Complex Gauss quadratures

Miroslav Pranić
Department of Mathematics, University of Banja Luka

joint work with
Stefano Pozza and Zdeněk Strakoš

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- **Mail goal**: Can we define Gauss quadratures for approximations of linear functionals which are not positive definite? How far we can go with generalization?

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- Complex Jacobi matrices

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- G3: The Gauss quadrature can be written in the form $m_0 \mathbf{e}_1^T f(J_n) \mathbf{e}_1$

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- J_n - complex Jacobi matrix: three-diagonal, symmetric, no zeros on sub-diagonal

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- The following properties are equivalent:
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- Should we call it Gauss quadrature? (G1, G2 and G3)

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- $m_0\mathbf{e}_1^T f(J_n)\mathbf{e}_1 = \mathcal{L}(f)$, for all $f \in \mathcal{P}_{2n-1}$

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- Problem 1: If we only know the moments $m_0, \ldots, m_{2n}$, we cannot determine the degree of exactness. If $m_{2n+1} = Q_n(x^{2n+1})$ then the degree of exactness of Q_n is at least $2n+1$. If $Q_n(x^{2n+1}) \neq m_{2n+1}$ then the degree of exactness is $2n$. And so on.

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- Problem 2: The construction of Q_{n+1}. Monic polynomial π_{n+1} of degree $n+1$ which is orthogonal to $\mathcal{P}_n$ either does not exist or there are infinitely many of them. So Q_{n+1} either does not exist or it is not unique.

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- J_3 is diagonalizable, J_2 is not diagonalizable.

Example

- Instead of 2-point we have 2-weight Gauss quadrature of the form: $A_1 f(2) + A_2 f'(2)$. It is easy to check that the nonlinear system

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\begin{aligned}
A_1 \cdot 1 + A_2 \cdot 0 &= 1 \\
A_1 z_1 + A_2 \cdot 1 &= 3 \\
A_1 z_1^2 + A_2(2z_1) &= 8 \\
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has unique solution (in $\mathbb{C}$): $A_1 = 1, A_2 = 1, z_1 = 2$. So the quadrature $f(2) + f'(2)$ has degree of exactness 3. Its degree of exactness would be higher if and only if $m_4 = 2^4 + 4 \cdot 2^3 = 48$. But in that case we would have $\Delta_2 = 0$, i.e. $\mathcal{L}$ would not be quasi definite on $\mathcal{P}_2$.

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- The functional $\mathcal{L}_1$ whose first five moments are

$$m_0 = 1,\ m_1 = 3,\ m_2 = 8,\ m_3 = 20,\ m_4 = 48,$$

  is not quasi-definite on $\mathcal{P}_2$. If $m_5 = 2^5 + 5 \cdot 2^4 = 112$ then the quadrature $f(2) + f'(2)$ would have degree of exactness at least 5.

THANK YOU FOR YOUR ATTENTION!