Control of Free Boundary Problems with Surface Tension Effects

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September 24, 2014

Modeling, Analysis and Computing in Nonlinear PDEs
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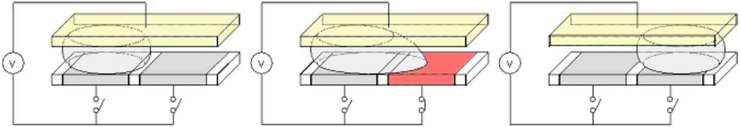
Motivation 1: Electrowetting on Dielectric (EWOD)



Courtesy: R. H. Nochetto, B. Shapiro, S. Walker (2009)

Motivation 2: Extrude-Swell Problem

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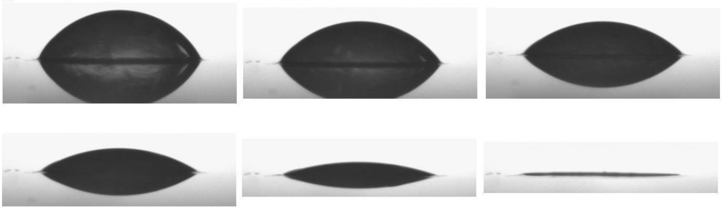
Motivation 3: Adaptive Optics and Ferrofluids



W. Xiao and S. Hardt, 2010

A micro-channel is filled with lens liquid and there is a ferrofluid plug in the channel whose position is controlled by the external magnetic fluid. By manipulating the location of the ferrofluid plug (in their case moving back and forth) one can get different lens shapes and the focal length of the lens can be manipulated. It seems to be more effective than electrowetting.

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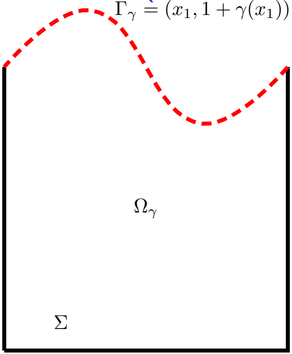
Outline

Model FBP (L.R. Scott and P. Saavedra (1991)



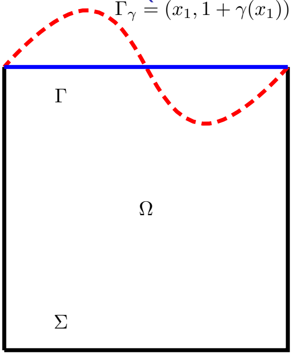
- Find $\gamma \in \mathring{W}^1_\infty(0,1), y \in h \oplus \mathring{W}^1_p(\Omega_\gamma)$, $p > 2$

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega_\gamma \\ y = h & \text{on } \partial\Omega_\gamma \\ -\kappa\mathcal{H}[\gamma] + \partial_\nu y = u & \text{on } \Gamma_\gamma \\ \gamma(0) = \gamma(1) = 0. & \end{cases}$$

- Curvature: $\mathcal{H}[\gamma] = \left(\frac{\gamma'}{\sqrt{1+\gamma'^2}}\right)' \approx \frac{\gamma''}{\sqrt{1+\gamma'^2}}$.

- Impose $|\gamma|_{W^1_\infty(0,1)} \leq 1$.

Model FBP (L.R. Scott and P. Saavedra (1991))



- Find $\gamma \in \mathring{W}^1_\infty(0,1)$, $y \in \mathring{W}^1_p(\Omega)$, $p > 2$.

$$\begin{cases} -\operatorname{div}\big(A[\gamma]\nabla(y+g)\big) = 0 & \text{in } \Omega \\ y = 0 & \text{in } \partial\Omega \\ -\kappa\gamma'' + \nu^T A[\gamma]\nabla(y+g) = u & \text{on } \Gamma \\ \gamma(0) = \gamma(1) = 0. \end{cases}$$

- Lifting operator: g such that $g\big|_{\partial\Omega} = h$.
- Variable diffusion:

$$A[\gamma] = \begin{bmatrix} 1+\gamma & -x_2\gamma' \\ -x_2\gamma' & \frac{1+(x_2\gamma')^2}{1+\gamma} \end{bmatrix}.$$

- Impose $|\gamma|_{W^1_\infty(0,1)} \le 1$.

Well-Posedness

- Spaces: $\mathbb{W} := \mathring{W}^1_\infty(\Gamma) \times \mathring{W}^1_p(\Omega) \qquad \mathbb{V} = \mathring{W}^1_1(\Gamma) \times \mathring{W}^1_{p'}(\Omega)$

- Find $(\gamma, y) \in \mathbb{W},\ p > 2$ such that

$$B_\Omega\left[y + g, z; A[\gamma]\right] = 0, \quad z \in \mathring{W}^1_{p'}(\Omega)$$
$$B_\Gamma[\gamma, \xi] + B_\Omega\left[y + g, \mathcal{E}\xi; A[\gamma]\right] = 0, \quad \xi \in \mathring{W}^1_1(\Gamma)$$

where

$$B_\Omega\left[y, z; A[\gamma]\right] = \int_\Omega \nabla y A[\gamma] \nabla z \; dx, \qquad B_\Gamma[\gamma, \xi] = \int_\Gamma \gamma' \xi' \; dx_1.$$

- Extension: $\mathcal{E} : \mathring{W}^1_1(\Gamma) \to W^1_{p'}(\Omega)$ is continuous provided $p' < 2$ $(p > 2)$.

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- Find $(\gamma, y) \in \mathbb{W},\ p > 2$ such that

$$B_\Omega\left[y + g, z; A[\gamma]\right] = 0,\ z \in \mathring{W}^1_q(\Omega)$$

$$B_\Gamma[\gamma, \xi] + B_\Omega\left[y + g, \mathcal{E}\xi; A[\gamma]\right] = \langle u, \xi\rangle,\ \xi \in \mathring{W}^1_1(\Gamma)$$

where

$$B_\Omega\left[y, z; A[\gamma]\right] = \int_\Omega \nabla y A[\gamma] \nabla z\ dx, \qquad B_\Gamma[\gamma, \xi] = \int_\Gamma \gamma' \xi'\ dx_1.$$

- Extension: $\mathcal{E} : \mathring{W}^1_1(\Gamma) \to W^1_{p'}(\Omega)$ is continuous provided $p' < 2$ $(p > 2)$.

- The above problem is well-posed if

$$|g|_{W^1_p(\Omega)} \le \epsilon \qquad \|u\|_{L^2(\Gamma)} \le C\left(\kappa, |g|_{W^1_p(\Omega)}\right). \tag{A}$$

- **Improved regularity**: We proved $\gamma \in W^{2-1/p}_p(\Gamma) \subset C^{1,1-\frac{2}{p}} \subset W^1_\infty(\Gamma)$.

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Contraction Argument

- Ball: $\mathbb{B}_g = \Big\{(\gamma, y) \in \mathbb{W} : |\gamma|_{W^1_\infty(\Gamma)} \le 1, |y|_{W^1_p(\Omega)} \le C|g|_{W^1_p(\Omega)}\Big\}$.

- Operator $T : \mathbb{B}_g \to \mathbb{W}$ gives $(\gamma, y) = T(\widetilde{\gamma}, \widetilde{y})$ where
  - $\gamma \in \mathring{W}^1_\infty(\Gamma)$ solves

$$B_\Gamma[\gamma, \xi] = -B_\Omega\left[\widetilde{y} + g, \mathcal{E}\xi; A[\widetilde{\gamma}]\right] + \langle u, \xi\rangle$$

  - $y \in \mathring{W}^1_p(\Omega)$ solves

$$B_\Omega\left[y + g, z; A[\gamma]\right] = 0.$$

- T maps $\mathbb{B}_g$ into $\mathbb{B}_g$ provided (A) holds;

- Equivalent norm: $\big\|(\gamma, y)\big\|_{\mathbb{W}} = \epsilon|\gamma|_{W^1_\infty(\Gamma)} + |y|_{W^1_p(\Omega)}$.

- T is a contraction in the norm $\big\|(\gamma, y)\big\|_{\mathbb{W}}$ provided (A) holds.

Solvability

Solvability for γ: We can find $\gamma \in \mathring{W}^1_\infty(\Gamma)$ which solves

$$B_\Gamma[\gamma, \xi] = -B_\Omega\left[\widetilde{y} + g, z; A[\widetilde{\gamma}]\right] + \langle u, \xi \rangle$$

because of Lax-Milgram and the estimate

$$|\gamma|_{W^1_\infty(\Gamma)} \leq \sup_{0 \neq \xi \in \mathring{W}^1_1(\Gamma)} \frac{\int_0^1 \gamma' \xi'}{|\xi|_{W^1_1(\Gamma)}}$$

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Solvability for y: We can find $y \in \mathring{W}^1_p(\Omega)$ which solves

$$B_\Omega\left[y + g, z; A[\gamma]\right] = 0$$

because Meyer's argument and Banach-Nečas theorem imply

$$0 < \beta = \inf_{0 \neq y \in \mathring{W}^1_p(\Omega)} \sup_{0 \neq z \in \mathring{W}^1_{p'}(\Omega)} \frac{B_\Omega\left[y + g, z; A[\gamma]\right]}{|y|_{W^1_p(\Omega)} |z|_{W^1_{p'}(\Omega)}}$$

for all $p \in (P', P)$, $P > 2$.

Optimal Control Problem

- Given $|g|_{W_p^1(\Omega)} \leq \epsilon$ and $\lambda > 0$, consider the minimization of functional

$$\mathcal{J}(\gamma, y, u) := \frac{1}{2}\|\gamma - \gamma_d\|^2_{L^2(\Gamma)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Gamma)}$$

subject to the state constraints that $(\gamma, y) \in \mathring{W}^1_\infty(\Gamma) \times \mathring{W}^1_p(\Omega)$

solve the previous free boundary problem

and the control constraints

$$u \in \mathcal{U}_{ad} := \left\{ u \in L^2(\Gamma) : \|u\|_{L^2(\Gamma)} \leq C\left(\kappa, |g|_{W_p^1(\Omega)}\right) \right\}.$$

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$$u \in \mathcal{U}_{ad} := \left\{ u \in L^2(\Gamma) : \|u\|_{L^2(\Gamma)} \leq C\left(\kappa, |g|_{W_p^1(\Omega)}\right) \right\}.$$

- Note $|g|_{W_p^1(\Omega)} \leq \epsilon$ implies

$$|\gamma|_{W^1_\infty(\Gamma)} \leq 1.$$

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Control-to-state map

- Control-to-state map:

$$\Lambda : \mathcal{U}_{ad} \to \mathbb{W} \qquad (\gamma, y) = \Lambda[u];$$

  - $\gamma \in \mathring{W}^1_\infty(\Gamma)$ solves

$$B_\Gamma[\gamma, \xi] + B_\Omega\left[y + g, \mathcal{E}\xi; A[\gamma]\right] = \langle u, \xi \rangle \quad \forall \xi \in \mathring{W}^1_1(\Gamma)$$

  - $y \in \mathring{W}^1_p(\Omega)$ solves

$$B_\Omega\left[y + g, z; A[\gamma]\right] = 0 \quad \forall z \in \mathring{W}^1_{p'}(\Omega);$$

- Using Λ, the reduced optimization problem:

$$\min_{u \in \mathcal{U}_{ad}} j(u) := \mathcal{J}(\Lambda[u], u).$$

  Then use infimizing sequence argument to show existence of optimal control.

- Λ is Fréchet differentiable;
- Λ is twice continuously Fréchet differentiable.

Adjoint Based Optimization

- The first order necessary optimality condition:

$$0 \leq \left\langle j'(\bar{u}), u - \bar{u} \right\rangle, \quad \forall u \in \mathcal{U}_{ad}$$

The definition of cost functional $j(u) = \frac{1}{2}\int_0^1 |\gamma(u) - \gamma_d|^2 + \frac{\lambda}{2}\int_0^1 u^2$ yields

$$\left\langle j'(\bar{u}), h \right\rangle = \int_0^1 (\bar{\gamma} - \gamma_d)\gamma_u(\bar{u})h + \lambda \int_0^1 \bar{u}h.$$

- The adjoint equations imply

$$j'(\bar{u}) = \bar{s} + \lambda\bar{u},$$

whence

$$0 \leq \langle \bar{s} + \lambda\bar{u}, u - \bar{u} \rangle, \quad \forall u \in \mathcal{U}_{ad}$$

Second Order Sufficient Conditions

- The control-to-state map Λ is twice continuously Fréchet differentiable:

$$j''(\bar{u})h^2 = \int_0^1 \big(\gamma_u(\bar{u})h\big)^2 + \int_0^1 (\gamma - \gamma_d)\gamma_{uu}(\bar{u})h^2 + \lambda \int_0^1 h^2;$$

- **Second order sufficient condition:** For $|g|_{W_p^1(\Omega)} \leq \epsilon$ small enough we get

$$\frac{\lambda}{2}\|u - \bar{u}\|^2_{L^2(\Gamma)} \leq j^{''}(\bar{u})(u - \bar{u})^2, \quad \forall u \in \mathcal{U}_{ad}.$$

- **Strict convexity (local uniqueness):** This further implies the existence of $\delta > 0$ such that for all $u \in \mathcal{U}_{ad}$, $\|u - \bar{u}\|_{L^2(\Gamma)} \leq \delta$,

$$\big\langle j'(u) - j'(\bar{u}), u - \bar{u} \big\rangle \geq \frac{\lambda}{4}\|u - \bar{u}\|^2_{L^2(\Gamma)}.$$

Strong Solutions: Second Order Regularity

- **State variables:**

$$\|\bar{y}\|_{W_p^2(\Omega)} \lesssim \|g\|_{W_p^2(\Omega)}, \qquad |\bar{\gamma}|_{W_\infty^2(0,1)} \leq 1;$$

- **Adjoint variables:**

$$\|\bar{s}\|_{W_1^2(0,1)} + \|\bar{r}\|_{W_{p'}^2(\Omega)} \lesssim \|\bar{\gamma} - \gamma_d\|_{L^2(0,1)}.$$

- **Contraction** argument in a suitable ball in $W_\infty^2(0,1) \times W_p^2(\Omega)$.

Finite element discretization

- **Discrete spaces:**
  - $\mathbb{V}_h$ space of C^0 piecewise linear finite elements over a shape-regular quasi uniform mesh of Ω of meshsize h;
  - $\mathbb{S}_h$ space of C^0 piecewise linear finite elements over a compatible partition of Γ of meshsize h;

- **Discrete optimal control problem:** find minimizers $(G, U) \in \mathbb{S}_h \times \mathbb{V}_h$ of

$$\mathcal{J}_h(G, U) := \frac{1}{2}\|G - \gamma_d\|^2_{L^2(\Gamma)} + \frac{\lambda}{2}\|U\|^2_{L^2(\Gamma)}$$

  subject to the

  discrete free boundary problem

  and the control constraints $u \in \mathbb{U}_{\text{ad}} = \mathcal{U}_{ad} \cap \mathbb{S}_h$.

- **State system:** The discrete inf-sup holds along with the existence of discrete state variables (Saavedra-Scott' 91);

- **Adjoint system:** The discrete inf-sup holds for the adjoint system which is thus well posed.

Finite element discretization (continued)

- **Discrete first order optimality condition:**

$$\left\langle \mathcal{J}_h'(\bar{U}), U - \bar{U} \right\rangle \geq 0 \quad \forall U \in \mathbb{U}_{ad};$$

- **Characterization of $\mathcal{J}_h'(\bar{U})$:**

$$\mathcal{J}_h'(\bar{U}) = \bar{S} + \lambda \bar{U};$$

- **Optimal control:** There exists a discrete optimal control $U \in \mathbb{U}_{\mathsf{ad}}$ which minimizes the discrete reduced functional.

A Priori Error Estimates for State and Adjoint Variables

- **State variables:**

$$|y - Y|_{W_p^1(\Omega)} + |\gamma - G|_{W_\infty^1(0,1)} \lesssim h\left(|\gamma|_{W_\infty^2(0,1)} + |y|_{W_p^2(\Omega)}\right) + \|u - U\|_{L^2(0,1)}$$

- **Adjoint variables:**

$$|r - R|_{W_q^1(\Omega)} + |s - S|_{W_1^1(0,1)} \lesssim h\left(|s|_{W_1^2(0,1)} + |r|_{W_q^2(\Omega)} + |\gamma|_{W_\infty^2(0,1)} + |y|_{W_p^2(\Omega)}\right) + \|u - U\|_{L^2(0,1)}$$

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A Priori Error Estimates for the Control

Theorem. Let both h_0 and $|g|_{W_p^1(\Omega)}$ be sufficiently small. If $h \leq h_0$, then

$$\left\|\bar{u} - \bar{U}\right\|_{L^2(0,1)} \leq \frac{4}{\lambda}\left\|s(\bar{U}) - S(\bar{U})\right\|_{L^2(0,1)},$$

where $s(\bar{U})$ is the solution of the continuous adjoint equation with $\big(\gamma(\bar{U}), y(\bar{U})\big)$ solutions of the continuous state equations with control $\bar{U}$, and $S(\bar{U})$ is the solution of the discrete adjoint equations.

A Priori Error Estimates for the Control

Theorem. Let both h_0 and $|g|_{W_p^1(\Omega)}$ be sufficiently small. If $h \le h_0$, then

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Proof.

- $\frac{\lambda}{4}\left\|\bar{U} - \bar{u}\right\|^2_{L^2(0,1)} \le \left\langle \mathcal{J}'(\bar{U}) - \mathcal{J}'(\bar{u}), \bar{U} - \bar{u}\right\rangle_{L^2(0,1)\times L^2(0,1)} = RHS$
- $RHS = \langle \mathcal{J}'(\bar{U}) - \mathcal{J}_h'(\bar{U}), \bar{U} - \bar{u}\rangle + \langle \mathcal{J}_h'(\bar{U}), \bar{U} - \bar{u}\rangle + \underbrace{\langle \mathcal{J}'(\bar{u}), \bar{u} - \bar{U}\rangle}_{\le 0}.$
- $\langle \mathcal{J}_h'(\bar{U}), \bar{U} - \bar{u}\rangle = \underbrace{\langle \mathcal{J}_h'(\bar{U}), P_h\bar{u} - \bar{u}\rangle}_{=0} + \underbrace{\langle \mathcal{J}_h'(\bar{U}), \bar{U} - P_h\bar{u}\rangle}_{\le 0}$
- $\frac{\lambda}{4}\left\|\bar{U} - \bar{u}\right\|^2_{L^2(0,1)} \le \langle \underbrace{\lambda\bar{U} + s(\bar{U})}_{=\mathcal{J}'(\bar{U})} - \underbrace{\lambda\bar{U} + S(\bar{U})}_{=\mathcal{J}_h'(\bar{U})}, \bar{U} - \bar{u}\rangle = \langle s(\bar{U}) - S(\bar{U})\bar{U} - \bar{u}\rangle$

Rates of Convergence

Corollary. If $h \le h_0$, then there is a constant $C_0 \ge 1$, depending on $|\gamma|_{W^2_\infty(0,1)}$, $|y|_{W^2_p(\Omega)}$, $\|s\|_{W^2_1(0,1)}$, $\|r\|_{W^2_q(\Omega)}$, $\|\gamma_d\|_{L^2(0,1)}$, such that

$$\left|\bar\gamma - \bar G\right|_{W^1_\infty(0,1)} + \left|\bar y - \bar Y\right|_{W^1_p(\Omega)} + \left|\bar s - \bar S\right|_{W^1_1(0,1)} + \left|\bar r - \bar R\right|_{W^1_q(\Omega)} + \lambda \left\|\bar u - \bar U\right\|_{L^2(0,1)} \le C_0 h.$$

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$$\left|\bar{\gamma} - \bar{G}\right|_{W^1_\infty(0,1)} + \left|\bar{y} - \bar{Y}\right|_{W^1_p(\Omega)} + \left|\bar{s} - \bar{S}\right|_{W^1_1(0,1)} + \left|\bar{r} - \bar{R}\right|_{W^1_q(\Omega)} + \lambda \left\|\bar{u} - \bar{U}\right\|_{L^2(0,1)} \leq C_0 h.$$

- $\left\|s(\bar{U}) - S(\bar{U})\right\|_{L^2(0,1)} \leq \left|s(\bar{U}) - S(\bar{U})\right|_{W^1_1(0,1)}$
- For $u = U = \bar{U}$ we get

$$\left\|s(\bar{U}) - S(\bar{U})\right\|_{L^2(0,1)} \leq C_1 h$$

Simulations: Data

- Target function γ_d:

Figure : The inverted hat function indicates the desired state γ_d and the colors indicate the state y corresponding to the configuration γ_d. This profile γ_d is not achievable because $\gamma \in W^2_\infty(0,1)$.

- Admissible controls:

$$\mathcal{U}_{ad} = \left\{ u \in L^2(0,1) : \|u\|_{L^2(0,1)} \leq 3 \right\}$$

- Surface tension: $\kappa \approx 1$

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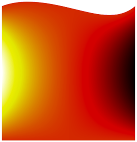
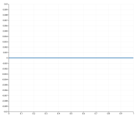
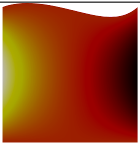
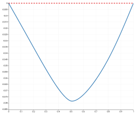
Simulations: Unconstrained Case

$(\bar{\gamma}, \bar{y})$	$\bar{u}$
$\lambda = \infty$, $\bar{u} \in [0, 0]$	
	
$\lambda = 1$, $\bar{u} \in (-0.0784971, 0)$	
	

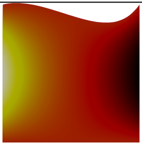
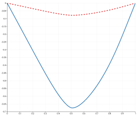
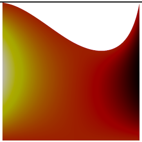
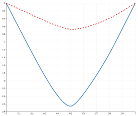
$(\bar{\gamma}, \bar{y})$	$\bar{u}$
$\lambda = 1e-1$, $\bar{u} \in (-0.675277, 0)$	
	
$\lambda = 1e-2$, $\bar{u} \in (-2.65675, 0)$	
	

Figure : The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. Each picture displays the corresponding value of λ from $\lambda = \infty$ to $\lambda =$ 1e-2.

Unconstrained Case (continued)

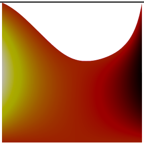
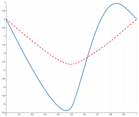
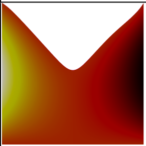
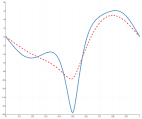
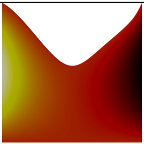
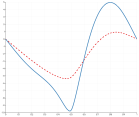
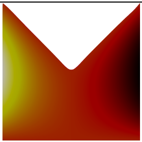
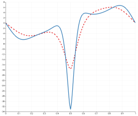
$(\bar{\gamma}, \bar{y})$	$\bar{u}$	$(\bar{\gamma}, \bar{y})$	$\bar{u}$
$\lambda = 1e-3,\ \bar{u} \in (-5.38, 0.93)$		$\lambda = 1e-5,\ \bar{u} \in (-17.63, 6.07)$	
			
$\lambda = 1e-4,\ \bar{u} \in (-9.78, 4.96)$		$\lambda = 1e-6,\ \bar{u} \in (-33.1146, 6.73)$	
			

Figure : The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. Each picture displays the corresponding value of λ from $\lambda = 1\text{e-}3$ to $\lambda = 1\text{e-}6$.

Simulations: Constrained Case

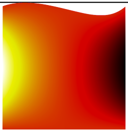
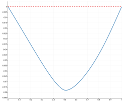
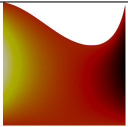
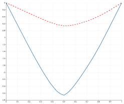
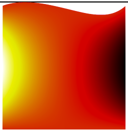
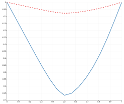
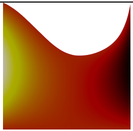
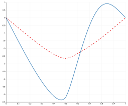
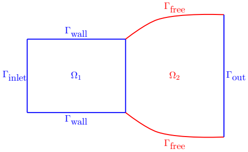
$(\bar{\gamma}, \bar{y})$	$\bar{u}$	$(\bar{\gamma}, \bar{y})$	$\bar{u}$
$\lambda = 1,\ \bar{u} \in (-.0780707, 0)$		$\lambda = 1e-2,\ \bar{u} \in (-2.65713, 0)$	
			
$\lambda = 1e-1,\ \bar{u} \in (-0.665452, 0)$		$\lambda = 1e-3,\ \bar{u} \in (-5.37, 0.93)$	
			

Figure : The optimal state solution $(\bar{\gamma}, \bar{y})$, the applied control $\bar{u}$ in solid blue, and the previous control in dashed red for comparison. The pictures show the corresponding value of λ, from $\lambda = 1$ to $\lambda =$ 1e-3, as well as the smallest and largest value of control. Notice that there is no visual difference between the optimal control for $\lambda =$ 1e-3, 1e-4 and 1e-5 (not displayed). This is because the control constraints are active.

H. Antil, R. H. Nochetto, and P. Sodré.
Optimal control of a free boundary problem: Analysis with second order sufficient conditions.
Accepted. SIAM Journal on Control and Optimization. arXiv:1210.0031, 2014.

H. Antil, R. H. Nochetto, and P. Sodré.
Optimal control of a free boundary problem with surface tension effects: A priori error analysis.
Submitted: arXiv:1402.5709, 2014.

Motivation: Stokes FBP



Courtesy: M. Verani (MOX, Politecnico di Milano)

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{\sigma}) = \boldsymbol{f}, \quad & \operatorname{div}(\boldsymbol{u}) = 0 && \text{in } \Omega \\
\boldsymbol{u} = \boldsymbol{g} \quad & && \text{on } \Gamma_{\text{inlet}} \cup \Gamma_{\text{wall}} \\
\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{0} \quad & && \text{on } \Gamma_{\text{out}} \\
\boldsymbol{u} \cdot \boldsymbol{\nu} = 0, \quad & \boldsymbol{\sigma}\boldsymbol{\nu} = \kappa \mathcal{H} \boldsymbol{\nu} && \text{on } \Gamma_{\text{free}},
\end{aligned}
$$

where $\boldsymbol{\sigma} = \eta\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right) - p\boldsymbol{I}$ is the stress tensor, η is viscosity, κ is surface tension.

Problem Formulation

- **Sobolev domain:** $\Omega \subset \mathbb{R}^d$ is of class $W_s^{2-1/s}$, with $s > d$.

- **Stokes equations:**

$$-\operatorname{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{u}, p\right)\right) = \boldsymbol{f}, \quad \operatorname{div}\left(\boldsymbol{u}\right) = g \quad \text{in } \Omega,$$

  where

$$\boldsymbol{\sigma} = 2\eta\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{I}p, \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}.$$

- **Navier boundary condition** (slip with friction):

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = \phi, \quad \beta \boldsymbol{T}\boldsymbol{u} + \boldsymbol{T}\boldsymbol{\sigma}\left(\boldsymbol{u}, p\right)\boldsymbol{\nu} = \boldsymbol{\psi} \quad \text{on } \partial\Omega,$$

  where $\beta \geq 0$ is the friction coefficient and $\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ is the projection operator into the tangent plane of $\partial\Omega$;

- **Slip boundary condition:** assume $\beta = 0$ (and $\phi = 0$).

References: Stokes with Slip BC

- **Solonnikov and Ščadilov** (1973): pioneering work in Hölder spaces;
- **Beirão da Veiga** (2004): $C^{1,1}$-domains and a Hilbert space setting;
- **Amrouche and Seloula** (2011): $C^{1,1}$-domains and reflexive Sobolev spaces;
- **Mitrea and Monniaux** (2009): time-dependent Navier-Stokes on C^2 domains;
- **Berselli** (2010): comprehensive survey.
- **Galdi, Simader and Sohr** (1994): pure slip boundary condition ($\boldsymbol{u} = \boldsymbol{0}$) but C^1 domains.

Sobolev Regularity of the Domain

- **Basic regularity:** Ω is of class $W_s^{2-1/s}$ for $s > d$ (dimension).
- **Regularity of unit normal:** $\boldsymbol{\nu} \in W_s^{1-1/s} \subset C^0$
- **Meaning of trace $\boldsymbol{u} \cdot \boldsymbol{\nu}$:** If $\boldsymbol{u} \in W_r^1(\Omega)$, with $s' \leq r \leq s$, then $\boldsymbol{u}|_{\partial\Omega} \in W_r^{1-1/r}(\partial\Omega)$ and

$$\boldsymbol{u} \cdot \boldsymbol{\nu} \in W_r^{1-1/r}(\partial\Omega);$$

- **Lipschitz regularity:** If Ω is Lipschitz, then $\boldsymbol{u} \cdot \boldsymbol{\nu}$ is not in a useful Sobolev space;
- **Piola transform $\boldsymbol{P}$:** This is instrumental to flatten the domain Ω and preserve the normals. The minimal regularity of $\boldsymbol{P}$ seems to be $W_s^2(\Omega)$ for $s > d$, which implies that the regularity $W_s^{2-1/s}$ for Ω is nearly optimal.

Function Spaces

- **Kernel of Stokes operator:**

  $$Z(\Omega) := \{\boldsymbol{z}(x) = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{b} : \boldsymbol{x} \in \Omega, \boldsymbol{A} = -\boldsymbol{A}^T \in \mathbb{R}^{d\times d}, \boldsymbol{b} \in \mathbb{R}^n, \boldsymbol{z}\cdot\boldsymbol{\nu}\big|_{\partial\Omega} = 0\};$$

- **Velocity space:**

  $$\mathring{V}_r(\Omega) := \left\{\boldsymbol{v} \in W_r^1(\Omega)/Z(\Omega) : \boldsymbol{v}\cdot\boldsymbol{\nu} = 0\right\};$$

- **Pressure space:**

  $$L_0^r(\Omega) := L^r(\Omega)/\mathbb{R};$$

- **Stokes space:** for $s' \leq r \leq s$, $s > d$, let

  $$\mathring{X}_r(\Omega) := \mathring{V}_r(\Omega) \times L_0^r(\Omega).$$

Variational Formulation of Slip BC: Main Result

Theorem. Let Ω be is class $W_s^{2-1/s}$ for $s > d$, and $s' \leq r \leq s$. Then, for every $\mathcal{F} \in \mathring{X}_{r'}(\Omega)^*$, there exists a unique $(\boldsymbol{u}, p) \in \mathring{X}_r(\Omega)$ such that

$$\mathcal{S}_\Omega\left(\boldsymbol{u}, p\right)\left(\boldsymbol{v}, q\right) = \mathcal{F}\left(\boldsymbol{v}, q\right) \quad \forall \left(\boldsymbol{v}, q\right) \in \mathring{X}_{r'}(\Omega)$$

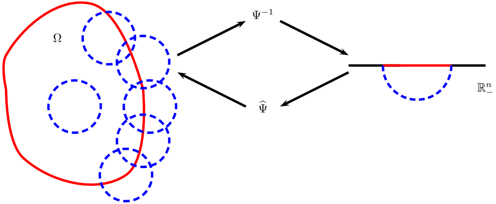
and

$$\left\|(\boldsymbol{u}, p)\right\|_{X_r(\Omega)} \leq C_{\Omega,\eta,n,r}\left(\left\|\mathcal{F}\right\|_{X_{r'}(\Omega)} + \left\|\phi\right\|_{W_r^{1-1/r}(\partial\Omega)}\right)$$

where the Stokes operator $\mathcal{S}_\Omega$ in Ω reads

$$\mathcal{S}_\Omega\left(\boldsymbol{u}, p\right)\left(\boldsymbol{v}, q\right) := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p\,\mathrm{div}\left(\boldsymbol{v}\right) + q\,\mathrm{div}\left(\boldsymbol{u}\right).$$

Step 1: Domain Decomposition Technique



- Cover the domain with finite number of balls $B(x_i, \delta_i/2)$:

$$\overline{\Omega} \subset \cup_{i=1}^{k} B(x_i, \delta_i/2).$$

- Associate to covering a smooth partition of unity $\{\varphi_i\}_{i=1}^{k}$.
- Let $\{\varrho_i\}_{i=1}^{k}$ be smooth cut-off functions so that supp $\varrho_i \subset B(x_i, \delta_i)$, $\rho_i = 1$ on $B(x_i, \delta_i/2)$.
- Use Piola transform $\boldsymbol{P} = \nabla\Psi / \det \nabla\Psi$ to write transformed vector fields:

$$(\mathsf{v}, \mathsf{q}) \mapsto (\boldsymbol{P}\mathsf{v}, \mathsf{q}) \circ \Psi^{-1} = (\boldsymbol{v}, q)$$

$$(\boldsymbol{v}, q) \mapsto \left(\boldsymbol{P}^{-1}\boldsymbol{v}, q\right) \circ \Psi = (\mathsf{v}, \mathsf{q})$$

- **Key property:** $\mathsf{v} \cdot \boldsymbol{\nu}_{\mathsf{x}} \, \mathrm{ds} = \boldsymbol{v} \cdot \boldsymbol{\nu}_{\boldsymbol{x}} \, \mathrm{d}s.$

Remaining Steps

- **Space decomposition:** This step requires Ω to be of class $W_s^{2-1/s}$ with $s > d$;
- **Operator localization:** Write the local operator formally as an invertible part plus a compact perturbation;
- **Pseudo-inverse:** Show that $\mathcal{S}_\Omega^\dagger \mathcal{S}_\Omega = I_{X_r(\Omega)}$ plus compact perturbation;
- **Injectivity:** Show that both $\mathcal{S}_\Omega$ and $\mathcal{S}_\Omega^*$ are injective;
- **Index theory of Fredholm operators:** Conclude that $\mathcal{S}_\Omega$ is bijective;

Space Decomposition

Projection map

$$\mathcal{R}_{\varphi_i} : \mathring{X}_r(\mathsf{Q}_i) \to \mathring{X}_r(\Omega), \qquad \mathsf{Q}_i = \mathbb{R}^d \text{ or } \mathbb{R}^d_-$$
$$(\mathsf{v}, \mathsf{q}) \mapsto \varphi_i \mathcal{P}_i (\mathsf{v}, \mathsf{q})$$

Restriction map

$$\mathcal{E}_{\varrho_i} : \mathring{X}_r(\Omega) \to \mathring{X}_r(\mathsf{Q}_i)$$
$$(\boldsymbol{v}, q) \mapsto \mathcal{P}_i^{-1} (\varrho_i \boldsymbol{v}, \varrho_i q)$$

continuous only when Piola matrix is in $W^2_s(\Omega)$.

- Given $(\boldsymbol{u}, p) \in \mathring{X}_r(\Omega)$, we have

$$(\boldsymbol{u}, p) = \sum_{i=1}^{k} \varphi_i (\boldsymbol{u}, p) = \sum_{i=1}^{k} \varphi_i (\varrho_i \boldsymbol{u}, \varrho_i p) = \sum_{i=1}^{k} \varphi_i \mathcal{P}_i \mathcal{P}_i^{-1} (\varrho_i \boldsymbol{u}, \varrho_i p)$$
$$= \sum_{i=1}^{k} \mathcal{R}_{\varphi_i} \underbrace{\mathcal{E}_{\varrho_i} (\boldsymbol{u}, p)}_{\in \mathring{X}_r(\mathsf{Q}_i)} .$$

  which implies $\mathring{X}_r(\Omega) = \sum_{i=1}^{k} \mathcal{R}_{\varphi_i} \mathring{X}_r(\mathsf{Q}_i)$.

- Similarly for the dual space

$$\mathring{X}_r(\Omega)^* = \sum^{k} \mathcal{R}^*_{\varphi_i} \mathring{X}_r(\mathsf{Q}_i)^* .$$

Operator Decomposition

$$\begin{aligned}\mathcal{S}_\Omega(\boldsymbol{u},p)\,\mathcal{R}_{\varphi_i}(\mathsf{v},\mathsf{q}) &= \big(\mathcal{S}_{\Omega_i}(\varphi_i\boldsymbol{u},\varphi_i p)+\mathcal{K}_{\varphi_i}(\boldsymbol{u},p)\big)\,\mathcal{P}_i(\mathsf{v},\mathsf{q})\\ &= \underbrace{\widetilde{\mathcal{S}}_i}_{\text{Invertible}}\quad \mathcal{E}_{\varphi_i}(\boldsymbol{u},p)(\mathsf{v},\mathsf{q})\\ &\quad+\underbrace{\mathcal{C}_i\mathcal{E}_{\varphi_i}(\boldsymbol{u})(\mathsf{v})+\mathcal{K}_{\varphi_i}(\boldsymbol{u},p)\,\mathcal{P}_i(\mathsf{v},\mathsf{q})}_{\text{Compact}}\end{aligned}$$

Fredholm operator

Consider the operator

$$\mathcal{S}_\Omega^\dagger := \sum_{i=1}^{k} \mathcal{R}_{\varrho_i} \widetilde{\mathcal{S}}_i^{-1} \mathcal{R}_{\varphi_i}^* .$$

Then

$$\mathcal{S}_\Omega^\dagger \mathcal{S}_\Omega = \mathcal{I}_{X_r(\Omega)} + \sum_{i=1}^{k} \underbrace{\mathcal{R}_{\varrho_i} \widetilde{\mathcal{S}}_i^{-1} \left(\mathcal{C}_i \mathcal{E}_{\varphi_i} + \mathcal{P}_i^* \mathcal{K}_i \right)}_{\text{compact}} .$$

Similarly

$$\mathcal{S}_\Omega \mathcal{S}_\Omega^\dagger = \text{identity} + \text{compact}.$$

Therefore $\mathcal{S}_\Omega$ has a pseudo-inverse (equivalently Fredholm), which implies

$$\dim N_{\mathcal{S}_\Omega} < \infty, \quad \operatorname{codim} R_{\mathcal{S}_\Omega} < \infty.$$

$\mathcal{S}_\Omega$ and $\mathcal{S}_\Omega^*$ are Injective

- Problem satisfies the Brezzi's theorem for Hilbert space case. This ensures the uniqueness of solution for

$$2 \le r \le s.$$

- Let $r_0 = s' < 2$ and $(\boldsymbol{u}, p) \in \mathring{X}_{r_0}(\Omega)$.
  - Consider the homogeneous problem

$$\mathcal{S}_\Omega(\boldsymbol{u}, p) = 0,$$

    we need to show that $(\boldsymbol{u}, p) = 0$.
  - Recall

$$\mathcal{S} = \underbrace{\widetilde{\mathcal{S}}_i}_{\text{invertible}} + \underbrace{\mathcal{C}_i + \mathcal{K}_{\varphi_i}}_{\text{compact}}.$$

    whence

$$\widetilde{\mathcal{S}}_i(\boldsymbol{u}, p) = -(\mathcal{C}_i + \mathcal{K}_{\varphi_i})(\boldsymbol{u}, p).$$

  - We improve the integrability of $(\boldsymbol{u}, p)$ to some $r_k \ge 2$, to conclude.

Index Theory of Fredholm Operators

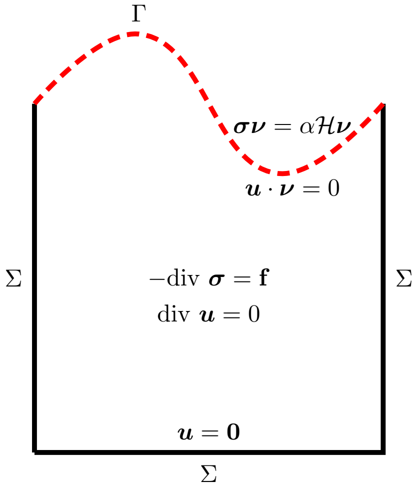
Let $\mathcal{A} : X \to Y$ has a pseudo-inverse. $\mathcal{A}$ is bijective if and only if $\mathcal{A}$ and $\mathcal{A}^*$ are injective.

Summary:

- Using index theory we have shown the well-posedness of the Stokes problem with slip boundary condition.
  - under mild domain regularity i.e. $C^{1,\epsilon}$, earlier result Amrouche '11 $C^{1,1}$ domain.
- We have provided a constructive approach based on domain decomposition.
- **Navier condition:** Extend to $\beta > 0$.

"dimension independent"

The Stokes Free Boundary Problem



- **Implicit Variational Formulation:**

$$\begin{aligned} \left\langle \mathcal{T}^*\mathcal{H}(\Omega) + \mathcal{S}\left(\boldsymbol{u}, p; \Omega\right) - \mathcal{F}(\Omega), (\boldsymbol{v}, q)\right\rangle &= 0 \qquad \forall\, (\boldsymbol{v}, q) \in D(\Omega) \\ \boldsymbol{u}\cdot\boldsymbol{\nu} &= 0 \qquad \text{on } \Gamma \\ \boldsymbol{u} &= 0 \qquad \text{on } \Sigma. \end{aligned}$$

where

$$\begin{aligned} \mathcal{T}^*\mathcal{H}(\Omega)(\boldsymbol{v}) = \left\langle \mathcal{H}(\Omega), \mathcal{T}\boldsymbol{v}\right\rangle_\Gamma &:= \kappa \int_\Gamma \mathcal{H}\boldsymbol{v}\cdot\boldsymbol{\nu} ds, \\ \mathcal{S}\left(\boldsymbol{u}, p; \Omega\right)(\boldsymbol{v}, q) &:= \int_\Omega \eta\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p\,\mathrm{div}\,(\boldsymbol{v}) + q\,\mathrm{div}\,(\boldsymbol{u})\, dx, \\ \mathcal{F}(\Omega)\,(\boldsymbol{v}, q) &:= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + gq dx. \end{aligned}$$

- **Reformulation in Reference Domain:** $\Omega \to \widehat{\Omega}, \quad \mathcal{S} \to \widehat{\mathcal{S}}$

Implicit Function Theory: IFT

- **Nonlinear PDE:** Write problem above as a nonlinear problem

$$\mathcal{N}\left(\hat{\boldsymbol{u}}, \hat{p}, \omega; f\right) = 0$$

  in suitable Sobolev spaces for velocity-pressure $(\hat{\boldsymbol{u}}, \hat{p})$ and boundary parametrization ω;

- **Differentiability:** Show that the modified equations are Fréchet differentiable.

- **Reference configuration:** Verify that the problem is invertible at $\omega = 0$ (flat interface).

- **Apply IFT:** This gives a differentiable control-to-state map for free.

Conclusions and Extensions

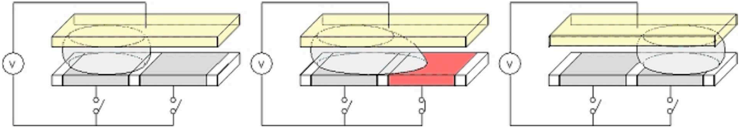
Conclusions:

- **Optimal control of model FBP with surface tension:** Show first and second order optimality conditions.
- **Discrete optimal control of a model FBP:** Show optimal error estimates for state, adjoint, and control variables.
- **Stokes with slip boundary condition:** Show existence of solution for domains of class $W_s^{2-1/s} \subset C^{1,\epsilon}$ (significant improvement over existing literature, i.e. $C^{1,1}$ regularity).
- **Stokes FBP:** Develop an abstract framework and apply it to Stokes.

Extensions and Open Problems:

- FEM analysis for Stokes FBP
- Optimal control for Stokes FBP
- Other FBPs.

Electrowetting on Dielectric (EWOD)



Courtesy: R. H. Nochetto, B. Shapiro, S. Walker (2009)

H. Antil, D. Wegner, M. Hintermüller, R. H. Nochetto, and T. M. Surowiec.
Instantaneous/optimal control of a semi-discrete EWOD with complementarity-based contact pinning conditions.
In Progress.