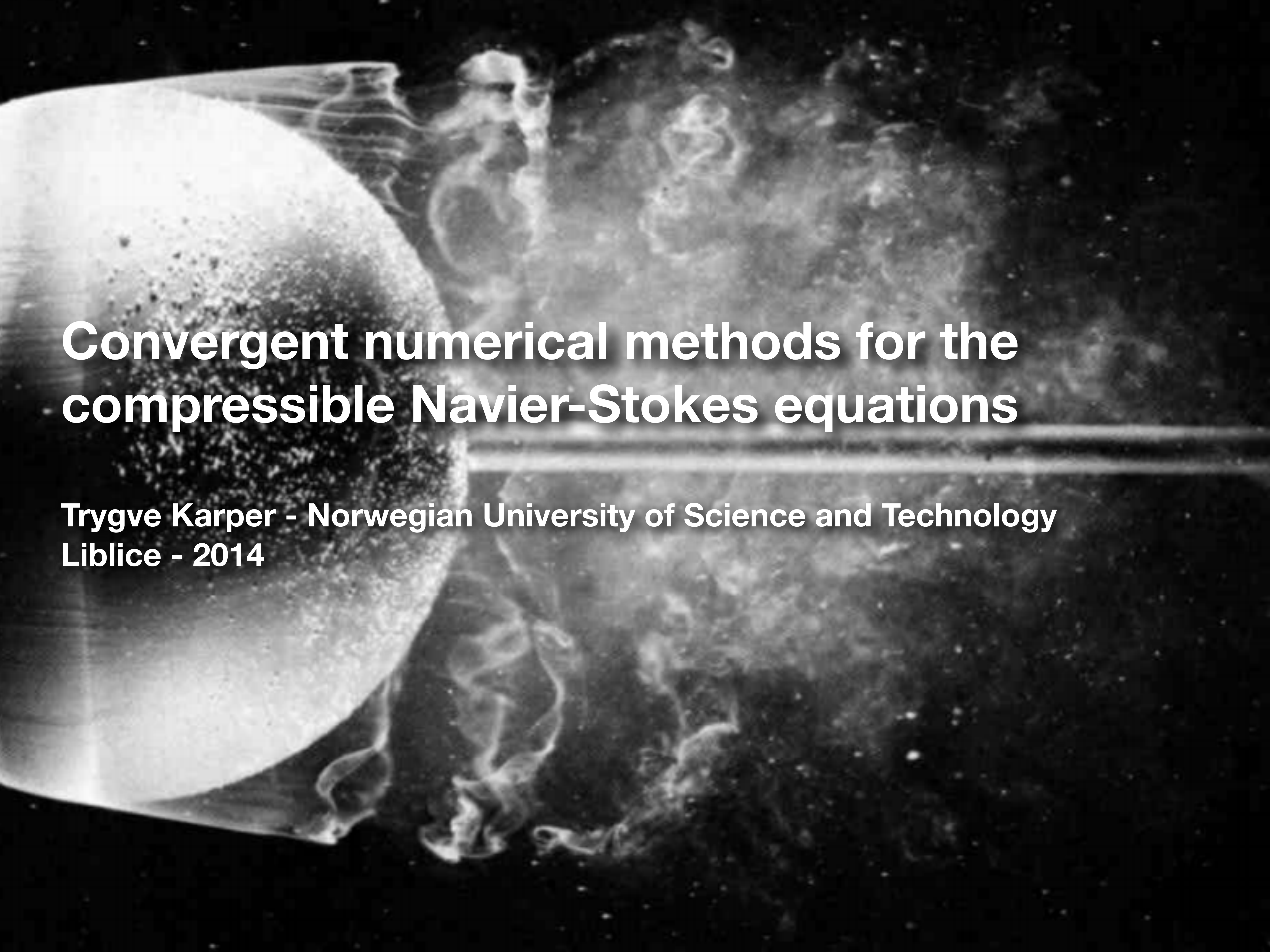

Convergent numerical methods for the compressible Navier-Stokes equations

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The equations

Viscous barotropic flow

$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$

$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \frac{1}{\mathrm{Ma}} \nabla p(\varrho) = \frac{1}{\mathrm{Re}} \operatorname{div} \sigma$$

Newtonian fluid

$$\sigma = \frac{\mu}{2} \left(\nabla u + \nabla u^T \right) + \lambda \operatorname{div} u \mathbb{I}$$

Isentropic pressure relation

$$p(\varrho) = a \varrho^{\gamma}$$

The equations

Lipschitz domain

$$\Omega \subset \mathbb{R}^3 \qquad t \in [0,T], \quad T - \text{finite}$$

Initial conditions

$$\varrho_0 \in L^{\gamma+1}(\Omega), \quad \int_\Omega \frac{a\varrho_0^\gamma}{\gamma - 1} + \varrho_0 |u_0|^2 \, dx \leq C$$

Goal: Construct and prove convergence of a numerical method

What is known analytically

$$\varrho_t + \mathrm{div}(\varrho u) = 0,$$
$$(\varrho u)_t + \mathrm{div}(\varrho u \otimes u) + \frac{1}{\mathrm{Ma}} \nabla p(\varrho) = \frac{1}{\mathrm{Re}} \Delta u$$

- Lions proved existence of global weak solutions for $\gamma > \frac{9}{5}$.
- Feireisl et. al. proved existence for $\gamma > \frac{3}{2}$.

Proof is accomplished by sending $\alpha, \delta \to 0$ in

$$\varrho_t + \mathrm{div}(\varrho u) = \alpha \Delta \varrho$$
$$(\varrho u)_t + \mathrm{div}(\varrho u \otimes u) + \nabla(p(\varrho) + \delta \varrho^4) = \Delta u$$

So, do the same for a numerical method and DONE?

What is known numerically

Gallouet et. al (2008-2012):

$$\begin{aligned} \operatorname{div}(\varrho u) &= 0 \\ -\Delta u + \nabla p(\varrho) &= f \end{aligned}$$

MAC Finite Volumes, Crouzeix-Raviart FEM, FVM for continuity

Karlsen-K. (2009-2011):

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho u) &= 0, \\ u_t - \Delta u + \nabla p(\varrho) &= f \end{aligned}$$

Nedelec elements, Crouzeix-Raviart, FVM for continuity

What is known numerically

K. (2013):

$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$
$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) = \Delta u$$

$\gamma > 3$

Crouzeix-Raviart finite elements

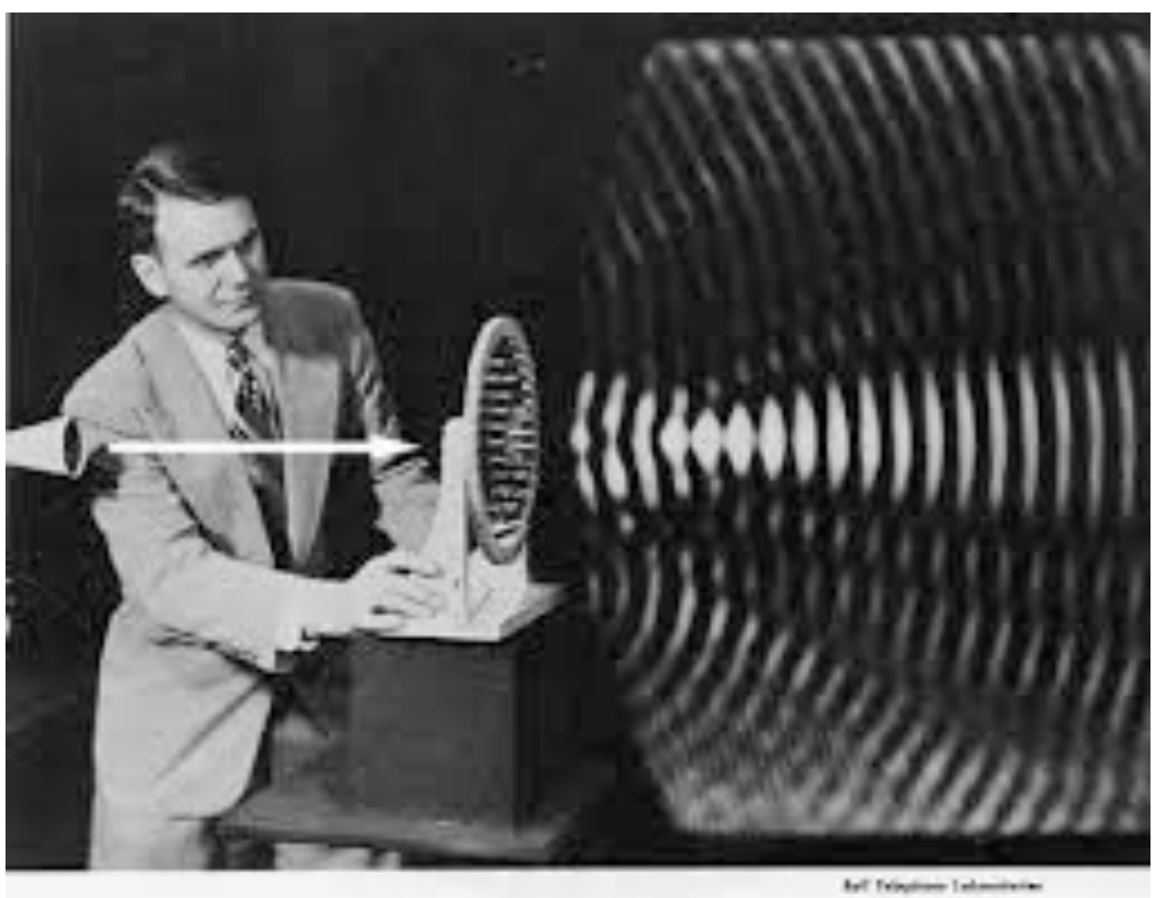
Now, things starts to become clear!

Why is this problem difficult?

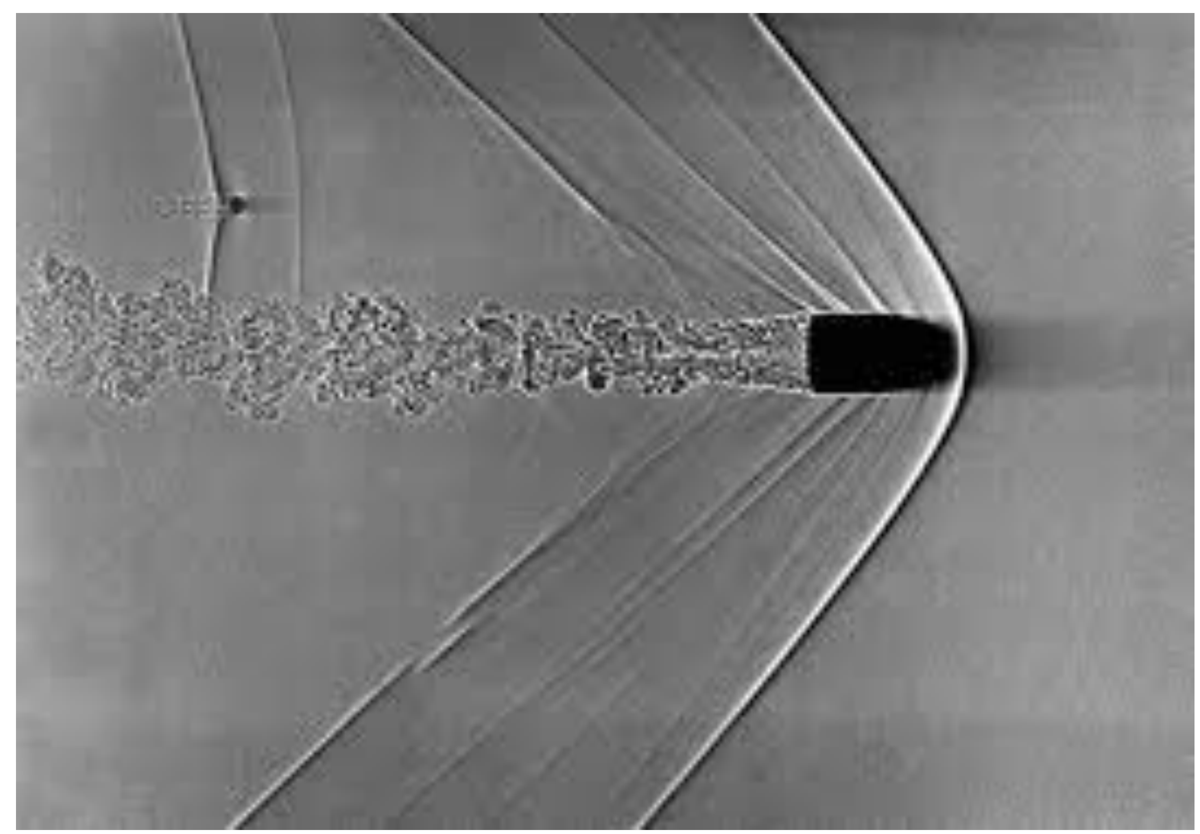
$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$

$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \frac{1}{\mathrm{Ma}} \nabla p(\varrho) = \frac{1}{\mathrm{Re}} \Delta u$$

Ma small



Re large



Discretization of the Euler equations

Let us look at the Euler equations:

$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$
$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) = 0$$

An Euler person will tell you that you should write

$$U_t + \operatorname{div} F(U) = 0$$

$$U = \begin{pmatrix} \varrho \\ \varrho u \end{pmatrix}$$

$$F(U) = \begin{pmatrix} 0 & \varrho u \\ p(\varrho) & \frac{\varrho u \otimes \varrho u}{\varrho} \end{pmatrix}$$

Discretization of the Euler equations

Now, you discretize this as any system of conservation laws

$$U_t + d_u F(U) \cdot \nabla_x U = 0$$

- Find eigenvalues and eigenvectors of $d_u F$ and upwind accordingly

In particular,

ϱ and u are approximated similarly

- Same order polynomials
- No dual mesh
- No staggered grids

Discretization of the Euler equations

Now, let us look at the low Ma case

$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$
$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \frac{1}{\epsilon}\nabla p(\varrho) = 0$$

Eigenvalues are of the form:

$$\lambda = u \pm \frac{1}{\epsilon}\sqrt{p'(\varrho)}$$

The usual strategy will fail!

Instead, people now use methods where

$$\operatorname{div} u_h \sim p(\varrho_h)$$

$$(u_h, \varrho_h) \in P^1 \times P^0, \quad \text{dual meshes}, \quad \text{staggered grid}$$

Pressure is IMPLICIT!

Discretization of the incompressible NS

If $Ma \to 0$, we get the incompressible NS

$$\operatorname{div} u = 0$$
$$u_t + \operatorname{div}(u \otimes u) + \nabla p = \Delta u$$

The whole finite element community will tell you that you need

$$\operatorname{div} V_h \sim Q_h$$

To satisfy something called the Babuska-Brezzi conditions

Finite Differences $\mapsto$ staggered grids
Finite Volumes $\mapsto$ dual meshes

"Concentration should match divergence"

Back to our equation

$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$
$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) = \Delta u$$

- Transport is Euler type of terms
- Pressure and viscosity are incompressible Navier-Stokes type of terms

This is reflected in the derivation of the Energy

This couples through u

$$\int \operatorname{div}(\varrho u \otimes u) u \, dx = \int \operatorname{div}(\varrho u) \frac{u^2}{2} \, dx = \int \varrho_t \frac{u^2}{2} \, dx$$

This couples through $\operatorname{div} u$

$$\int \nabla p(\varrho) u \, dx = -\int p(\varrho) \operatorname{div} u \, dx = -\int \frac{1}{\gamma - 1} p'(\varrho) \varrho_t \, dx$$

Back to our equation

Nearly all discretizations in the literature gives up one of the two

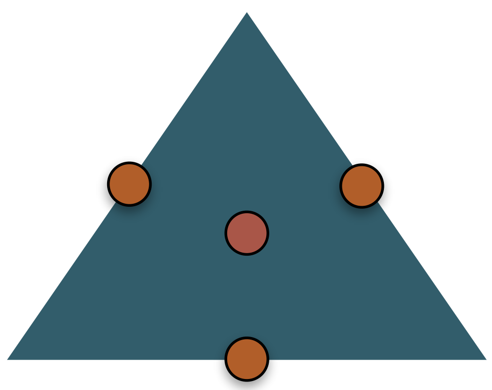
Euler: The first is easy, the second requires work
NS: The second is designed to work, the first requires work

To perform a convergence proof, you will need both!

$$\varrho_t + \operatorname{div}(\varrho u) = 0,$$
$$(\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) = \Delta u$$

The convergent method!

We will approximate



$\varrho_h \in Q_h$- space of piecewise constants

$u_h \in V_h$- Crouzeix-Raviart finite element space

We will use the following upwind flux

$$\text{Up}(m_h u_h) = (m_h u_h \cdot \nu)|_\Gamma = m_+(u_h \cdot \nu)^+ + m_-(u_h \cdot \nu)^-$$

$$(u_h \cdot \nu)^\pm = \max / \min \left\{ 0, \int_\Gamma u_h \cdot \nu \; dS \right\}$$

The continuity method

Find $\varrho_h^k \in Q_h$ such that

$$\int_\Omega (D_t \varrho_h^k)\phi_h \, dx - \sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho_h^k u_h^k)[\phi_h] \, dS(x) = 0$$

for all $\phi_h \in Q_h$.

For the convenience of analysis, the method is implicit

$$D_t \varrho_h^k = \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t}$$

Average upwinding

Fundamental idea:

$$\begin{aligned} \partial_t \varrho_h + \operatorname{div}_{\mathrm{up}}(\varrho_h u_h) &= 0, \\ \partial_t m_h + \operatorname{div}_{\mathrm{up}}(m_h \otimes u_h) + \nabla_h p(\varrho_h) &= \Delta_h u_h \\ m_h &= \varrho_h \Pi_h^Q[u_h] \end{aligned}$$

Hence, momentum transport is handled exactly as density transport.

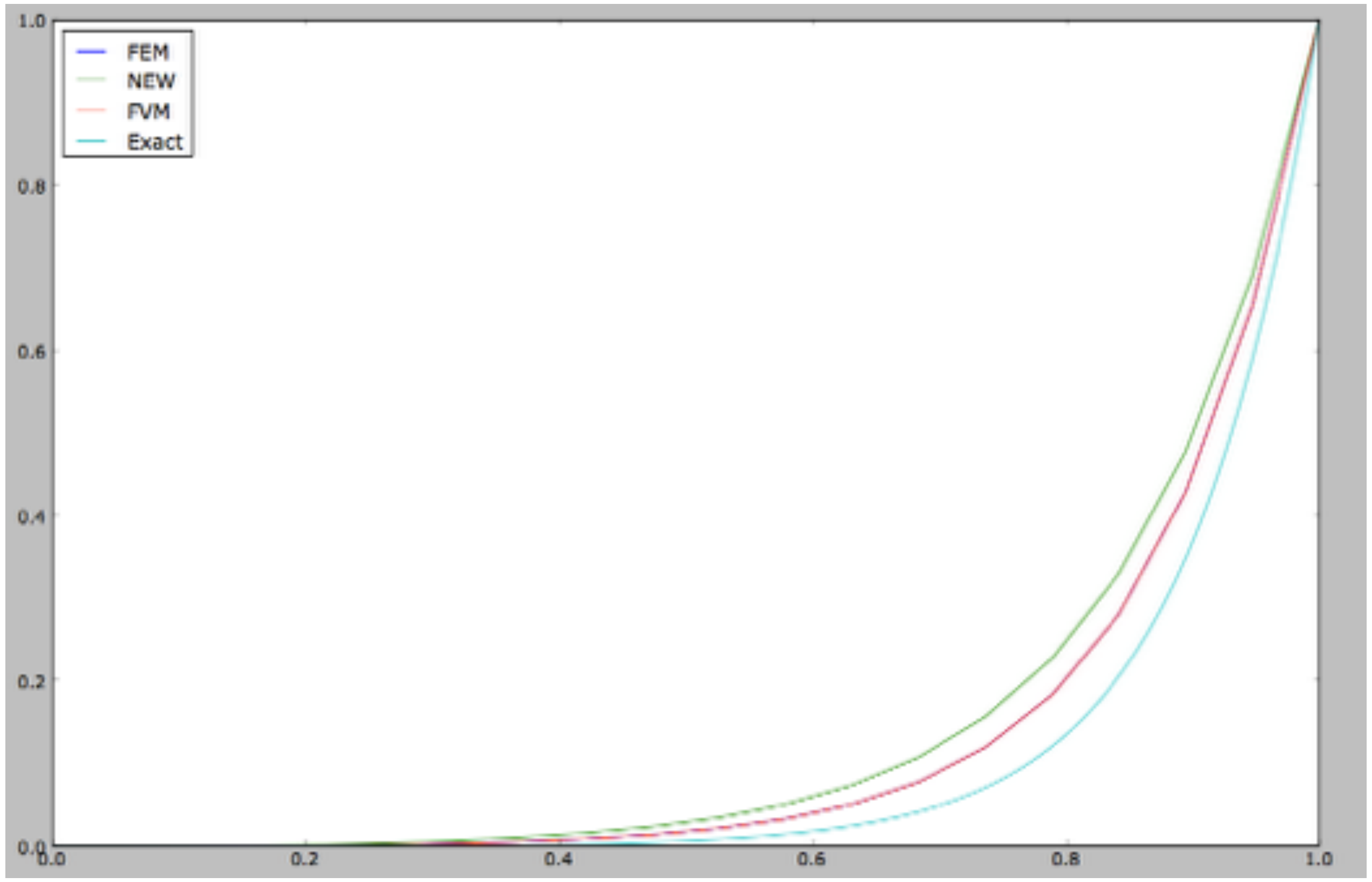
$$\int \operatorname{div}_{\mathrm{up}}(m_h \otimes u_h) u_h \, dx = \int \operatorname{div}_{\mathrm{up}}(\varrho_h u_h) \frac{\left(\Pi_h^Q[u_h]\right)^2}{2} \, dx$$
$$+ \text{Numerical diffusion}$$

On the down-side

$$\partial_t \varrho_h + \mathrm{div}_{\mathrm{up}}(\varrho_h u_h) = 0,$$
$$\partial_t m_h + \mathrm{div}_{\mathrm{up}}(m_h \otimes u_h) + \nabla_h p(\varrho_h) = \Delta_h u_h$$
$$m_h = \varrho_h \Pi_h^Q [u_h]$$

Transport operator is singular (if $\mathrm{div}\, V_h \sim Q_h$)

More diffusive



The method reads

Find $\varrho_h^k \in Q_h$ such that

$$\int_\Omega (D_t \varrho_h^k)\phi_h \; dx - \sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho_h^k u_h^k)[\phi_h] \; dS(x) = 0$$

for all $\phi_h \in Q_h$.

Find $u_h^k \in V_h$ such that

$$\int_\Omega (D_t^h m_h^k) v_h \; dx - \sum_\Gamma \int_\Gamma \mathrm{Up}(m_h^k u_h^k)[\Pi_h^Q[v_h]]$$

$$+ \int_\Omega \nabla_h u_h \nabla_h v_h - p(\varrho_h^k) \operatorname{div} v_h \; dx = 0$$

$$m_h^k = \varrho_h^k \Pi_h^Q[u_h^k]$$

for all $v_h \in V_h$

To prove convergence….

Due to a technical issue:

$$\mathrm{Up}(\varrho_h u_h) = \frac{\varrho_+}{2}\left[(u_h\cdot\nu + c)^+ + (u_h\cdot\nu - c)^+\right] + \frac{\varrho_-}{2}\left[(u_h\cdot\nu + c)^- + (u_h\cdot\nu - c)^-\right]$$

If $u_h \cdot \nu$ is small, we still add diffusion

It is needed to control

$$\sum_\Gamma \int_\Gamma [\varrho_h]^2 \, dS(x)$$

Needed only for convergence...

Main result

Let $\{(\varrho_h, u_h)\}_{h>0}$ be a sequence of solutions of the method.

As, $h \to 0$, $\varrho_h \to \varrho$ and $u_h \rightharpoonup u$, where (ϱ, u) satisfies

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho u) &= 0, \\ (\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) &= \Delta u \end{aligned}$$

in the sense of distributions.

The energy

With the design of the method, it is rather easy to prove

$$E^k = \int_\Omega \varrho_h^k \left|\Pi_h^Q u_h^k\right|^2 + \frac{p(\varrho_h^k)}{\gamma - 1}\ dx$$

the energy identity

$$D_t E^k + \int_\Omega |\nabla_h u_h^k|^2\ dx + \text{Numerical diff} = 0$$

Hence, the method is unconditionally stable

How to prove convergence?

I like to write the methods in consistency form

$$\int_\Omega (D_t \varrho_h)\phi - \varrho_h u_h \nabla\phi \; dxdt = P_1(\phi)$$

$$\int_\Omega (D_t m_h) v - m_h \otimes u_h : \nabla v + \nabla_h u_h \nabla v - p(\varrho_h)\,\mathrm{div}\, v \; dx = P_2(v)$$

... and then try to control the weak error terms P_1 and P_2.

This is extremely convenient for analysis of nonlinear problems

Consistency formulation of the continuity equation

Start by setting $\phi_h = \Pi_h^Q[\phi]$ in

$$\int_\Omega (D_t \varrho_h)\phi_h \, dx - \sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho_h u_h)[\phi_h] \, dS(x) = 0$$

Now, observe that

$$\int_\Omega (D_t \varrho_h)\phi_h \, dx = \int_\Omega (D_t \varrho_h)\phi \, dx$$

Next, we write

$$\begin{aligned}
\sum_\Gamma \int_\Gamma \mathrm{Up}(\varrho_h u_h)[\phi_h] \, dS &= -\sum_E \int_{\partial E} \mathrm{Up}(\varrho_h u_h)\phi_+ \, dS(x) \\
&= -\sum_E \int_{\partial E} \mathrm{Up}(\varrho_h u_h)(\phi_+ - \phi) \, dS(x) \\
&= -\sum_E \int_{\partial E} \varrho_h (u_h \cdot \nu)(\phi_+ - \phi) - [\varrho_h](u_h \cdot \nu)^-(\phi_+ - \phi) \, dS(x) \\
&= -\int_\Omega \varrho_h u_h \nabla\phi \, dx + \sum_\Gamma \int_\Gamma [\varrho_h](u_h \cdot \nu)^-(\phi_+ - \phi) \, dS(x)
\end{aligned}$$

Consistency formulation of the continuity equation

Thus, we see that

$$\int_\Omega (D_t \varrho_h)\phi - \varrho_h u_h \nabla\phi \; dx = \underbrace{\boxed{\sum_E \int_{\partial E} [\varrho_h](u_h \cdot \nu)^-(\Pi_h^Q[\phi] - \phi)\; dS(x)}}_{P_1(\phi)}$$

To control this error, you will need the numerical diffusion

$$\sum_k \int_0^T \sum_\Gamma \int_\Gamma P''(\varrho_\dagger)[\varrho_h]^2 |u_h \cdot \nu| \; dS(x)dt \leq C$$

The easy bound: $P_1(\phi) \leq \sqrt{h}\|\nabla\phi\|_{L^\infty}$

The better bound: $P_1(\phi) \leq h^{\frac{1}{2} - 3\frac{4-\gamma}{4\gamma}} \|\nabla\phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega)}$

For this method, one can prove

$$\int_\Omega (D_t \varrho_h)\phi - \varrho_h u_h \nabla\phi \; dxdt = P_1(\phi)$$

$$\int_\Omega (D_t m_h) v - m_h \otimes u_h : \nabla v + \nabla_h u_h \nabla v - p(\varrho_h) \operatorname{div} v \; dx = P_2(v)$$

where the weak errors satisfy

$$P_1(\phi) \leq h^{\frac{1}{2} - 3\frac{4-\gamma}{4\gamma}} \|\nabla\phi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega)}$$

$$P_2(v) \leq h^{\alpha} \|\nabla v\|_{L^\infty(0,T;L^\gamma(\Omega))}$$

Now, we can try to use the existence framework on our method

Convergence of the continuity equation

We have that

$$\int_\Omega (D_t \varrho_h)\phi - \varrho_h u_h \nabla\phi \; dx = P_1(\phi)$$

Control in time on ϱ_h + control in space on u_h gives

$$\varrho_h u_h \rightharpoonup \varrho u \quad \text{as } h \to 0$$

Thus, there is no problems with passing to the limit

$$\int_0^T \int_\Omega \varrho(\phi_t + u\nabla\phi) \; dxdt = \int_\Omega \varrho_0 \phi(0,\cdot) \; dx$$

The momentum equation

We have that

$$\int_\Omega (D_t m_h) v - m_h \otimes u_h : \nabla v + \nabla_h u_h \nabla v - p(\varrho_h) \operatorname{div} v \ dx = P_2(v)$$

Again, control in time on ϱ_h + space on u_h,

$$m_h \rightharpoonup \varrho u$$
$$m_h \otimes u_h \rightharpoonup \varrho u \otimes u$$

We can pass to the limit to conclude

$$- \int_0^T \int_\Omega \varrho u v_t + \varrho u \otimes u : \nabla v \ dxdt + \int_0^T \int_\Omega \nabla u \nabla v \ dxdt$$
$$= \int_0^T \int_\Omega \overline{p(\varrho)} \operatorname{div} v \ dxdt - \int_\Omega \varrho_0 u_0 v(0, \cdot) \ dx$$

Here is where things become difficult

$$\text{Is } \overline{p(\varrho)} = p(\varrho) \text{ ??}$$

Well, what do we know about ϱ_h?

$$\varrho_h \in L^\infty(0,T;L^\gamma(\Omega)) \Rightarrow p(\varrho_h) \in L^\infty(0,T;L^1(\Omega))$$

Hence, we do not even know if $\overline{p(\varrho_h)} \in L^\infty(0,T;L^1(\Omega))$

and we definitely don't know that $\varrho_h \to \varrho$

There is no time to show you both, let us do the first!

Higher integrability

Let us return to the consistency formulation

$$\int_\Omega (D_t m_h)v - m_h \otimes u_h : \nabla v + \nabla_h u_h \nabla v - p(\varrho_h) \operatorname{div} v \; dx = P_2(v)$$

Set $v = B[\varrho_h]$, where $B[]$ is the Bogovskii operator

$$\operatorname{div} B[\varrho_h] = \varrho_h, \qquad \|\nabla B[\varrho_h]\|_{L^p} \leq \|\varrho_h\|_{L^p}, \qquad \|B[q_h]\|_{L^q} \leq \|q_h\|_{W^{-1,q}}$$

Then, we have that

$$\begin{aligned}\int_0^T \int_\Omega p(\varrho_h)\varrho_h \; dxdt &= \text{bounded terms} + \int_0^T \int_\Omega D_t m_h v \; dxdt \\ &= \text{bounded terms} - \int_0^T \int_\Omega m_h B[D_t \varrho_h] \; dxdt \leq C\end{aligned}$$

Thank you!!