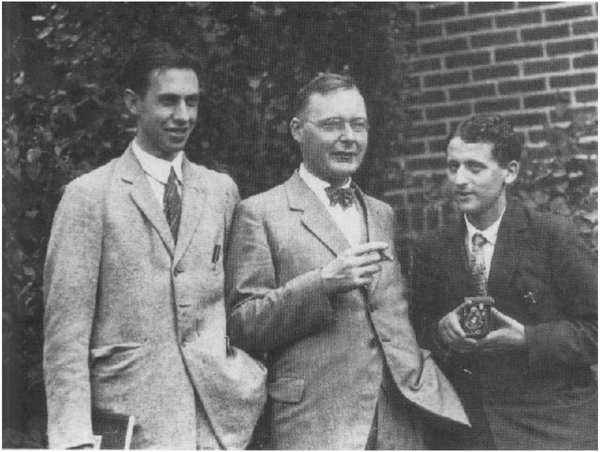

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Existence of global weak solutions to compressible isentropic finitely extensible nonlinear bead-spring chain models for dilute polymers

Part 1.

The mathematical model: kinetic theory of dilute polymers

Problem (P): The solvent is a compressible, isentropic, viscous, isothermal Newtonian fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and $T \in \mathbb{R}_{>0}$. Find:

$$\rho \,:\, (\underset{\sim}{x}, t) \in \Omega \times [0, T] \mapsto \rho(\underset{\sim}{x}, t) \in \mathbb{R},$$
$$\underset{\sim}{u} \,:\, (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0, T] \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d, \qquad \text{such that}$$

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Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho\, \underset{\sim}{u}) = \underset{\sim}{0} \qquad \text{in } \Omega \times (0, T],$$
$$\rho(\underset{\sim}{x}, 0) = \rho_0(\underset{\sim}{x}) \quad \forall \underset{\sim}{x} \in \Omega,$$

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Conservation of momentum (Navier–Stokes equation):

$$\frac{\partial(\rho\, \underset{\sim}{u})}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho\, \underset{\sim}{u} \otimes \underset{\sim}{u}) - \underset{\sim}{\nabla}_x \cdot \underset{\approx}{S}(\underset{\sim}{u}, \rho) + \underset{\sim}{\nabla}_x\, p(\rho) = \rho\, \underset{\sim}{f} \qquad \text{in } \Omega \times (0,T],$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0,T],$$

$$(\rho\, \underset{\sim}{u})(\underset{\sim}{x}, 0) = (\rho_0\, \underset{\sim}{u}_0)(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega.$$

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Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho \underset{\sim}{u}) = \underset{\sim}{0} \qquad \text{in } \Omega \times (0, T],$$
$$\rho(\underset{\sim}{x}, 0) = \rho_0(\underset{\sim}{x}) \quad \forall \underset{\sim}{x} \in \Omega,$$

Conservation of momentum (Navier–Stokes equation + elastic effects):

$$\frac{\partial (\rho \underset{\sim}{u})}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho \underset{\sim}{u} \otimes \underset{\sim}{u}) - \underset{\sim}{\nabla}_x \cdot \underset{\approx}{S}(\underset{\sim}{u}, \rho) + \underset{\sim}{\nabla}_x p(\rho) = \rho \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \quad \text{in } \Omega \times (0, T],$$
$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0, T],$$
$$(\rho \underset{\sim}{u})(\underset{\sim}{x}, 0) = (\rho_0 \underset{\sim}{u}_0)(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega.$$

ρ: nondimensional solvent density,

$\underset{\sim}{u}$: nondimensional solvent velocity,

$\underset{\approx}{S}(\underset{\sim}{u}, \rho)$ is the Newtonian part of the viscous stress tensor, defined by

$$\underset{\approx}{S}(\underset{\sim}{u}, \rho) := \mu^S(\rho) \, [\underset{\approx}{D}(\underset{\sim}{u}) - \frac{1}{d} \, (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}) \, \underset{\approx}{I}] + \mu^B(\rho) \, (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}) \, \underset{\approx}{I},$$

where $\underset{\approx}{I}$ is the $d \times d$ identity tensor and

$$\underset{\approx}{D}(\underset{\sim}{v}) := \frac{1}{2} \, (\underset{\approx}{\nabla}_x \, \underset{\sim}{v} + (\underset{\approx}{\nabla}_x \, \underset{\sim}{v})^{\mathrm{T}})$$

is the rate of strain tensor.

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is the rate of strain tensor.

The shear viscosity, $\mu^S(\cdot) \in \mathbb{R}_{>0}$, and the bulk viscosity, $\mu^B(\cdot) \in \mathbb{R}_{\geq 0}$, of the solvent are generally, density-dependent; here both will be assumed to be constant.

p is the nondimensional pressure satisfying the isentropic equation of state

$$p(\rho) = c_p \, \rho^{\gamma},$$

where $c_p \in \mathbb{R}_{>0}$, and the constant γ is such that $\gamma > \frac{3}{2}$.

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Remark

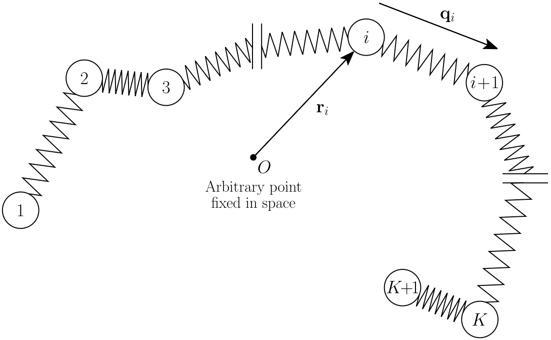
Our analysis applies, without alterations, to some other familiar equations of state, such as the (Kirkwood-modified) Tait equation of state

$$p(\rho) = A_0 \left(\frac{\rho}{\rho_*} \right)^\gamma - A_1,$$

where $\gamma > \frac{3}{2}$, A_0 and A_1 are constants, $A_0 - A_1 = p_*$ is the equilibrium reference pressure, and ρ_* is the equilibrium reference density.

For distilled water: $\gamma \in [5.16, 7.11]$ (depending on the ambient temperature); for glycerine (at $20°\mathrm{C}$) $\gamma = 9.80$; for carbon tetrachloride (at $30°\mathrm{C}$) $\gamma = 12.54$.

Definition of the elastic extra stress tensor $\underset{\approx}{\tau}$



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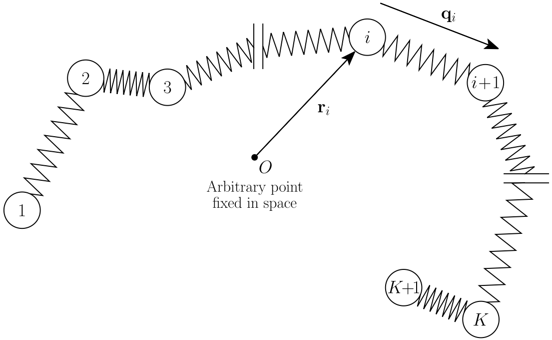

In the absence of external forces and neglecting inertial effects Langevin's equation for the i-th bead in this model is, for $i = 1, \ldots, K+1$:

$$0 = \underbrace{-\zeta \left(\mathrm{d}\underset{\sim}{r}_i - \underset{\sim}{u}(\underset{\sim}{r}_i, \cdot)\,\mathrm{d}t \right)}_{\text{Hydrodynamic drag force}} + \underbrace{\sum_{j=1}^{K} G_{ij}\, \underset{\sim}{F}_j(\underset{\sim}{q}_j)\,\mathrm{d}t}_{\text{Intramolecular force}} + \underbrace{\sqrt{2\,k_B\,\mathtt{T}\,\zeta}\;\mathrm{d}\underset{\sim}{W}_i}_{\text{Brownian force}}.$$

- $\zeta > 0$ is a characteristic *drag coefficient*;
- $\underset{\approx}{G} \in \mathbb{R}^{(K+1)\times K}$ is the *graph incidence matrix*:

$$\underset{\approx}{G} := \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix} \in \mathbb{R}^{(K+1)\times K},$$

where

$$G_{ij} := \begin{cases} +1 & \text{if spring } j \text{ starts from bead } i, \\ -1 & \text{if spring } j \text{ teminates in bead } i, \\ 0 & \text{otherwise;} \end{cases}$$

By defining

$$\underset{\sim}{Z}(t) := \begin{bmatrix} \underset{\sim}{x}_1(t) \\ \underset{\sim}{x}_2(t) \\ \vdots \\ \underset{\sim}{x}_{K+1}(t) \end{bmatrix}, \qquad \underset{\sim}{W}(t) := \begin{bmatrix} \underset{\sim}{W}_1(t) \\ \underset{\sim}{W}_2(t) \\ \vdots \\ \underset{\sim}{W}_{K+1}(t) \end{bmatrix}, \qquad \underset{\approx}{\sigma} := \sqrt{\frac{2k_B\mathtt{T}}{\zeta}}\, \underset{\approx}{I},$$

$$\underset{\sim}{b}(\underset{\sim}{Z}(t)) := \begin{bmatrix} \underset{\sim}{u}(\underset{\sim}{x}_1(t),t) \\ \underset{\sim}{u}(\underset{\sim}{x}_2(t),t) \\ \vdots \\ \underset{\sim}{u}(\underset{\sim}{x}_{K+1}(t),t) \end{bmatrix} + \zeta^{-1}\, \underset{\approx}{G} \begin{bmatrix} \underset{\sim}{F}_1(\underset{\sim}{x}_2(t) - \underset{\sim}{x}_1(t)) \\ \underset{\sim}{F}_2(\underset{\sim}{x}_3(t) - \underset{\sim}{x}_2(t)) \\ \vdots \\ \underset{\sim}{F}_K(\underset{\sim}{x}_{K+1}(t) - \underset{\sim}{x}_K(t)) \end{bmatrix},$$

we get the Itō stochastic differential equation (SDE):

$$\mathrm{d}\underset{\sim}{Z}(t) = \underset{\sim}{b}(\underset{\sim}{Z}(t))\,\mathrm{d}t + \underset{\approx}{\sigma}\,\mathrm{d}\underset{\sim}{W}(t), \qquad \underset{\sim}{Z}(0) = \underset{\sim}{Z},$$

for the $(K+1)d$-component vectorial random variable $\underset{\sim}{Z}(t)$, $t \in [0,T]$.

Theorem (Kolmogorov (1931))

Let the $(K+1)d$*-component vectorial random variable* $\underset{\sim}{Z}(t)$ *have a probability density function* $(\underset{\sim}{z},t) \mapsto \psi(\underset{\sim}{z},t)$ *in* $C^{2,1}(\mathbb{R}^{(K+1)d} \times [0,T])$*, and let* $\underset{\sim}{Z}(0) = \underset{\sim}{Z}$ *be a square-integrable random variable with probability density function* $\psi_0 \in C^2(\mathbb{R}^{(K+1)d})$*. Also, let* b *and* σ *in the above SDE be globally Lipschitz continuous, and* $c(\underset{\sim}{z}) := \sigma(\underset{\sim}{z})\,\sigma(\underset{\sim}{z})^{\mathrm{T}}$*.*

Then,

$$\frac{\partial \psi}{\partial t} + \sum_{j=1}^{(K+1)d} \frac{\partial}{\partial z_j}(b_j \psi) = \frac{1}{2} \sum_{i,j=1}^{(K+1)d} \frac{\partial^2}{\partial z_i\, \partial z_j}(c_{ij}\psi),$$

in $\mathbb{R}^{(K+1)d} \times [0,\infty)$ *where* $\psi(\underset{\sim}{z},0) = \psi_0(\underset{\sim}{z})$*,* $\underset{\sim}{z} = (\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_{K+1}) \in \mathbb{R}^{(K+1)d}$*.*

Thus, after nondimensionalization and the linear change of variables:

$$\underset{\sim}{x} := \frac{1}{K+1}\sum_{i=1}^{K+1} \underset{\sim}{r}_i, \qquad \underset{\sim}{q}_i := \underset{\sim}{r}_{i+1} - \underset{\sim}{r}_i, \quad i = 1, \ldots, K,$$

we arrive at the Fokker–Planck equation:

$$\frac{\partial \psi}{\partial t} + \nabla_x \cdot (\underset{\sim}{u}\,\psi) + \sum_{i=1}^{K} \underset{\sim}{\nabla}_{q_i} \cdot \left((\underset{\sim}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}_i\, \psi - \frac{1}{4\lambda}\sum_{j=1}^{K} A_{ij}\, \underset{\sim}{F}_i(\underset{\sim}{q}_j)\psi \right)$$

$$= \varepsilon\, \Delta_x \psi + \frac{1}{4\,\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij} \underset{\sim}{\nabla}_{q_i} \cdot \left(\underset{\sim}{\nabla}_{q_j}\, \psi\right).$$

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$\varepsilon := \frac{1}{4\,\lambda\,(K+1)}\left(\frac{\ell_0}{L_0}\right)^2$ is the centre-of-mass diffusion coefficient;

$\lambda := (\zeta/4\mathtt{H})(U_0/L_0) = \mathrm{De}$ is the Deborah number;

$\underset{\sim}{F}_i(\underset{\sim}{q}_i) = U_i'(\frac{1}{2}|\underset{\sim}{q}_i|^2)\underset{\sim}{q}_i$, $i = 1, \dots, K$: nondimensional spring forces;

$A := G^{\mathrm{T}} G \in \mathbb{R}^{K\times K}_{\text{symm}}$: Rouse matrix.

The Maxwellian

The normalized (partial) Maxwellian M_i is:

$$M_i(\underset{\sim}{q}_i) := \frac{\mathrm{e}^{-U_i(\frac{1}{2}|\underset{\sim}{q}_i|^2)}}{\int_{D_i} \mathrm{e}^{-U_i(\frac{1}{2}|\underset{\sim}{q}_i|^2)}\,\mathrm{d}\underset{\sim}{q}_i}, \qquad i = 1, \dots, K.$$

The Maxwellian in the model is then defined by

$$M(\underset{\sim}{q}) := \prod_{i=1}^{K} M_i(\underset{\sim}{q}_i) \qquad \forall \underset{\sim}{q} := (\underset{\sim}{q}_1, \dots, \underset{\sim}{q}_K) \in D := \bigtimes_{i=1}^{K} D_i.$$

$$D_i := \{\underset{\sim}{q}_i \in \mathbb{R}^d : |\underset{\sim}{q}_i|^2 < b_i\}, \qquad b_i \in (0, \infty).$$

The U_i are finitely extensible nonlinear elastic (FENE) type potentials:

$$U_i(\tfrac{1}{2}|\underset{\sim}{q}_i|^2) \to +\infty \quad \text{as} \quad |\underset{\sim}{q}_i|^2 \to b_i.$$

Letting

$$\widehat{\psi} := \frac{\psi}{M}$$

gives the following, transformed,

Fokker–Planck equation:

$$\frac{\partial}{\partial t}(M\,\widehat{\psi}) + \nabla_x \cdot (\underset{\sim}{u}\,M\,\widehat{\psi}) + \sum_{i=1}^{K} \underset{\sim}{\nabla}_{q_i} \cdot \Big((\underset{\sim}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}_i\,M\,\widehat{\psi}\Big)$$

$$= \varepsilon\,\Delta_x M\,\widehat{\psi} + \frac{1}{4\,\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}\underset{\sim}{\nabla}_{q_i} \cdot \Big(M\,\underset{\sim}{\nabla}_{q_j}\,\widehat{\psi}\Big) \text{ on } \Omega \times D \times (0,T].$$

degenerate parabolic PDE; ∂D Feller-natural boundary

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degenerate parabolic PDE; ∂D Feller-natural boundary

The elastic extra-stress tensor $\underset{\approx}{\tau}$ is defined by

$$\underset{\approx}{\tau}(\psi)(\underset{\sim}{x},t) := \underset{\approx}{\tau}_1(\psi)(\underset{\sim}{x},t) - \left(\int_{D\times D} \gamma(\underset{\sim}{q},\underset{\sim}{q}')\,\psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\psi(\underset{\sim}{x},\underset{\sim}{q}',t)\;\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{q}' \right) \underset{\approx}{I}.$$

Here, $\gamma : D \times D \to \mathbb{R}_{\geq 0}$ is a smooth, time-independent, $\underset{\sim}{x}$-independent and ψ-independent interaction kernel, which we shall henceforth consider to be

$$\gamma(\underset{\sim}{q}, \underset{\sim}{q}') \equiv \mathfrak{z},$$

where $\mathfrak{z} \in \mathbb{R}_{>0}$, and

Kramers' expression (1944):

$$\underset{\approx}{\tau}_1(\psi)(\underset{\sim}{x}, t) := k\Bigg(\underbrace{\sum_{i=1}^{K} \int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t) \underset{\sim}{q}_i\, \underset{\sim}{q}_i^{\mathrm{T}} U_i' \left(\tfrac{1}{2}|\underset{\sim}{q}_i|^2\right) \mathrm{d}\underset{\sim}{q}}_{=:\underset{\approx}{C}(\psi)} - (K+1)\underset{\approx}{I} \underbrace{\int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q}}_{=:\varrho} \Bigg)$$

Here, $\gamma : D \times D \to \mathbb{R}_{\geq 0}$ is a smooth, time-independent, $\underset{\sim}{x}$-independent and ψ-independent interaction kernel, which we shall henceforth consider to be

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Remark

We thus have the decomposition of the Cauchy stress $\underset{\approx}{\pi}$ as the sum of a contribution from the solvent, $\underset{\approx}{\pi}_s$, and the polymeric extra stress, $\underset{\approx}{\pi}_p$:

$$\underset{\approx}{\pi} \quad = \quad \underset{\approx}{\pi}_s + \underset{\approx}{\pi}_p = \left(\underset{\approx}{S}(\underset{\sim}{u}, \rho) - p_s\, \underset{\approx}{I}\right) + \left(k\, \underset{\approx}{C}(\psi) - p_p\, \underset{\approx}{I}\right),$$

where $p_s = p = c_p\, \rho^\gamma$ is the fluid pressure, and $p_p = k\,(K+1)\,\varrho + \mathfrak{z}\,\varrho^2$ is the polymeric contribution to the total pressure, defined as $p_s + p_p$.

Part 2.

Mathematical analysis of the model: existence of global weak solutions

Assumptions

$$\partial\Omega \in C^{2,\theta}, \quad \theta \in (0,1), \qquad \rho_0 \in L^\infty_{\geq 0}(\Omega), \qquad \underset{\sim}{u}_0 \in \underset{\sim}{L}^2(\Omega);$$

$$\mu^S \in \mathbb{R}_{>0},\ \mu^B \in \mathbb{R}_{\geq 0}, \qquad \underset{\sim}{f} \in L^2(0,T;\underset{\sim}{L}^\infty(\Omega));$$

$\psi_0 \geq 0$ a.e. on $\Omega \times D$ with $\int_D \psi_0(\cdot, \underset{\sim}{q})\ \mathrm{d}\underset{\sim}{q} \in L^\infty_{\geq 0}(\Omega)$;

$\mathcal{F}(\widehat{\psi}_0) \in L^1_M(\Omega \times D)$ where $\mathcal{F}(s) = s(\log s - 1) + 1, \quad s \in \mathbb{R}_{\geq 0}$;

the Rouse matrix $\underset{\approx}{A} \in \mathbb{R}^{K\times K}_{\mathrm{symm}}$ and there exists $a_0 > 0$ s.t. $\underset{\approx}{A} \geq a_0\, \underset{\approx}{I}$;

there exist constants $\gamma_i > 1$, $i = 1, \ldots, K$, s.t.:

$$M_i(\underset{\sim}{q}_i) \asymp [\mathsf{dist}(\underset{\sim}{q}_i,\, \partial D_i)]^{\gamma_i} \qquad \text{as } \underset{\sim}{q}_i \to \partial D_i,$$

where

$$D_i = B(\underset{\sim}{0}, b_i^{\frac{1}{2}}), \quad b_i > 0, \quad i = 1, \ldots, K.$$

Theorem (Existence of a global weak solution to (P))

There exist functions $(\rho, \underset{\sim}{u}, \widehat{\psi})$, *such that*

$$\begin{aligned} &\rho \in C_w([0,T]; L^{\gamma}_{\geq 0}(\Omega)) \cap H^1(0,T; W^{1,6}(\Omega)'), \\ &\underset{\sim}{u} \in L^2(0,T; \underset{\sim}{H}^1_0(\Omega)), \\ &\widehat{\psi} := \tfrac{\psi}{M} \in L^{\upsilon}(0,T; L^1_{\geq 0,M}(\Omega \times D)) \cap H^1(0,T; M^{-1}(H^s(\Omega \times D))'), \end{aligned}$$

for any $\gamma > \frac{3}{2}$, $\upsilon \in [1, \infty)$ *and* $s > 1 + \frac{1}{2}(K+1)d$,

$$\mathcal{F}(\widehat{\psi}) \in L^{\infty}(0,T; L^1_M(\Omega \times D)) \qquad \textit{and} \qquad \sqrt{\widehat{\psi}} \in L^2(0,T; H^1_M(\Omega \times D)),$$

$$\underset{\approx}{\tau}(M\,\widehat{\psi}) \in \underset{\approx}{L}^r(\Omega_T) \qquad \textit{for any } r \in \left[1, \tfrac{4(d+2)}{3d+4}\right);$$

and,

$$\varrho := \int_D M\,\widehat{\psi}\,\mathrm{d}\underset{\sim}{q} \in L^{\infty}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)),$$

Theorem (continued)

$$\rho \underset{\sim}{u} \in C_w([0,T]; \underset{\sim}{L}^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^2(0,T; \underset{\sim}{H}^1(\Omega)') \cap L^2(0,T; \underset{\sim}{L}^{\frac{6\gamma}{\gamma+6}}(\Omega)),$$
$$\rho \underset{\sim}{u} \otimes \underset{\sim}{u} \in L^2(0,T; \underset{\approx}{L}^{\frac{6\gamma}{4\gamma+3}}(\Omega)),$$

letting $\Gamma := \max(\gamma, 8)$, $s := \max\{4, \frac{6\gamma}{2\gamma-3}\}$, $r := \max\{s, \frac{\Gamma+\vartheta}{\vartheta}\}$,

$$\vartheta(\gamma) := \begin{cases} \frac{2\gamma-3}{3} & \textit{for } \frac{3}{2} < \gamma \leq 4, \\ \frac{5}{12}\gamma & \textit{for } 4 \leq \gamma, \end{cases}$$

and letting $\Omega_T := \Omega \times (0,T]$,

$$\rho \underset{\sim}{u} \in \underset{\sim}{L}^{\frac{10\gamma-6}{3(\gamma+1)}}(\Omega_T) \cap W^{1,\frac{\Gamma+\vartheta}{\Gamma}}(0,T; \underset{\sim}{W}_0^{1,r}(\Omega)'),$$
$$\rho \in L^{\gamma+\vartheta}(\Omega_T),$$
$$\rho^\gamma \in L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T),$$

such that,

Theorem (continued)

$(\rho, \underset{\sim}{u}, \widehat{\psi})$ *is a global weak solution to the problem, in the sense that*

$$\int_0^T \left\langle \frac{\partial \rho}{\partial t}, \eta \right\rangle_{W^{1,6}(\Omega)} \mathrm{d}t - \int_0^T \int_\Omega \rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x \eta \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t = 0 \quad \forall \eta \in L^2(0,T; W^{1,6}(\Omega)),$$

with $\rho(\cdot, 0) = \rho_0(\cdot)$,

$$\int_0^T \left\langle \frac{\partial(\rho \underset{\sim}{u})}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,r}(\Omega)} \mathrm{d}t + \int_0^T \int_\Omega \left[\underset{\approx}{S}(\underset{\sim}{u}) - \rho \underset{\sim}{u} \otimes \underset{\sim}{u} - c_p \rho^\gamma \underset{\approx}{I} \right] : \underset{\approx}{\nabla}_x \underset{\sim}{w} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \left[\rho \underset{\sim}{f} \cdot \underset{\sim}{w} - \left(\underset{\approx}{\tau}_1(M \widehat{\psi}) - \mathfrak{z} \varrho^2 \underset{\approx}{I} \right) : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$\forall \underset{\sim}{w} \in L^{\frac{\gamma+\vartheta}{\vartheta}}(0,T; \underset{\sim}{W}_0^{1,r}(\Omega)),$$

with $(\rho \underset{\sim}{u})(\cdot, 0) = (\rho_0 \underset{\sim}{u}_0)(\cdot)$, *and*

Theorem (continued)

$$\int_0^T \left\langle M \frac{\partial \widehat{\psi}}{\partial t}, \varphi \right\rangle_{H^s(\Omega\times D)} \mathrm{d}t + \frac{1}{4\lambda} \sum_{i,j=1}^{K} A_{ij} \int_0^T \int_{\Omega\times D} M \, \underset{\sim}{\nabla}_{q_j} \widehat{\psi} \cdot \underset{\sim}{\nabla}_{q_i} \varphi \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega\times D} M \left[\varepsilon \, \underset{\sim}{\nabla}_x \widehat{\psi} - \underset{\sim}{u} \, \widehat{\psi} \right] \cdot \underset{\sim}{\nabla}_x \varphi \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$- \int_0^T \int_{\Omega\times D} M \sum_{i=1}^{K} \left[\underset{\approx}{\sigma}(\underset{\sim}{u}) \, \underset{\sim}{q}_i \right] \widehat{\psi} \cdot \underset{\sim}{\nabla}_{q_i} \varphi \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t = 0$$

$$\forall \varphi \in L^2(0,T; H^s(\Omega \times D)),$$

with $\widehat{\psi}(\cdot, 0) = \widehat{\psi}_0(\cdot)$ *and* $s > 1 + \frac{1}{2}(K+1)d$.

Theorem (continued)

In addition, the following energy inequality holds for a.e. $t \in [0, T]$*:*

$$
\begin{aligned}
&\tfrac{1}{2} \int_\Omega \rho(t) \, |\underset{\sim}{u}(t)|^2 \, \mathrm{d}\underset{\sim}{x} + \int_\Omega P(\rho(t)) \, \mathrm{d}\underset{\sim}{x} + k \int_{\Omega \times D} M \, \mathcal{F}(\widehat{\psi}(t)) \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \\
&\quad + \mu^S c_0 \int_0^t \|\underset{\sim}{u}\|^2_{H^1(\Omega)} \, \mathrm{d}s + k \int_0^t \int_{\Omega \times D} M \left[\frac{a_0}{2\lambda} \left| \underset{\sim}{\nabla}_q \sqrt{\widehat{\psi}} \right|^2 + 2\varepsilon \left| \underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}} \right|^2 \right] \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}s \\
&\quad + \mathfrak{z} \, \|\varrho(t)\|^2_{L^2(\Omega)} + 2 \, \mathfrak{z} \, \varepsilon \int_0^t \|\underset{\sim}{\nabla}_x \, \varrho\|^2_{L^2(\Omega)} \, \mathrm{d}s \\
&\leq \mathrm{e}^t \Bigg[\tfrac{1}{2} \int_\Omega \rho_0 \, |\underset{\sim}{u}_0|^2 \, \mathrm{d}\underset{\sim}{x} + \int_\Omega P(\rho_0) \, \mathrm{d}\underset{\sim}{x} + k \int_{\Omega \times D} M \, \mathcal{F}(\widehat{\psi}_0) \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \\
&\qquad\qquad + \mathfrak{z} \int_\Omega \left(\int_D M \, \widehat{\psi}_0 \, \mathrm{d}\underset{\sim}{q} \right)^2 \mathrm{d}\underset{\sim}{x} + \tfrac{1}{2} \int_0^s \|\underset{\sim}{f}\|^2_{L^\infty(\Omega)} \, \mathrm{d}t \int_\Omega \rho_0 \, \mathrm{d}\underset{\sim}{x} \Bigg],
\end{aligned}
$$

with

$$
P(\rho) := \frac{p(\rho)}{\gamma - 1}, \quad k := k_B \, \mathtt{T}, \quad \mathfrak{z} > 0.
$$

Key to the proof of the existence of global weak solutions

Formal energy identity:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\rho\,|\underset{\sim}{u}|^2 + P(\rho) + \mathfrak{z}\,\varrho^2 + k\int_D M\,\mathcal{F}\left(\frac{\psi}{M}\right)\,\mathrm{d}\underset{\sim}{q}\right]\mathrm{d}\underset{\sim}{x}\\
&\quad + \mu^S\int_\Omega \left|\underset{\approx}{D}(\underset{\sim}{u}) - \frac{1}{d}\,(\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u})\,\underset{\approx}{I}\right|^2\mathrm{d}\underset{\sim}{x} + \mu^B\int_\Omega |\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x}\\
&\quad + 2\varepsilon\,\mathfrak{z}\int_\Omega |\underset{\sim}{\nabla}_x\,\varrho|^2\,\mathrm{d}\underset{\sim}{x} + \varepsilon\,k\int_{\Omega\times D} M\left|\underset{\sim}{\nabla}_x\left(\frac{\psi}{M}\right)\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&\quad + \frac{k}{4\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_{q_j}\left(\frac{\psi}{M}\right)\cdot\underset{\sim}{\nabla}_{q_i}\left(\frac{\psi}{M}\right)\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&= \int_\Omega \rho\,\underset{\sim}{f}\cdot\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x}, \qquad \text{for all } t\in(0,T].
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\end{aligned}
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Idea: construct an approximating sequence obeying an energy inequality

Key to the proof of the existence of global weak solutions

Formal energy identity:

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\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\rho\,|\underset{\sim}{u}|^2 + P(\rho) + \mathfrak{z}\,\varrho^2 + k\int_D M\,\mathcal{F}\left(\frac{\psi}{M}\right)\,\mathrm{d}\underset{\sim}{q}\right]\mathrm{d}\underset{\sim}{x} \\
&\quad + \mu^S\int_\Omega \left|\underset{\approx}{D}(\underset{\sim}{u}) - \frac{1}{d}\,(\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u})\,\underset{\approx}{I}\right|^2\mathrm{d}\underset{\sim}{x} + \mu^B\int_\Omega |\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x} \\
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Idea: construct an approximating sequence obeying an energy inequality

$\longrightarrow$ Energy inequality yields weak convergence of the approximating sequence

Key to the proof of the existence of global weak solutions

Formal energy identity:

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&\quad + \mu^S\int_\Omega \left|\underset{\approx}{D}(\underset{\sim}{u}) - \frac{1}{d}\,(\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u})\,\underset{\approx}{I}\right|^2 \mathrm{d}\underset{\sim}{x} + \mu^B\int_\Omega |\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x} \\
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&\quad = \int_\Omega \rho\,\underset{\sim}{f}\cdot\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x}, \qquad \text{for all } t\in(0,T].
\end{aligned}
$$

Idea: construct an approximating sequence obeying an energy inequality
$\longrightarrow$ Energy inequality yields weak convergence of the approximating sequence
$\longrightarrow$ Most difficult step: passage to limit in nonlinear terms requires strong convergence

weak convergence $\longrightarrow$ strong convergence

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- incompressible NS with variable ρ, $\mu(\rho)$
  Aubin–Lions–Simon compactness theorem works:
  Antontsev, Kazhikhov & Monakhov (1990), Simon (1990), P.-L. Lions (1996).

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  neither Aubin–Lions–Simon nor Dubinskiĭ's theorem works.

A much more complicated argument had to be used:

- entropy estimates, together with
- Vitali's theorem,
- Feireisl & Novotný theorem on weak lower-semicontinuity of convex functions,
- a compensated compactness argument based on the Div-Curl lemma,
- interior estimates based on function space interpolation.
  M. Bulíček, J. Málek & E. Süli (Communications in PDEs, 2013);
  Barrett & Süli (J. Diff. Eqs., 2012).

The proof

The proof ... is long and extremely technical.

Perform several regularizations, discretize in time the resulting regularized system of PDEs, show existence of solutions to the discretized regularized system of PDEs, and then pass to the natural limits with the regularization parameters and the temporal discretization parameter, with the aim to show that the limit is a weak solution, that satisfies the energy inequality.

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- 1st regularization (cut-off in the Fokker–Planck equation): introduce cut-off in the transport terms in FP by the cut-off function:

$$\widehat{\psi} \rightsquigarrow M\beta^{L}\left(\frac{\widehat{\psi}}{M}\right) \quad \text{where} \quad \beta^{L}(s) := \min(s, L), \quad L > 1.$$

- 2nd regularization (continuity eq. regularization): add to the continuity equation the term

$$-\alpha\Delta\rho, \qquad \alpha > 0.$$

- 3rd regularization (pressure regularization):

$$p_{\kappa}(\rho) := p(\rho) + \kappa(\rho^{4} + \rho^{\Gamma}), \qquad \kappa > 0, \quad \Gamma := \max\{\gamma, 8\}.$$

STEP 1. [Problem $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L})$]
Discretize in time the (κ, α, L)-regularized system using a time step Δt while retaining the cancellation between the regularized continuity, momentum and FP equations resulting in a crucial energy inequality.

Define an upper-truncated entropy $\mathcal{F}^L$ using the cut-off parameter $L > 1$ as we need to cut off $\widehat{\psi}^{\Delta t}_{\kappa,\alpha,L}$ from above in the transport term.

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STEP 2. [Problem $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L,\delta})$]
Since the solution of the resulting problem is not regular enough to deduce the necessary energy inequality, a sixth-order hyperviscosity term is added, in weak form, to the momentum equation, with coefficient $\delta \in (0, 1)$.

In the argument that follows we need to truncate the upper-truncated entropy $\mathcal{F}^L$ from below also, using a $\delta \in (0, 1)$ as a lower cut-off parameter; call the resulting upper-lower truncated entropy function $\mathcal{F}^L_\delta$.

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STEP 3.
We use *Schaefer's fixed point theorem* to show that the nonlinear system resulting at each time step in $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L,\delta})$ has a solution.

STEP 4.

We pass to the limit $\delta \to 0$ in problem $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L,\delta})$ using the available bounds on the solution, to deduce the existence of a solution to $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L})$.

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We wish to pass to the limits $\Delta t \to 0_+$ and $L \to +\infty$. To this end,...

we require bounds, independent of L and Δt, on $\rho^{\Delta t}_{\kappa,\alpha,L}$, $\underset{\sim}{u}^{\Delta t}_{\kappa,\alpha,L}$, $\widehat{\psi}^{\Delta t}_{\kappa,\alpha,L}$.

We obtain these by passing to the limit $\delta \to 0$ in the available norm bounds on the solution of $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L,\delta})$ obtained by testing based on the entropy $\mathcal{F}^L_\delta$.

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We obtain these by passing to the limit $\delta \to 0$ in the available norm bounds on the solution of $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L,\delta})$ obtained by testing based on the entropy $\mathcal{F}^L_\delta$.

Hence, from the time-discrete equations we derive L and Δt independent bounds on the temporal difference quotients of $\rho^{\Delta t}_{\kappa,\alpha,L}$, $\underset{\sim}{u}^{\Delta t}_{\kappa,\alpha,L}$, $\widehat{\psi}^{\Delta t}_{\kappa,\alpha,L}$.

Passage to the limit requires linking Δt to L:

$$\Delta t = o(L^{-1}), \qquad \text{with } \Delta t \to 0 \text{ (as } L \to \infty).$$

STEP 6.

We use *Vitali's theorem* to deduce strong convergence of the sequence $\rho^{\Delta t}_{\kappa,\alpha,L}$ as $\Delta t \to 0$ (and $L \to \infty$).

We use *Simon's compactness theorem* to deduce strong convergence of the sequence $\underset{\sim}{u}^{\Delta t}_{\kappa,\alpha,L}$ as $\Delta t \to 0$ (and $L \to \infty$).

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STEP 7.

We prove strong converge of $\underset{\approx}{\tau}(M\widehat{\psi}^{\Delta t}_{\kappa,\alpha,L})$ in $L^1(\Omega_T)$.

Using this and boundedness of $\underset{\approx}{\tau}(M\widehat{\psi}^{\Delta t}_{\kappa,\alpha,L})$ in $\underset{\approx}{L}^{\frac{4(d+2)}{3d+4}}(\Omega_T)$, we deduce strong convergence in $\underset{\approx}{L}^r(\Omega_T)$ for $r \in [1, \frac{4(d+2)}{3d+4})$.

STEP 8. [Level 1 passage to the limit]

We pass to the limit $\Delta t \to 0$ (and $L \to \infty$) in $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L})$ to deduce the existence of a solution to $(\mathrm{P}_{\kappa,\alpha})$, and we pass to the same limit in the energy inequality satisfied by the solution of $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L})$ to deduce the associated energy inequality for the solution of $(\mathrm{P}_{\kappa,\alpha})$.

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We pass to the limit $\Delta t \to 0$ (and $L \to \infty$) in $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L})$ to deduce the existence of a solution to $(\mathrm{P}_{\kappa,\alpha})$, and we pass to the same limit in the energy inequality satisfied by the solution of $(\mathrm{P}^{\Delta t}_{\kappa,\alpha,L})$ to deduce the associated energy inequality for the solution of $(\mathrm{P}_{\kappa,\alpha})$.

STEP 9. [Notation: $\in_\alpha :=$ belongs to, and is bdd w.r.t. α]

P.-L. Lions (1998), E. Feireisl (2002, 2004), A. Novotný & I. Straškraba (2004)

$$\rho_{\kappa,\alpha} \in_\alpha L^\infty(0,T;L^\Gamma(\Omega)) \cap H^1(0,T;H^1(\Omega)'), \qquad \Gamma := \max(\gamma, 8),$$

$$\underset{\sim}{u}_{\kappa,\alpha} \in_\alpha L^2(0,T;\underset{\sim}{H}^1_0(\Omega)),$$

$$\rho_{\kappa,\alpha}\,\underset{\sim}{u}_{\kappa,\alpha} \in_\alpha L^\infty(0,T;\underset{\sim}{L}^{\frac{2\Gamma}{\Gamma+1}}(\Omega)) \cap L^2(0,T;\underset{\sim}{L}^{\frac{6\Gamma}{\Gamma+6}}(\Omega))$$

$$\cap\ \underset{\sim}{L}^{\frac{10\Gamma-6}{3(\Gamma+1)}}(\Omega_T) \cap W^{1,\frac{8\Gamma-12}{7\Gamma-6}}(0,T;\underset{\sim}{W}^{1,4}_0(\Omega)'),$$

$$\rho_{\kappa,\alpha}\,|\underset{\sim}{u}_{\kappa,\alpha}|^2 \in_\alpha L^\infty(0,T;\underset{\sim}{L}^1(\Omega)) \cap L^2(0,T;\underset{\sim}{L}^{\frac{6\Gamma}{4\Gamma+3}}(\Omega)).$$

STEP 10.

$$\widehat{\psi}_{\kappa,\alpha} \in_\alpha L^{\upsilon}(0,T;L^1_{\geq 0,M}(\Omega \times D)) \cap H^1(0,T;M^{-1}(H^s(\Omega \times D))'),$$

where $\upsilon \in [1,\infty)$ and $s > 1 + \frac{1}{2}(K+1)d$, with

$$\mathcal{F}(\widehat{\psi}_{\kappa,\alpha}) \in_\alpha L^\infty(0,T;L^1_M(\Omega \times D)) \quad \text{and} \quad \sqrt{\widehat{\psi}_{\kappa,\alpha}} \in_\alpha L^2(0,T;H^1_M(\Omega \times D)).$$

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STEP 11.

In addition

$$\varrho_{\kappa,\alpha} := \int_D M\, \widehat{\psi}_{\kappa,\alpha} \, \mathrm{d}\underset{\sim}{q} \in_\alpha L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

$$\varrho_{\kappa,\alpha} \in_\alpha L^{\frac{2(d+2)}{d}}(\Omega_T) \cap L^4(0,T;L^{\frac{2d}{d-1}}(\Omega)).$$

Hence,

$$\underset{\approx}{\tau}(M\, \widehat{\psi}_{\kappa,\alpha}) \in_\alpha L^2(0,T;L^{\frac{4}{3}}(\Omega)) \cap L^{\frac{4(d+2)}{3d+4}}(\Omega_T).$$

STEP 12.

From the NS momentum equation:

$$\left| \int_0^T \left\langle \frac{\partial(\rho_{\kappa,\alpha}\, \underset{\sim}{u}_{\kappa,\alpha})}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,4}(\Omega)} \mathrm{d}t - \int_0^T \int_\Omega \underset{\sim}{p}_\kappa(\rho_{\kappa,\alpha})\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w}\, \mathrm{d}\underset{\sim}{x}\ \mathrm{d}t \right|$$

$$\leq C\, \|\underset{\sim}{w}\|_{L^{s'}(0,T;W^{1,4}(\Omega))} \qquad \forall \underset{\sim}{w} \in L^{s'}(0,T;\underset{\sim}{W}_0^{1,4}(\Omega)), \quad s' := \frac{5\Gamma - 3}{\Gamma - 3}.$$

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Consider the Bogovskiĭ operator $\underset{\sim}{\mathcal{B}} : L_0^r(\Omega) \to \underset{\sim}{W}_0^{1,r}(\Omega)$, $r \in (1, \infty)$, s.t.:

$$\int_\Omega \left(\nabla_x \cdot \underset{\sim}{\mathcal{B}}(\zeta) - \zeta \right) \eta \, \mathrm{d}\underset{\sim}{x} = 0 \qquad \forall \eta \in L^{\frac{r}{r-1}}(\Omega),$$

which satisfies

$$\|\underset{\sim}{\mathcal{B}}(\zeta)\|_{W^{1,r}(\Omega)} \leq C\, \|\zeta\|_{L^r(\Omega)} \qquad \forall \zeta \in L_0^r(\Omega),$$
$$\|\underset{\sim}{\mathcal{B}}(\underset{\sim}{\nabla}_x \cdot \underset{\sim}{w})\|_{L^r(\Omega)} \leq C\, \|\underset{\sim}{w}\|_{L^r(\Omega)} \qquad \forall \underset{\sim}{w} \in \underset{\sim}{L}^r(\Omega),\ \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} \in L^s(\Omega).$$

STEP 13.

Take $\underset{\sim}{w} = \eta \, \underset{\sim}{\mathcal{B}}((I - \fint)\rho_{\kappa,\alpha})$, $\eta \in C_0^\infty(0,T)$, in the inequality in STEP 12.

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Using the bounds from STEP 9 $\rightarrow$ STEP 11:

$$\left| \int_0^T \eta \int_\Omega \left(c_p \, \rho_{\kappa,\alpha}^{\gamma+1} + \kappa \left(\rho_{\kappa,\alpha}^5 + \rho_{\kappa,\alpha}^{\Gamma+1} \right) \right) \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t \right| \leq C \, \left[\|\eta\|_{L^\infty(0,T)} + \left\| \frac{\mathrm{d}\eta}{\mathrm{d}t} \right\|_{L^1(0,T)} \right].$$

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We take $\eta = \eta_m \in C_0^\infty(0,T)$, $m \in \mathbb{N}$, where $\eta_m \in [0,1]$ with $\eta_m(t) = 1$ for $t \in [\frac{1}{m}, T - \frac{1}{m}]$ and $\|\frac{\mathrm{d}\eta_m}{\mathrm{d}t}\|_{L^\infty(0,T)} \leq 2m$ yielding $\|\frac{\mathrm{d}\eta_m}{\mathrm{d}t}\|_{L^1(0,T)} \leq 4$.

As $\eta_m \rightarrow 1$ pointwise in $(0,T)$, as $m \rightarrow \infty$, we obtain the crucial bound

$$\rho_{\kappa,\alpha} \in_\alpha L^{\Gamma+1}(\Omega_T).$$

STEP 14.

With this information we return to the inequality in STEP 12, noting that $4 < s' < \Gamma + 1$ to deduce that

$$\left| \int_0^T \left\langle \frac{\partial(\rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha})}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} \mathrm{d}t \right| \leq C \, \|\underset{\sim}{w}\|_{L^{\Gamma+1}(0,T;W^{1,\Gamma+1}(\Omega))}$$

for all $\underset{\sim}{w} \in L^{\Gamma+1}(0,T;\underset{\sim}{W}_0^{1,\Gamma+1}(\Omega))$, and hence

$$\frac{\partial(\rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha})}{\partial t} \in_\alpha L^{\frac{\Gamma+1}{\Gamma}}(0,T;W_0^{1,\Gamma+1}(\Omega)').$$

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$$\frac{\partial(\rho_{\kappa,\alpha}\underset{\sim}{u}_{\kappa,\alpha})}{\partial t} \in_\alpha L^{\frac{\Gamma+1}{\Gamma}}(0,T;W_0^{1,\Gamma+1}(\Omega)').$$

STEP 15. [Level 2 passage to the limit]

With the information from STEP 9 $\rightarrow$ STEP 11, STEP 13, STEP 14, we pass to the limit $\alpha \rightarrow 0$ in $(\mathrm{P}_{\kappa,\alpha})$ to deduce the existence of a solution to (P_{κ}), and we pass to the same limit in the energy inequality satisfied by the solution of $(\mathrm{P}_{\kappa,\alpha})$ to deduce the associated energy inequality for the solution of (P_{κ}).

STEP 16.

Unfortunately, all that can be claimed at this point is that

$$p_\kappa(\rho_{\kappa,\alpha}) \rightharpoonup \overline{p_\kappa(\rho_{\kappa,\alpha})} \qquad \text{in } L^{\frac{\Gamma+1}{\Gamma}}(\Omega_T) \text{ as } \alpha \to 0.$$

It is yet to be proved that $\overline{p_\kappa(\rho_{\kappa,\alpha})} = p_\kappa(\rho_\kappa)$.

In the case of the compressible NS system this can be proved by using P.-L. Lions' effective viscous flux technique. The presence of the extra stress term on the r.h.s. of the momentum equation here complicates the analysis.

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Nevertheless, it is possible to show that, as $\alpha \to 0$,

$$\begin{aligned} \rho_{\kappa,\alpha} &\to \rho_\kappa && \text{strongly in } L^r(\Omega_T), && \text{for any } r \in [1, \Gamma+1), \\ p_\kappa(\rho_{\kappa,\alpha}) &\to p_\kappa(\rho_\kappa) && \text{weakly in } L^{\frac{\Gamma+1}{\Gamma}}(\Omega_T), && \text{that is, } \overline{p_\kappa(\rho_{\kappa,\alpha})} = p_\kappa(\rho_\kappa). \end{aligned}$$

STEP 17. [Notation: $\in_\kappa :=$ belongs to, and is bdd w.r.t. κ]

$$\widehat{\psi}_\kappa \in_\kappa L^\infty(0,T;L^1_{\geq 0,M}(\Omega\times D)) \cap H^1(0,T;M^{-1}(H^s(\Omega\times D))'),$$

$$\widehat{\psi}_\kappa \in_\kappa L^\infty(0,T;L^1_M(\Omega\times D)), \qquad \sqrt{\widehat{\psi}_\kappa} \in_\kappa L^2(0,T;H^1_M(\Omega\times D)),$$

$$\varrho_\kappa \in_\kappa L^\infty(0,T;L^2(\Omega)) \cap L^{\frac{2(d+2)}{d}}(\Omega_T) \cap L^2(0,T;L^6(\Omega)) \cap L^4(0,T;L^{\frac{2d}{d-1}}(\Omega)).$$

Hence,

$$\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa) \in_\kappa L^2(0,T;L^{\frac{4}{3}}(\Omega)) \cap L^{\frac{4(d+2)}{3d+4}}(\Omega_T) \cap L^{\frac{4}{3}}(0,T;L^{\frac{12}{7}}(\Omega)).$$

Further, for any $\Gamma = \max(\gamma, 8)$ and $\gamma > \frac{3}{2}$,

$$\rho_\kappa \in_\kappa L^\infty(0,T;L^\gamma(\Omega)) \cap L^2(0,T;W^{1,6}(\Omega)'),$$

$$\underset{\sim}{u}_\kappa \in_\kappa L^2(0,T;\underset{\sim}{H}^1_0(\Omega)),$$

$$\kappa^{\frac{1}{\Gamma}}\rho_\kappa \in_\kappa L^\infty(0,T;L^\Gamma(\Omega)),$$

$$\rho_\kappa\,|\underset{\sim}{u}_\kappa|^2 \in_\kappa L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;L^{\frac{6\gamma}{3(\gamma+1)}}(\Omega)),$$

$$\rho_\kappa \underset{\sim}{u}_\kappa \in_\kappa \underset{\sim}{L}^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap L^2(0,T;\underset{\sim}{L}^{\frac{6\gamma}{\gamma+6}}(\Omega)) \cap \underset{\sim}{L}^{\frac{10\gamma-6}{3(\gamma+1)}}(\Omega_T).$$

STEP 18.

Using an analogous argument as in STEP 12 and STEP 13, we deduce that

$$\rho_\kappa \in_\kappa L^{\gamma+\vartheta(\gamma)}(\Omega_T),$$

where

$$\vartheta(\gamma) := \begin{cases} \frac{2\gamma-3}{3} & \text{for } \frac{3}{2} < \gamma \leq 4, \\ \frac{5}{12}\gamma & \text{for } 4 \leq \gamma. \end{cases}$$

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STEP 19. Using an analogous argument as in STEP 14, we deduce that

$$\frac{\partial(\rho_\kappa \underset{\sim}{u_\kappa})}{\partial t} \in_\kappa L^{\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W_0^{1,r}(\Omega)'),$$

where $r = \max\{s, \frac{\Gamma+\theta}{\theta}\}$ and $s = \max\{4, \frac{6\gamma}{2\gamma-3}\}$.

STEP 20. [Level 3 passage to the limit]

Unfortunately, all that can be claimed at this point is that

$$p_\kappa(\rho_\kappa) \rightharpoonup \overline{p_\kappa(\rho_\kappa)} \qquad \text{in } L^{\frac{\gamma+\vartheta(\gamma)}{\gamma}}(\Omega_T) \text{ as } \kappa \to 0.$$

It is yet to be proved that $\overline{p_\kappa(\rho_\kappa)} = p(\rho) = \rho^\gamma$.

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Nevertheless it is possible to show that, as $\kappa \to 0$,

$$\rho_\kappa \to \rho \qquad \text{strongly in } L^s(\Omega_T), \qquad \text{for any } s \in [1, \gamma + \vartheta(\gamma)),$$

$$p_\kappa(\rho_\kappa) \to \rho^\gamma \qquad \text{weakly in } L^{\frac{\gamma+\vartheta(\gamma)}{\gamma}}(\Omega_T), \qquad \text{that is, } \overline{p_\kappa(\rho_\kappa)} = \rho^\gamma.$$

That completes our passage to the limit $\kappa \to 0$. $\square$

Comments

Elsewhere, in the case of incompressible flows:

1. we proved the existence and equilibration of global weak solutions to general classes of both FENE-type and Hookean-type models in the case of constant density, viscosity and drag and as well as in the case of variable density, and density-dependent viscosity and drag (see p.4).

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The extensions of these to compressible flows are open.