Complete fluid systems, the state of art

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Model reduction in continuum thermodynamics:
Modeling, analysis and computation
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Model description

STATE VARIABLES:

Mass density

$$\varrho = \varrho(t, x)$$

Absolute temperature

$$\vartheta = \vartheta(t, x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

THERMODYNAMIC FUNCTIONS:

Pressure

$$p = p(\varrho, \vartheta)$$

Internal energy

$$e = e(\varrho, \vartheta)$$

Entropy

$$s = s(\varrho, \vartheta)$$

Gibbs' law

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} = c_v(\varrho, \vartheta) > 0$$

Transport and conservative boundary conditions

TRANSPORT:

Newton's law

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

CONSERVATIVE BOUNDARY CONDITIONS:

No-slip

$$\mathbf{u}|_{\partial\Omega} = 0$$

No-flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Navier-Stokes-Fourier system (classical formulation)

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Thermal energy equation

$$\varrho c_v(\varrho, \vartheta) \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \right) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}$$

Navier-Stokes-Fourier system (weak formulation)

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Entropy equation

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \, \mathrm{d}x = 0$$

Initial state

Initial data (regular)

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

- $$\varrho_0 \in W^{3,2}(\Omega),\ 0 < \underline{\varrho} \leq \varrho_0 < \overline{\varrho}$$
- $$\vartheta_0 \in W^{3,2}(\Omega),\ 0 < \underline{\vartheta} \leq \vartheta_0 < \overline{\vartheta}$$
- $$\mathbf{u}_0 \in W^{3,2}(\Omega; R^3)$$

Compatibility conditions

$$\nabla_x \vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{u}_0|_{\partial\Omega} = 0$$

$$\nabla_x p(\varrho_0, \vartheta_0) = \mathrm{div}_x \mathbb{S}(\vartheta_0, \nabla_x \mathbf{u}_0) - \varrho_0 \nabla_x F|_{\partial\Omega} = 0$$

Existence of smooth solutions (classical theory)

A. Valli [1982] Existence of classical *local-in-time* solutions in the class:

$$\varrho \in C([0, T_{\max}); W^{3,2}(\Omega)), \ \vartheta_0 \in C([0, T_{\max}); W^{3,2}(\Omega))$$
$$\mathbf{u} \in C([0, T_{\max}); W^{3,2}(\Omega; R^3))$$

A.Matsumura, T.Nishida [1980,1983] Existence of classical *global-in-time* solutions in the same class for the initial data sufficiently close to a static state

Static states

$$\mathbf{u} \equiv 0, \ \vartheta = \tilde{\vartheta} > 0 - \text{a positive constant}, \ \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F$$

Weak solutions

- Equation of continuity holds in the sense of distributions (renormalized equation also satisfied)
- Momentum balance holds in the sense of distributions
- Entropy production equation holds in the sense of distributions, entropy production rate satisfies the inequality
- The system is augmented by the total energy balance

Global existence of weak solutions

Hypotheses

Pressure

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4 \ , \ a > 0$$

$$\lim_{Y\to\infty} \frac{P(Y)}{Y^{5/3}} = p_\infty > 0$$

Internal energy

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4 \ , \ a > 0$$

Viscosity coefficients

$$\mu(\vartheta) \approx (1+\vartheta^\alpha), \ 0 \le \eta(\vartheta) \le c(1+\vartheta^\alpha), \ \alpha \in [2/5, 1]$$

$$\kappa(\vartheta) \approx (1+\vartheta^3)$$

Total dissipation balance

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big)$$

$$\partial^2_{\varrho,\varrho} H_\Theta(\varrho,\Theta) = \frac{1}{\varrho}\partial_\varrho p(\varrho,\Theta)$$

$$\partial_\vartheta H_\Theta(\varrho,\vartheta) = \varrho(\vartheta - \Theta)\partial_\vartheta s(\varrho,\vartheta)$$

Coercivity

$\varrho \mapsto H_\Theta(\varrho,\Theta)$ is convex

$\vartheta \mapsto H_\Theta(\varrho,\vartheta)$ attains its global minimum (zero) at $\vartheta = \Theta$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\tilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) \right) \ \mathrm{d}x$$

$$+ \int_\Omega \tilde{\vartheta} \sigma \ \mathrm{d}x = 0$$

$\tilde{\varrho}, \ \tilde{\vartheta}$ – static solution

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = \int_\Omega \varrho \ \mathrm{d}x = M_0, \ \int_\Omega \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \ \mathrm{d}x = E_0$$

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Results

Global existence of weak solutions

Dissipative (weak) solutions exist (under the constitutive restrictions specified above) globally in time for any choice of the initial data.

Stability

Any (weak) solution of the Navier-Stokes-Fourier system stabilizes to an equilibrium (static) solution for $t \to \infty$.

Weak $\Rightarrow$ dissipative

Any weak solution is a dissipative solution

Weak-strong uniqueness

Dissipative (weak) and strong solutions emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions.

Conditional regularity criterion

Theorem (Conditional regularity)

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Under the structural hypotheses specified above, suppose that $\{\varrho, \vartheta, \mathbf{u}\}$ is a dissipative (weak) solution of the Navier-Stokes-Fourier system on the set $(0, T) \times \Omega$ emanating from regular initial data satisfying the relevant compatibility conditions.
Assume, in addition, that

$$\mathrm{ess} \sup_{t \in (0,T)} \|\nabla_x \mathbf{u}(t, \cdot)\|_{L^\infty(\Omega; R^{3\times 3})} < \infty$$

The $\{\varrho, \vartheta, \mathbf{u}\}$ is a classical solution determined uniquely in the class of all dissipative (weak) solutions to the problem.

Other applications

- Inviscid incompressible limits for the system with Navier-type boundary conditions
- Inviscid vanishing viscosity and/or heat conductivity, convergence to (inviscid) Boussinesq system

Scaled Navier-Stokes-Fourier system

EQUATION OF CONTINUITY

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

BALANCE OF MOMENTUM

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}$$

ENTROPY PRODUCTION

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S} : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Target system

INCOMPRESSIBILITY

$$\mathrm{div}_x \mathbf{v} = 0$$

EULER SYSTEM

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

TEMPERATURE TRANSPORT

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

BASIC ASSUMPTION

The incompressible Euler system possesses a strong solution $\mathbf{v}$ *on a time interval* $(0, T_{\rm max})$ *for the initial data* $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$.

Prepared data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),\ \mathbf{v}_0 \in W^{k,2}(\Omega; R^3),\ k > \frac{5}{2}$$

Boundary conditions

NAVIER'S COMPLETE SLIP CONDITION

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Convergence

$$b > 0, \ 0 < a < \frac{10}{3}$$

$$\mathrm{ess}\sup_{t \in (0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \ \text{in} \ \boxed{L^\infty_{\rm loc}((0, T]; L^2_{\rm loc}(\Omega; R^3))}$$

$$\text{and weakly-(*) in } L^\infty(0, T; L^2(\Omega; R^3))$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \ \text{in} \ \boxed{L^\infty_{\rm loc}((0, T]; L^q_{\rm loc}(\Omega; R^3)), \ 1 \leq q < 2},$$

$$\text{and weakly-(*) in } L^\infty(0, T; L^2(\Omega))$$

Linearization

$$\varepsilon\partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right)} + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \varepsilon \mathbf{f}_1$$

$$\partial_t \left(\overline{\varrho}\partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho}\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+\mathrm{div}_x \left[\left(\overline{\varrho}\partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho}\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = \varepsilon f_2$$

Stability

Application of the relative entropy inequality

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon \mathcal{T}_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Lighthill's acoustic equation

Wave equation

$$\varepsilon\partial_t Z + \Delta\Phi = 0, \ \varepsilon\partial_t\Phi + Z = 0,$$

Neumann boundary condition

$$\nabla_x\Phi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial conditions

$$\Phi(0\cdot) = \Phi_0, \ Z(0,\cdot) = Z_0,$$