Mixed Least Squares Finite Element Methods Based on Inverse Stress - Strain Relations in Hyperelasticity

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Outline

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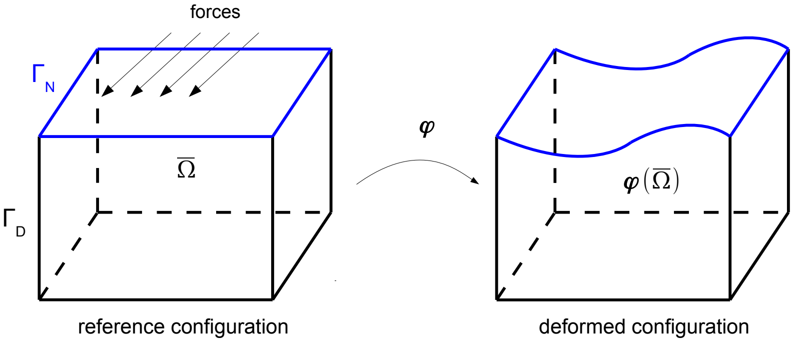
Initial situation for a given body $\Omega \subset \mathbb{R}^3$



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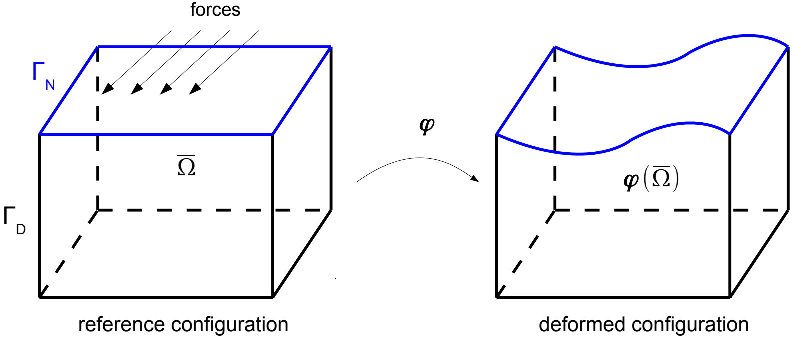

- body force: given by a density $\mathbf{f} : \Omega \to \mathbb{R}^3$
- surface force: given by a density $\mathbf{g} : \Gamma_N \to \mathbb{R}^3$
- deformation: $\varphi : \bar{\Omega} \to \mathbb{R}^3$
- $\varphi = \mathbf{id} + \mathbf{u}$ with the pointwise displacement $\mathbf{u} : \bar{\Omega} \to \mathbb{R}^3$
- stress tensor: mapping from $\bar{\Omega}$ to $\mathbb{R}^{3\times 3}$

Notations:

- $\lambda, \mu > 0$: material dependend Lamé-constants (alternatively: Young's modulus E, Poisson's ratio ν)
- deformation gradient: $\mathbf{F} := \nabla \varphi = \mathbf{I} + \nabla \mathbf{u} =: \mathbf{F}(\mathbf{u})$
- $\mathbf{Cof\,A} := (\det \mathbf{A})\mathbf{A}^{-T}$ for regular $\mathbf{A} \in \mathbb{R}^{3\times 3}$
- $\mathbf{dev\,A} := \mathbf{A} - \frac{1}{3}\mathrm{tr}\,(\mathbf{A})\mathbf{I}$ for $\mathbf{A} \in \mathbb{R}^{3\times 3}$

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- **Strain tensors:**
  - $\mathbf{B} := \mathbf{F}\mathbf{F}^T$ (left Cauchy - Green)
  - $\mathbf{C} := \mathbf{F}^T\mathbf{F}$ (right Cauchy - Green)
  - linear elasticity:

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

LSFEM for linear elasticity [Cai, Starke (2004)]

- Stress - strain relation:

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\mathrm{tr}\,(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} =: \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{(invertible for finite } \lambda)$$

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- Inverse stress - strain relation:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{3\lambda + 2\mu}\mathrm{tr}\,(\boldsymbol{\sigma})\mathbf{I}\right) =: \mathcal{A}_{lin}(\boldsymbol{\sigma}) \overset{\lambda \to \infty}{\to} \frac{1}{2\mu}\mathrm{dev}\ \boldsymbol{\sigma}$$

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- Minimize the (linear) least squares functional:

$$\mathcal{F}_{lin}(\boldsymbol{\sigma},\mathbf{u};\mathbf{f}) = \|\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f}\|^2_{L^2(\Omega)} + \|\mathcal{A}_{lin}(\boldsymbol{\sigma}) - \varepsilon(\mathbf{u})\|^2_{L^2(\Omega)}$$

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- proof of continuity and ellipticity of $\mathcal{F}_{lin}(\boldsymbol{\tau},\mathbf{v};\mathbf{0})$ in $\mathcal{V} := H_{\Gamma_N}(\mathrm{div},\Omega)^3 \times H^1_{\Gamma_D}(\Omega)^3$ is given:

$$\mathcal{F}_{lin}(\boldsymbol{\tau},\mathbf{v};\mathbf{0}) \lesssim \|\varepsilon(\mathbf{v})\|^2_{L^2(\Omega)} + \|\boldsymbol{\tau}\|^2_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{\tau}\|^2_{L^2(\Omega)} \lesssim \|(\boldsymbol{\tau},\mathbf{v})\|^2_{\mathcal{V}}$$

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Purposes:

- Extension of the approach of linear to nonlinear elasticity, i.e. in particular
  - using the Least Squares Finite Element Method
  - mixed formulation (approximate displacements **and** stresses)
  - using nonlinear kinematics (strain tensor is nonlinear in $\mathbf{u}$)
  - using nonlinear material laws (stress - strain relation is nonlinear)

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  - mixed formulation (approximate displacements **and** stresses)
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  - using nonlinear material laws (stress - strain relation is nonlinear)
- Desirable properties:
  - reliability in the incompressible limit (no Poisson locking as Poisson's ratio $\nu \to \frac{1}{2}$)
  - equivalence of the (nonlinear and linearized) least squares functional to the error, i.e.

$$\mathcal{F}(\mathbf{P}_h, \mathbf{u}_h) \eqsim \|(\mathbf{P} - \mathbf{P}_h, \mathbf{u} - \mathbf{u}_h)\|_V^2.$$

Outline

First order system in hyperelasticity:

Given:

frame - indifferent material with stored energy function
$\psi : \bar{\Omega} \times \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$

Strong formulation:

Seek $\mathbf{u}$ and the first Piola - Kirchhoff stress tensor $\mathbf{P}$ with

$$-\operatorname{div} \mathbf{P} = \mathbf{f} \text{ in } \Omega$$
$$\mathbf{P} = \partial_{\mathbf{F}} \psi(\mathbf{x}, \mathbf{C}) \text{ in } \Omega \text{ (hyperelastic material law)}$$

and boundary conditions

$$\mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D, \quad \mathbf{P} \cdot \mathbf{n} = \mathbf{g} \text{ on } \Gamma_N.$$

Homogeneous isotropic materials:

It exists $\tilde{\psi} : \mathbb{R}^3 \to \mathbb{R}$ with

$$\psi(\mathbf{C}) = \tilde{\psi}(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}))$$

and $I_1(\mathbf{C}) = \mathrm{tr}\,(\mathbf{C})$, $I_2(\mathbf{C}) = \mathrm{tr}\,(\mathbf{Cof\,C})$, $I_3(\mathbf{C}) = \det \mathbf{C}$, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

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$$\begin{aligned}\mathbf{P} = \partial_{\mathbf{F}}\psi(\mathbf{C}) &= \frac{\partial\tilde{\psi}}{\partial I_1}\partial_{\mathbf{F}} I_1(\mathbf{C}) + \frac{\partial\tilde{\psi}}{\partial I_2}\partial_{\mathbf{F}} I_2(\mathbf{C}) + \frac{\partial\tilde{\psi}}{\partial I_3}\partial_{\mathbf{F}} I_3(\mathbf{C}) \\ &= 2\frac{\partial\tilde{\psi}}{\partial I_1}\mathbf{F} + 2\frac{\partial\tilde{\psi}}{\partial I_2}\mathbf{F}\left(\operatorname{tr}(\mathbf{C})\mathbf{I} - \mathbf{C}\right) + 2(\det\mathbf{F})^2\frac{\partial\tilde{\psi}}{\partial I_3}\mathbf{F}^{-T}.\end{aligned}$$

$$\Rightarrow \boldsymbol{\tau} = \mathbf{P}\mathbf{F}^T = 2\frac{\partial\tilde{\psi}}{\partial I_1}\mathbf{B} + 2\frac{\partial\tilde{\psi}}{\partial I_2}\left(\operatorname{tr}(\mathbf{B})\mathbf{B} - \mathbf{B}^2\right) + 2\det\mathbf{B}\frac{\partial\tilde{\psi}}{\partial I_3}\mathbf{I} =:= \mathcal{G}(\mathbf{B}).$$

Inverse stress - strain relation

- Assumptions:
  - $\mathbf{P}(\mathbf{u} = \mathbf{0}) = \mathbf{0}$ (no displacement $\Rightarrow$ no stress)
  - $\mathbf{P}'(\mathbf{0})[\mathbf{v}] = 2\mu\, \boldsymbol{\varepsilon}(\mathbf{v}) + \lambda\, \mathrm{tr}\,(\boldsymbol{\varepsilon}(\mathbf{v}))\mathbf{I} = \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v})$
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- Consequences for $\mathcal{G}(\mathbf{B}(\mathbf{u})) = \mathbf{P}(\mathbf{u})(\mathbf{F}(\mathbf{u}))^T$:

$$\mathcal{G}'(\mathbf{B}(\mathbf{0}))[\mathbf{B}'(\mathbf{0})[\mathbf{v}]] = \mathbf{P}'(\mathbf{0})[\mathbf{v}](\mathbf{F}(\mathbf{0}))^T + \mathbf{P}(\mathbf{0})(\nabla\mathbf{v})^T$$
$$\Leftrightarrow \mathcal{G}'(\mathbf{I})[2\boldsymbol{\varepsilon}(\mathbf{v})] = \mathbf{P}'(\mathbf{0})[\mathbf{v}] = \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v})$$

for all $\mathbf{v}$ in a neighborhood of $\mathbf{u} = \mathbf{0}$.

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- $\mathcal{G}^{-1}(\boldsymbol{\tau})=\mathcal{G}^{-1}(\mathbf{P}\mathbf{F}^T)=\mathbf{B}$ is well-defined in a neighborhood of $\boldsymbol{\tau}=\mathbf{0}$.

Outline

Inverse first order system and least squares formulation

Inverse formulation of the FOS:

$$\mathcal{R}(\mathbf{P},\mathbf{u}) := \begin{pmatrix} \omega_1 \left(\operatorname{div} \mathbf{P} + \mathbf{f}\right) \\ \omega_2 \left(\mathcal{A}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u})\right) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \text{ in } \Omega$$

with scaling parameters $\omega_1, \omega_2 > 0$ and $\mathcal{A} := \mathcal{G}^{-1}$ for compressible materials (case $\lambda < \infty$).

Least squares functional:

For $(\mathbf{P},\mathbf{u}) \in W^q(\operatorname{div};\Omega)^3 \times W^{1,p}(\Omega)^3$ we define

$$\begin{aligned} \mathcal{F}(\mathbf{P},\mathbf{u}) &:= \|\mathcal{R}(\mathbf{P},\mathbf{u})\|^2_{L^2(\Omega)} \\ &= \omega_1^2 \|\operatorname{div} \mathbf{P} + \mathbf{f}\|^2_{L^2(\Omega)} + \omega_2^2 \|\mathcal{A}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u})\|^2_{L^2(\Omega)}. \end{aligned}$$

Sketch of further steps

- Linearized least squares functional for fixed $(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})$:

  Seek correction term $(\mathbf{Q}^{(k)}, \mathbf{v}^{(k)}) \in W^q_{\Gamma_N}(\mathrm{div};\Omega)^3 \times W^{1,p}_{\Gamma_D}(\Omega)^3$, such that

  $$\mathcal{F}^{\mathrm{lin}}(\mathbf{Q}^{(k)}, \mathbf{v}^{(k)}) := \|\mathcal{R}(\mathbf{P}^{(k)}, \mathbf{u}^{(k)}) + \mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\mathbf{Q}^{(k)}, \mathbf{v}^{(k)}]\|^2_{L^2(\Omega)}$$

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- Necessary condition $\frac{d}{dt}\mathcal{F}^{\text{lin}}(\mathbf{Q}^{(k)} + t\hat{\mathbf{Q}}, \mathbf{v}^{(k)} + t\hat{\mathbf{v}})|_{t=0} = 0$ leads to a (standard) variational formulation.

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  - $\left(\mathbf{P}_h^{(k+1)}, \mathbf{u}_h^{(k+1)}\right) = \left(\mathbf{P}_h^{(k)}, \mathbf{u}_h^{(k)}\right) + \alpha^{(k)} \left(\mathbf{Q}_h^{(k)}, \mathbf{v}_h^{(k)}\right)$

    ($\alpha^{(k)}$: parameter in backtracking line search damping strategy)

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$$\Rightarrow \mathbf{P}\mathbf{F}^T = \partial_{\mathbf{F}} \psi_{NH}(\mathbf{C})\mathbf{F}^T = \mu \mathbf{B} + \left(\frac{\lambda}{2}(\det \mathbf{B} - 1) - \mu\right) \mathbf{I} =: \mathcal{G}_{NH}(\mathbf{B})$$

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- Open questions:
  - Direct evaluation of $\mathcal{A}_{NH}(\boldsymbol{\tau})$ for given $\boldsymbol{\tau} = \mathbf{PF}(\mathbf{u})^T$ possible ?
  - Well - posedness of $\mathcal{A}_{NH}(\boldsymbol{\tau})$ for $\lambda \to \infty$?

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For finite λ and given stress $\boldsymbol{\tau}$ we seek the corresponding strain $\mathbf{B}$ with $\mathcal{G}_{NH}(\mathbf{B}) = \boldsymbol{\tau}$.

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$$\mu\, \mathbf{dev}\, \mathbf{B} = \mathbf{dev}\, \boldsymbol{\tau}$$

$$\mu \left(\frac{1}{3} \mathrm{tr}\, \mathbf{B} - 1 \right) + \frac{\lambda}{2} (\det \mathbf{B} - 1) = \frac{1}{3} \mathrm{tr}\, \boldsymbol{\tau}. \qquad (*)$$

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Using
$\det(\mathbf{A}_1 + \mathbf{A}_2) = \det\,\mathbf{A}_1 + \mathbf{Cof}\,\mathbf{A}_1 : \mathbf{A}_2 + \mathbf{A}_1 : \mathbf{Cof}\,\mathbf{A}_2 + \det\,\mathbf{A}_2$ and $(*)_1$ results in

$$\det\mathbf{B} = \frac{1}{27}(\mathrm{tr}\,\mathbf{B})^3 + \frac{1}{3\mu^2}(\mathrm{tr}\,\mathbf{B})\mathrm{tr}\,(\mathbf{Cof}\,(\mathbf{dev}\,\boldsymbol{\tau})) + \frac{1}{\mu^3}\det(\mathbf{dev}\,\boldsymbol{\tau})$$

Well-posedness of $\mathcal{A}_{NH}$ for $\lambda \to \infty$:

Inserting det **B** into $(*)_2$ results in a cubic equation

$$(\mathrm{tr}\,\mathbf{B})^3 + \mathrm{S}\,\mathrm{tr}\,\mathbf{B} + \mathrm{T} = 0$$

with

$$S := \frac{9}{\mu^2}\mathrm{tr}\left(\mathbf{Cof}\left(\mathbf{dev}\,\boldsymbol{\tau}\right)\right) + \frac{18\mu}{\lambda}$$

$$T := 27\left(\frac{1}{\mu^3}\det(\mathbf{dev}\,\boldsymbol{\tau}) - 1 - \frac{2\mu}{\lambda} - \frac{2}{3\lambda}\mathrm{tr}\,\boldsymbol{\tau}\right).$$

For the discriminant $D := \left(\frac{S}{3}\right)^3 + \left(\frac{T}{2}\right)^2 > 0$ the cubic equation is uniquely solvable through

$$\mathrm{tr}\,\mathbf{B} = \sqrt[3]{-\frac{T}{2} + \sqrt{D}} + \sqrt[3]{-\frac{T}{2} - \sqrt{D}}.$$

Well-posedness of $\mathcal{A}_{NH}$ for $\lambda \to \infty$:

Theorem (M., Starke)

Assume that the discriminant D of the cubic equation is positive. Then the mapping $\mathbf{B} = \mathcal{A}_{NH}(\boldsymbol{\tau})$, defined by $(*)$, is well-defined in the incompressible limit $\lambda \to \infty$. Its inverse does not exist in this case.

Well - posedness of $\mathcal{A}_{NH}$ for $\lambda \to \infty$:

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Proof:

- Well - posedness of $\mathcal{A}_{NH}$ for $\lambda \to \infty$: ✓

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Proof:

- Well-posedness of $\mathcal{A}_{NH}$ for $\lambda \to \infty$: ✓
- For $\boldsymbol{\tau}_2 := \boldsymbol{\tau}_1 + c\mathbf{I}$ with given matrix $\boldsymbol{\tau}_1$ and $c \in \mathbb{R} \setminus \{0\}$ it holds $\boldsymbol{\tau}_1 \neq \boldsymbol{\tau}_2$ and $\mathbf{dev}\,\boldsymbol{\tau}_1 = \mathbf{dev}\,\boldsymbol{\tau}_2$

$$\Rightarrow \mathbf{dev}\,\mathbf{B}_1 = \mathbf{dev}\,\mathbf{B}_2 \text{ and } \operatorname{tr}\mathbf{B}_1 = \operatorname{tr}\mathbf{B}_2 \Rightarrow \mathbf{B}_1 = \mathbf{B}_2.$$

$\Rightarrow \mathcal{A}_{NH}$ is not injective and therefore not invertible for $\lambda = \infty$

□

Outline

Continuity and ellipticity of $\mathcal{F}_{NH}(\mathbf{P}, \mathbf{u})$ near the origin

For mixed boundary conditions and small $\delta > 0$ we set:

$$\mathbf{\Pi}^\infty := \{\mathbf{Q} \in W^\infty(\text{div};\Omega)^3 : \|\mathbf{Q}\|_{L^\infty(\Omega)} \leq \delta\} \cap (\mathbf{P}^N + W^4_{\Gamma_N}(\text{div};\Omega))^3$$
$$\mathbf{U}^\infty := \{\mathbf{u} \in W^{1,\infty}(\Omega)^3 : \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \leq \delta\} \cap (\mathbf{u}_D + W^{1,4}_{\Gamma_D}(\Omega)^3)$$

with $\mathbf{P}^N \in W^\infty(\text{div};\Omega)^3$ and $\mathbf{u}_D \in W^{1,\infty}(\Omega)^3$.

Lemma (M., Starke)

If $\delta > 0$ is chosen sufficiently small, then there is a $\rho \in [0, 1)$ such that

$$\|\mathcal{R}'(\mathbf{Q}, \mathbf{v})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}] - \mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \leq \rho \|\mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}$$

for all $(\mathbf{Q}, \mathbf{v}) \in \mathbf{\Pi}^\infty \times \mathbf{U}^\infty$ and $(\hat{\mathbf{Q}}, \hat{\mathbf{v}}) \in H_{\Gamma_N}(div;\Omega)^3 \times H^1_{\Gamma_D}(\Omega)^3$.

Continuity and ellipticity of $\mathcal{F}_{NH}(\mathbf{P}, \mathbf{u})$ near the origin

Theorem (M., Starke)

If $\delta > 0$ is chosen sufficiently small, then

$$\|\mathcal{R}(\hat{\mathbf{Q}}, \hat{\mathbf{v}}) - \mathcal{R}(\mathbf{Q}, \mathbf{v})\|^2_{L^2(\Omega)} \lesssim \|\hat{\mathbf{Q}} - \mathbf{Q}\|^2_{H(div\,;\,\Omega)} + \|\hat{\mathbf{v}} - \mathbf{v}\|^2_{H^1(\Omega)}$$
$$\|\mathcal{R}(\hat{\mathbf{Q}}, \hat{\mathbf{v}}) - \mathcal{R}(\mathbf{Q}, \mathbf{v})\|^2_{L^2(\Omega)} \gtrsim \|\hat{\mathbf{Q}} - \mathbf{Q}\|^2_{H(div\,;\,\Omega)} + \|\hat{\mathbf{v}} - \mathbf{v}\|^2_{H^1(\Omega)}$$

holds uniformly for $\lambda \to \infty$ and all $\mathbf{Q}, \hat{\mathbf{Q}} \in \mathbf{\Pi}^\infty$ and $\mathbf{v}, \hat{\mathbf{v}} \in \mathbf{U}^\infty$.

Consequence:

$$\mathcal{F}_{NH}(\mathbf{P}_h, \mathbf{u}_h) \lesssim \|\mathbf{P}_h - \mathbf{P}\|^2_{H(\mathrm{div};\,\Omega)} + \|\mathbf{u}_h - \mathbf{u}\|^2_{H^1(\Omega)}$$
$$\mathcal{F}_{NH}(\mathbf{P}_h, \mathbf{u}_h) \gtrsim \|\mathbf{P}_h - \mathbf{P}\|^2_{H(\mathrm{div};\,\Omega)} + \|\mathbf{u}_h - \mathbf{u}\|^2_{H^1(\Omega)}$$

i.e. $\mathcal{F}_{NH}(\mathbf{P}_h, \mathbf{u}_h)$ is an a-posteriori error estimator.

Well-posedness of the linearized problem

Linearized least squares functional:

$$\mathcal{F}_{NH}^{\text{lin}}(\mathbf{Q},\mathbf{v};\mathcal{R}(\mathbf{P}^{(k)},\mathbf{u}^{(k)})) := \|\mathcal{R}(\mathbf{P}^{(k)},\mathbf{u}^{(k)}) + \mathcal{R}'(\mathbf{P}^{(k)},\mathbf{u}^{(k)})[\mathbf{Q},\mathbf{v}]\|_{L^2(\Omega)}^2$$

Corresponding bilinear form and linear form:

$$a((\mathbf{Q},\mathbf{v}),(\hat{\mathbf{Q}},\hat{\mathbf{v}})) = \Big(\mathcal{R}'(\mathbf{P}^{(k)},\mathbf{u}^{(k)})[\mathbf{Q},\mathbf{v}],\mathcal{R}'(\mathbf{P}^{(k)},\mathbf{u}^{(k)})[\hat{\mathbf{Q}},\hat{\mathbf{v}}]\Big)_{L^2(\Omega)}$$

$$F((\hat{\mathbf{Q}},\hat{\mathbf{v}})) = -\Big(\mathcal{R}(\mathbf{P}^{(k)},\mathbf{u}^{(k)}),\mathcal{R}'(\mathbf{P}^{(k)},\mathbf{u}^{(k)})[\hat{\mathbf{Q}},\hat{\mathbf{v}}]\Big)_{L^2(\Omega)}$$

Wanted property:

$$\mathcal{F}_{NH}^{\text{lin}}(\hat{\mathbf{Q}},\hat{\mathbf{v}};\mathbf{0}) = \|\mathcal{R}'(\mathbf{P}^{(k)},\mathbf{u}^{(k)})[\hat{\mathbf{Q}},\hat{\mathbf{v}}]\|_{L^2(\Omega)}^2 \eqsim \|(\hat{\mathbf{Q}},\hat{\mathbf{v}})\|_{\mathcal{V}}^2$$

for all $\hat{\mathbf{Q}} \in H_{\Gamma_N}(\text{div};\Omega)^3$ and $\hat{\mathbf{v}} \in H^1_{\Gamma_D}(\Omega)^3$.

Proof of „wanted property“

Let $(\mathbf{P}^{(k)}, \mathbf{u}^{(k)}) \in \mathbf{\Pi}^\infty \times \mathbf{U}^\infty$, $(\hat{\mathbf{Q}}, \hat{\mathbf{v}}) \in H_{\Gamma_N}(\text{div};\Omega)^3 \times H^1_{\Gamma_D}(\Omega)^3$.
Then it follows by our lemma

$$\|\mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}] - \mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \leq \rho \|\mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}$$

with $\rho < 1$ and therefore

$$\|\mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \leq (1+\rho) \|\mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}$$
$$\|\mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \geq (1-\rho) \|\mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}.$$

Proof of „wanted property“

Let $(\mathbf{P}^{(k)}, \mathbf{u}^{(k)}) \in \mathbf{\Pi}^\infty \times \mathbf{U}^\infty$, $(\hat{\mathbf{Q}}, \hat{\mathbf{v}}) \in H_{\Gamma_N}(\text{div};\Omega)^3 \times H^1_{\Gamma_D}(\Omega)^3$.
Then it follows by our lemma

$$\|\mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}] - \mathcal{R}'(\mathbf{0},\mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \leq \rho \|\mathcal{R}'(\mathbf{0},\mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}$$

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$$\|\mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \geq (1-\rho) \|\mathcal{R}'(\mathbf{0},\mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}.$$

Since $\mathcal{R}'(\mathbf{0},\mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}] = \begin{pmatrix} \text{div}\,\mathbf{Q} \\ 2\left(\mathcal{A}_{lin}(\hat{\mathbf{Q}}) - \varepsilon(\hat{\mathbf{v}})\right) \end{pmatrix}$ it holds

$$\|\mathcal{R}'(\mathbf{0},\mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)} \eqsim \mathcal{F}_{lin}(\hat{\mathbf{Q}}, \hat{\mathbf{v}}; \mathbf{0}).$$

Proof of „wanted property“

Altogether

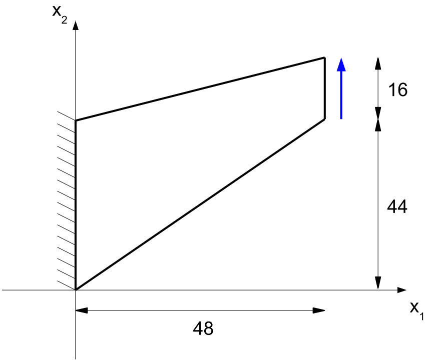
$$\begin{aligned}\mathcal{F}_{NH}^{\text{lin}}(\hat{\mathbf{Q}}, \hat{\mathbf{v}}; \mathbf{0}) &= \|\mathcal{R}'(\mathbf{P}^{(k)}, \mathbf{u}^{(k)})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}^2 \\ &\eqsim \|\mathcal{R}'(\mathbf{0}, \mathbf{0})[\hat{\mathbf{Q}}, \hat{\mathbf{v}}]\|_{L^2(\Omega)}^2 \\ &\eqsim \mathcal{F}_{lin}(\hat{\mathbf{Q}}, \hat{\mathbf{v}}; \mathbf{0}) \\ &\eqsim \|(\hat{\mathbf{Q}}, \hat{\mathbf{v}})\|_{\mathcal{V}}^2\end{aligned}$$

with $\mathcal{V} := H_{\Gamma_N}(\text{div}; \Omega)^3 \times H^1_{\Gamma_D}(\Omega)^3$.

Outline

Numerical example 1: Cook's membrane 2d (plane strain)

Densities of forces: $\mathbf{f} \equiv \mathbf{0}$, $\mathbf{g} = (0, \gamma)^T$ with $\gamma \in \mathbb{R}$



constraints on Γ_D:

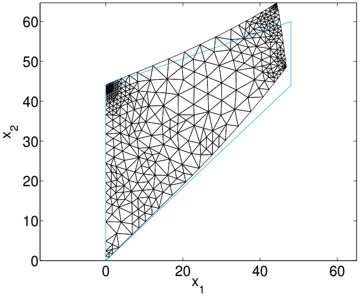
- $\mathbf{u} = \mathbf{0}$ (left)

constraints on Γ_N:

- $\mathbf{P} \cdot \mathbf{n} = \mathbf{0}$ (top, bottom)
- $\mathbf{P} \cdot \mathbf{n} = \mathbf{g}$ (right)

Numerical example 1

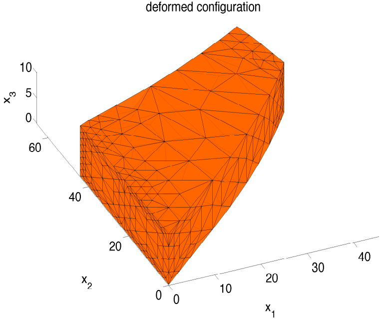
- Lamé constants: $\mu = 1, \lambda = \infty$, load value: $\gamma = 0.05$
- scaling parameters: $\omega_1 = 10^2, \omega_2 = 1$
- reference solution: $u_2(48, 60) = 4.7010$ for $nt = 47616$



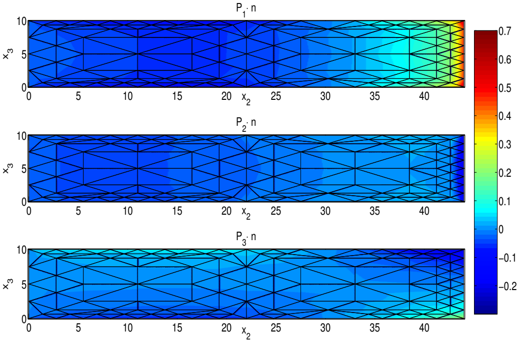
nt	$\mathcal{F}_{NH}(\mathbf{P}_h, \mathbf{u}_h)$	(order)	$\mathbf{u}_2(48, 60)$
186	$2.9972 \cdot 10^{-2}$		4.5092
275	$1.4042 \cdot 10^{-2}$	(1.94)	4.6120
390	$6.7178 \cdot 10^{-3}$	(2.11)	4.6586
559	$3.2427 \cdot 10^{-3}$	(2.02)	4.6810
821	$1.5525 \cdot 10^{-3}$	(1.92)	4.6921
1211	$7.3322 \cdot 10^{-4}$	(1.93)	4.6974
1796	$3.3695 \cdot 10^{-4}$	(1.97)	4.6999
2622	$1.4855 \cdot 10^{-4}$	(2.16)	4.7011

Numerical example 2: Cook's membrane 3d

- boundary conditions, Lamé constants and forces as in the first example
- Ω now expanded in x_3 - direction (thickness $d = 10$)



nt	880	1550	3435
order		1.4187	0.9164
$u_2(48, 60, 10)$	5.3001	5.5505	5.6559



Plot of the normal components of the stress tensor on the left clamped boundary

Outline

Summary

- **General:**
  - extension of a LSFEM approach from linear elasticity to geometrically nonlinear elasticity
  - first Piola-Kirchhoff stress tensor is approximated next to the displacement
    $\Rightarrow$ no post-processing is necessary and better stress approximations can be achieved (cp. talk of Prof. Starke)
- **Neo-Hooke:**
  - one approach covers compressible, quasi-incompressible and full incompressible materials
  - analysis under strong regularity assumptions and close to the origin (Neo-Hooke)

References

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Thank you for your attention