LMU Munich, Germany • Lars Diening

Finite elements for electrorheological fluids

based on joint work with

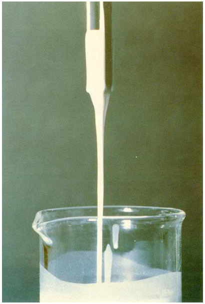
Berselli, Breit, Schwarzacher

Electrorheological fluids

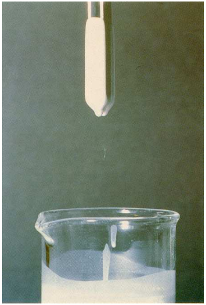
Discovered by Winslow '49

Electrical field changes viscosity significantly! [play movie]

(1000 V/mm $\Leftrightarrow$ viscosity$\times$1000)



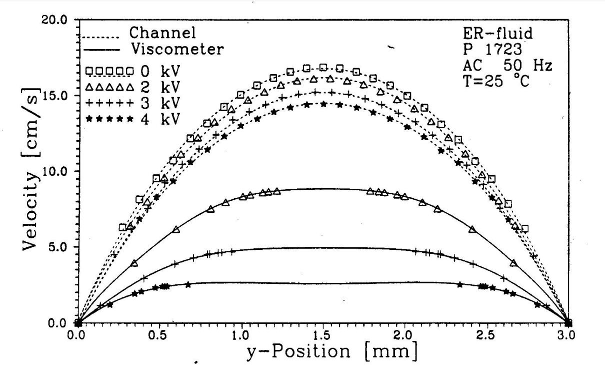
electrical field is OFF



electrical field is ON

Electrorheological fluids – plug flow

Experiment by Wunder



Conclusion: Extra stress (friction) depends on electrical field.

Electrorheological fluids – model

Model by Rajagopal and Růžička '96:

$$\partial_t \mathbf{v} - \operatorname{div} \mathbf{S} + \nabla q + [\nabla \mathbf{v}]\mathbf{v} = \mathbf{f},$$
$$\operatorname{div} \mathbf{v} = 0,$$

with velocity $\mathbf{v}$, pressure q, extra stress $\mathbf{S}$, electric field $\mathbf{E}$.

The equation for $\mathbf{E}$ decouples.

Moreover, $\mathbf{S} = \mathbf{S}(\mathbf{E}, \varepsilon(\mathbf{v}))$ behaves like

$$\mathbf{S}\big(\mathbf{E}, \varepsilon(\mathbf{v})\big) = (1 + |\varepsilon(\mathbf{v})|)^{p(t,x)-2}\varepsilon(\mathbf{v})$$

with symmetric gradient $\varepsilon(\mathbf{v}) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2}$ and $p(t,x) = p(|\mathbf{E}(t,x)|)$.

Goal

Numerical analysis for finite element solutions.

Aspects of the (stationary) model:

$$
\begin{aligned}
-\operatorname{div} \mathbf{S} + \nabla q + [\nabla \mathbf{v}]\mathbf{v} &= \mathbf{f},\\
\operatorname{div} \mathbf{v} &= 0,\\
\mathbf{S}\big(\mathbf{E}, \varepsilon(\mathbf{v})\big) &= (1 + |\varepsilon(\mathbf{v})|)^{p(x)-2}\varepsilon(\mathbf{v}).
\end{aligned}
$$

Simplifications (to be removed later)

- Assume constant electric field $p(x) = p \in \mathbb{R}$. ($\Rightarrow$ power law fluids)
- Neglect convection $[\nabla \mathbf{v}]\mathbf{v}$.
- Remove pressure ∇q and incompressibility $\operatorname{div} \mathbf{v}$.
- Replace symmetric gradients $\varepsilon(\mathbf{v})$ by gradients $\nabla \mathbf{v}$.

Reduced model

$$
-\operatorname{div}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big) = \mathbf{f} \qquad (p\text{-Laplacian})
$$

Finite elements for the p-Laplacian

Simplified problem: (constant p, no convection, no pressure, gradients)

$$-\mathrm{div}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big) = \mathbf{f} \qquad (p\text{-Laplacian})$$
$$\mathbf{v} = 0 \qquad \text{on } \partial\Omega$$

Weak solutions

Find $\mathbf{v} \in W_0^{1,p}$ with $\langle \mathbf{S}(\nabla \mathbf{v}), \nabla \boldsymbol{\xi}\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle$ for all $\boldsymbol{\xi} \in W_0^{1,p}$, with $\mathbf{S}(\mathbf{Q}) = |\mathbf{Q}|^{p-2}\mathbf{Q}$.

Discrete setting:

- $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) polyhedral.
- $\mathcal{T}$ shape regular triangulation with mesh size h.
- $X_h := \{w_h \in W_0^{1,p} \,:\, w_h \text{ piecewise linear}\}$.

Discrete solutions

Find $\mathbf{v}_h \in X_h$ with $\langle \mathbf{S}(\nabla \mathbf{v}_h), \nabla \xi_h\rangle = \langle \mathbf{f}, \xi_h\rangle$ for all $\xi_h \in X_h$.

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Discrete solutions

Find $\mathbf{v}_h \in X_h$ with $\langle \mathbf{S}(\nabla \mathbf{v}_h), \nabla \boldsymbol{\xi}_h\rangle = \langle \mathbf{f}, \boldsymbol{\xi}_h\rangle$ for all $\boldsymbol{\xi}_h \in X_h$.

Measuring the error

Equation of the error

$$\left\langle \mathbf{S}(\nabla \mathbf{v}_h) - \mathbf{S}(\nabla \mathbf{v}), \nabla \xi_h \right\rangle = 0 \qquad \text{for all } \xi_h \in X_h.$$

Natural measure for error is $\left\langle \mathbf{S}(\nabla \mathbf{v}_h) - \mathbf{S}(\nabla \mathbf{v}), \nabla \mathbf{v}_h - \nabla \mathbf{v} \right\rangle \geq 0$.

Observation

$\|\nabla \mathbf{v} - \nabla \mathbf{w}\|_p$ is no good measure for the error,

Reason: $-\operatorname{div} \mathbf{S}(\nabla \cdot)$ is not uniformly elliptic with respect to $\|\cdot\|_{1,p}$.

For $p \geq 2$:

$$\|\nabla \mathbf{v} - \nabla \mathbf{w}\|_p^p \lesssim \langle \mathbf{S}(\nabla \mathbf{v}) - \mathbf{S}(\nabla \mathbf{w}), \nabla \mathbf{v} - \nabla \mathbf{w} \rangle \lesssim \frac{\|\nabla \mathbf{v} - \nabla \mathbf{w}\|_p^2}{(\|\nabla \mathbf{v}\|_p + \|\nabla \mathbf{w}\|_p)^{2-p}}.$$

For $p \leq 2$:

$$\frac{\|\nabla \mathbf{v} - \nabla \mathbf{w}\|_p^2}{(\|\nabla \mathbf{v}\|_p + \|\nabla \mathbf{w}\|_p)^{2-p}} \lesssim \langle \mathbf{S}(\nabla \mathbf{v}) - \mathbf{S}(\nabla \mathbf{w}), \nabla \mathbf{v} - \nabla \mathbf{w} \rangle \lesssim \|\nabla \mathbf{v} - \nabla \mathbf{w}\|_p^p.$$

Natural quantities

Recall: $\mathbf{S}(\mathbf{A}) = |\mathbf{A}|^{p-2}\mathbf{A}$. Define $\mathbf{F}(\mathbf{A}) := |\mathbf{A}|^{\frac{p-2}{2}}\mathbf{A}$. Then

$$\mathbf{S}(\mathbf{A}) \cdot \mathbf{A} = |\mathbf{F}(\mathbf{A})|^2, \qquad \frac{\mathbf{F}(\mathbf{A})}{|\mathbf{F}(\mathbf{A})|} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{S}(\mathbf{A})}{|\mathbf{S}(\mathbf{A})|}.$$

Moreover, $\quad (\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \eqsim |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2.$

Natural distance

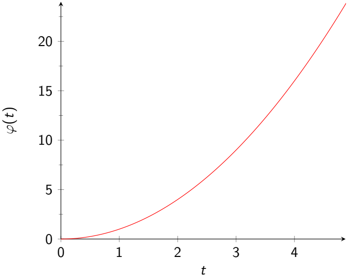
$$\langle \mathbf{S}(\nabla \mathbf{v}_h) - \mathbf{S}(\nabla \mathbf{v}), \nabla \mathbf{v}_h - \nabla \mathbf{v} \rangle \eqsim \|\mathbf{F}(\nabla \mathbf{v}_h) - \mathbf{F}(\nabla \mathbf{v})\|_2^2.$$

Natural regularity

Difference quotient technique gives $\mathbf{F}(\nabla \mathbf{v}) \in W^{1,2}_{\mathrm{loc}}$: (not $\mathbf{v} \in W^{2,p}$!!!)

$$h^{-2}\langle \tau_h \mathbf{S}(\nabla \mathbf{v}), \tau_h \nabla \mathbf{v} \rangle \eqsim h^{-2}\|\tau_h \mathbf{F}(\nabla \mathbf{v})\|_2^2 \quad \xrightarrow{h \to 0} \quad \|\nabla \mathbf{F}(\nabla \mathbf{v})\|_2^2.$$

Useful Orlicz functions



Shifted N-functions

Define $\varphi_a(t) := \int_0^t (\max\{a,s\})^{p-2}\, s\, ds \eqsim (\max\{a,t\})^{p-2}\, t^2$. Then

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \eqsim \varphi_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|) \eqsim |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2$$
$$|\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})| \eqsim \varphi'_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|)$$

Best-Approximation (Cea's Lemma)

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \eqsim \varphi_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|) \eqsim |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2$$
$$|\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})| \eqsim \varphi'_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|)$$

Theorem (Liu, Barrett '94; Diening, Růžička '07)

$$\|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2 \lesssim \inf_{\mathbf{w}_h \in X_h} \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{w}_h)\|_2.$$

Proof: Young's inequality $\varphi'_a(s)\, t \leq \delta \varphi_a(s) + c_\delta \varphi_a(t)$ (generalizes $st \leq \frac{1}{2}s^2 + \frac{1}{2}t^2$)

$$\begin{aligned}
\|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2^2 &\eqsim \big\langle \mathbf{S}(\nabla \mathbf{v}) - \mathbf{S}(\nabla \mathbf{v}_h), \nabla \mathbf{v} - \nabla \mathbf{v}_h \big\rangle \\
&= \big\langle \mathbf{S}(\nabla \mathbf{v}) - \mathbf{S}(\nabla \mathbf{v}_h), \nabla \mathbf{v} - \nabla \mathbf{w}_h \big\rangle \qquad \text{equation of error} \\
&\leq \ldots \text{Young's inequality for } \varphi_{|\nabla \mathbf{v}|} \ \ldots \\
&\leq \delta\, \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2^2 + c_\delta \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{w}_h)\|_2^2.
\end{aligned}$$

Projection error (Interpolation error)

Theorem (Ebmeyer, Liu '05; Diening, Růžička '07)

$$\|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2 \leq \|\mathbf{F}(\nabla \mathbf{w}) - \mathbf{F}(\nabla \Pi_h \mathbf{v})\|_2 \lesssim h \left\|\nabla \mathbf{F}(\nabla w)\right\|_2,$$

where $\Pi_h : X \to X_h$ is Scott-Zhang interpolation (defined by local means).

Proof:

Due to $|\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2 \eqsim \varphi_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|)$ problem reduces to stability of Π_h in Orlicz spaces.

Use $W^{1,1}$-stability $\fint_T |\nabla \mathbf{w} - \nabla \Pi_h \mathbf{w}|\, dx \lesssim \fint_{S_T} |\nabla \mathbf{w}|\, dx$ with $S_T = T \cup$ neighbors

and Jensen's inequality $\varphi_a\left(\fint_{S_T} |\nabla \mathbf{w}|\, dx\right) \leq \fint_{S_T} \varphi_a(|\nabla \mathbf{w}|)\, dx$.

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Numerical experiments for p-Laplacian

$\mathbf{v} = |\mathbf{x}|^{\alpha-1}\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \delta = 0.00001, \Omega = (-1,1)^2$, $\mathcal{P}_1$-element
with α such that $\mathbf{F}(\nabla \mathbf{v}) \in W^{1,2}$

Table with EOC of $\|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2$:

h	p									
	1.1	1.2	1.33	1.5	1.8	2.0	3.0	4.0	5.0	6.0
1.76 e-01	0.84	0.85	0.84	0.84	0.84	0.84	0.86	0.88	0.90	0.93
8.83 e-02	0.88	0.87	0.87	0.87	0.87	0.87	0.89	0.91	0.92	0.92
4.41 e-02	0.90	0.89	0.89	0.89	0.89	0.89	0.91	0.92	0.94	0.95
2.20 e-02	0.92	0.91	0.90	0.90	0.91	0.91	0.92	0.93	0.95	0.96
1.10 e-02	0.93	0.92	0.92	0.92	0.92	0.92	0.93	0.94	0.95	0.89
5.52 e-03	**0.93**	**0.93**	**0.93**	**0.93**	**0.93**	**0.93**	**0.94**	**0.95**	**0.96**	**0.95**
Theory	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**

Finite elements for p-Stokes

Include **pressure**, **incompressibility** and **symmetric gradients**:

$$\begin{aligned} -\operatorname{div}\big(\mathbf{S}(\varepsilon(\mathbf{v}))\big) + \nabla q &= \mathbf{f}, \\ \operatorname{div}\mathbf{v} &= 0. \end{aligned} \qquad (p\text{-Stokes})$$

Weak formulation

Find $\mathbf{v} \in X := W_0^{1,p}$ and $q \in Q \in L_0^{p'}$ with

$$\begin{aligned} \langle \mathbf{S}(\varepsilon(\mathbf{v})), \varepsilon(\boldsymbol{\xi})\rangle - \langle q, \operatorname{div}\boldsymbol{\xi}\rangle &= \langle \mathbf{f}, \boldsymbol{\xi}\rangle && \text{for } \boldsymbol{\xi} \in X, \\ \langle \operatorname{div}\mathbf{v}, \eta\rangle &= 0 && \text{for } \eta \in Q. \end{aligned}$$

Discrete setting formulation

For suitable $X_h \subset X$ and $Q_h \subset Q$ find $\mathbf{v}_h \in X_h$ and $q_h \in Q_h$ with

$$\begin{aligned} \langle \mathbf{S}(\varepsilon(\mathbf{v}_h)), \varepsilon(\boldsymbol{\xi}_h)\rangle - \langle q_h, \operatorname{div}\boldsymbol{\xi}_h\rangle &= \langle \mathbf{f}, \boldsymbol{\xi}_h\rangle && \text{for } \boldsymbol{\xi}_h \in X_h, \\ \langle \operatorname{div}\mathbf{v}_h, \eta_h\rangle &= 0 && \text{for } \eta_h \in Q_h. \end{aligned}$$

Attention: $\operatorname{div}\mathbf{v}_h \neq 0$ pointwise!

Inf-sup condition (pressure reconstruction)

Classical inf-sup condition: $\sup_{\boldsymbol{\xi} \in X} \frac{|\langle q, \operatorname{div} \boldsymbol{\xi} \rangle|}{\|\boldsymbol{\xi}\|_X} \eqsim \|q\|_Q$ for all $q \in Q$.

Discrete inf-sup condition:

$$\sup_{\boldsymbol{\xi}_h \in X_h} \frac{|\langle q_h, \operatorname{div} \boldsymbol{\xi}_h \rangle|}{\|\boldsymbol{\xi}_h\|_X} \eqsim \|q_h\|_Q \qquad \text{for all } q_h \in Q_h.$$

Examples:

X_h (large)	Q_h (small)	Name
$\mathcal{P}_2$	$\mathcal{P}_0$	$\mathcal{P}_2 - \mathcal{P}_0$
$\mathcal{P}_2 + \mathcal{B}_3$-bubble	discont.-$\mathcal{P}_1$	Crouziex-Raviart
$\mathcal{P}_1 + \mathcal{B}_3$-bubble	$\mathcal{P}_1$	MINI element
$\mathcal{P}_2$	$\mathcal{P}_1$	Taylor-Hood

Divergence preserving projection

The existence of $\Pi_h^{\mathrm{div}} : X \to X_h$ with $\langle \operatorname{div} \Pi_h^{\mathrm{div}} \mathbf{w}, \eta_h \rangle = \langle \operatorname{div} \mathbf{w}, \eta_h \rangle$ for $\eta_h \in Q_h$ implies the discrete inf-sup condition.

Best Approximation (Strang type)

Theorem (Berselli, Diening, Růžička '12)

$$\|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))\|_2^2 \lesssim \inf_{\mathbf{w}_h \in X_{h,\mathrm{div}}} \|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{w}_h))\|_2^2 + \inf_{\mu_h \in Q_h} \int_\Omega (\varphi_{|\varepsilon(\mathbf{v})|})^*(|q - \mu_h|)\,dx,$$

with $X_{h,\mathrm{div}} := \{\mathbf{w}_h \in X_h : \langle \operatorname{div} \mathbf{w}_h, \eta_h \rangle = 0 \text{ for all } \eta_h \in Q_h\}$.

Recall: $|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))|^2 \eqsim \varphi_{|\varepsilon(\mathbf{v})|}(|\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{v}_h)|)$

Note that $\operatorname{div} \mathbf{v} = 0$ implies $\Pi_h^{\mathrm{div}} \mathbf{v} \in X_{h,\mathrm{div}}$, so

$$\inf_{\mathbf{w}_h \in X_{h,\mathrm{div}}} \|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{w}_h))\|_2 \le \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \Pi_h^{\mathrm{div}} \mathbf{v})\|_2.$$

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with $X_{h,\mathrm{div}} := \{\mathbf{w}_h \in X_h \,:\, \langle \operatorname{div} \mathbf{w}_h, \eta_h \rangle = 0$ *for all* $\eta_h \in Q_h\}$.

Recall: $|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))|^2 \eqsim \varphi_{|\boldsymbol{\varepsilon}(\mathbf{v})|}(|\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{v}_h)|)$

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Divergence preserving projection

Theorem (Berselli, Diening, Růžička '12)

There exists $\Pi_h^{\mathrm{div}} : X \to X_h$ *with* $\langle \mathrm{div} \Pi_h^{\mathrm{div}} \mathbf{w}, \eta_h \rangle = \langle \mathrm{div} \mathbf{w}, \eta_h \rangle$ *for* $\eta_h \in Q_h$

$$\|\mathbf{F}(\varepsilon(\mathbf{w})) - \mathbf{F}(\varepsilon(\Pi_h^{\mathrm{div}} \mathbf{w}))\|_2 \lesssim h \, \|\nabla \mathbf{F}(\varepsilon(\mathbf{w}))\|_2.$$

Local correction of divergence:
Define $\Pi_h^{\mathrm{div}} \mathbf{w} := \Pi_h \mathbf{w} - \Pi_{\mathrm{local}}^{\mathrm{div}}(\mathbf{w} - \Pi_h \mathbf{w})$ like in [Brezzi-Fortin].

Show $$\fint_T \varphi_a(|\nabla \Pi_h^{\mathrm{div}} \mathbf{w}|)\, dx \lesssim \fint_{S_T} \varphi_a(|\nabla \mathbf{w}|)\, dx$$ (based on Jensen's inequality)

Local Korn inequality [Diening, Růžička, Schumacher '10]:

$$\fint_{S_T} \varphi_a(|\nabla \mathbf{w} - \langle \nabla \mathbf{w} \rangle_{S_T}|)\, dx \lesssim \fint_{S_T} \varphi_a(|\varepsilon(\mathbf{w}) - \langle \varepsilon(\mathbf{w}) \rangle_{S_T}|)\, dx.$$

Velocity error

Theorem (Berselli, Diening, Růžička '12)

$$\|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))\|_2^2 \lesssim h \, \|\nabla \mathbf{F}(\varepsilon(\mathbf{v}))\|_2^2 + \inf_{\mu_h \in Q_h} \int_\Omega (\varphi_{|\varepsilon(\mathbf{v})|})^*(|q - \mu_h|)\, dx.$$

Note that $(\varphi_a)^*(t) \eqsim (\max\{a,t\})^{p'-2} t^2$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Corollary (Berselli, Diening, Růžička '12)

If $\mathbf{F}(\varepsilon(\mathbf{v})) \in W^{1,2}$ *and* $q \in W^{1,p'}$, *then*

$$\|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))\|_2 \lesssim h^{\min\{1, \frac{p'}{2}\}}.$$

Pressure error

$$-\operatorname{div}\big(\mathbf{S}(\varepsilon(\mathbf{v}))\big) + \nabla q = \mathbf{f}, \qquad \operatorname{div}\mathbf{v} = 0. \qquad (p\text{-Stokes})$$

Pressure error

$$\|q - q_h\|_{p'} \lesssim \inf_{\mu_h \in Q_h} \|q - \mu_h\|_{p'} + \|\mathbf{S}(\varepsilon(\mathbf{v})) - \mathbf{S}(\varepsilon(\mathbf{v}_h))\|_{p'}.$$

Problem:

Error of $\mathbf{F}(\varepsilon(\mathbf{v}))$ does not match error of $\mathbf{S}(\varepsilon(\mathbf{v}))$:

$$p \le 2: \qquad \|\mathbf{S}(\varepsilon(\mathbf{v})) - \mathbf{S}(\varepsilon(\mathbf{v}_h))\|_{p'}^{p'} \lesssim \|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))\|_2^2 \lesssim h^2,$$

$$p \ge 2: \qquad \|\mathbf{S}(\varepsilon(\mathbf{v})) - \mathbf{S}(\varepsilon(\mathbf{v}_h))\|_{p'} \lesssim c(\dots)\, \|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))\|_2 \lesssim h^{\frac{p'}{2}}.$$

Corollary (Berselli, Diening, Růžička '12)

For $\mathbf{F}(\varepsilon(\mathbf{v})) \in W^{1,2}$ and $q \in W^{1,p'}$ we have $\|q - q_h\|_{p'} \lesssim h^{\min\{\frac{2}{p'}, \frac{p'}{2}\}}$.

Velocity as predicted

$\mathbf{v} = |\mathbf{x}|^{\alpha-1}\binom{x_2}{-x_1}, q = |\mathbf{x}|^{\gamma}, \delta = 0.00001, \Omega = (-1,1)^2$, MINI element

with α, γ such that $\mathbf{F}(\varepsilon(\mathbf{v})) \in W^{1,2}, q \in W^{1,p'}$

Table of EOC of $\|\mathbf{F}(\varepsilon(\mathbf{v})) - \mathbf{F}(\varepsilon(\mathbf{v}_h))\|_2$:

h	p							
	1.25	1.33	1.5	1.67	1.8	2.0	2.5	3.0
1.77e-01	0.86	0.90	0.85	0.91	0.86	0.85	0.26	0.66
8.84e-02	0.88	0.90	0.87	0.91	0.88	0.88	0.84	0.74
4.42e-02	0.89	0.91	0.89	0.92	0.90	0.90	0.85	0.75
2.21e-02	0.91	0.92	0.91	0.93	0.91	0.91	0.86	0.76
1.10e-02	0.92	0.93	0.92	0.94	0.92	0.92	0.85	0.76
5.52e-03	**0.93**	**0.94**	**0.93**	**0.94**	**0.93**	**0.93**	**0.85**	**0.76**
$\min\{1, \frac{p'}{2}\}$	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**0.83**	**0.75**

Pressure is better than predicted

$\mathbf{v} = |\mathbf{x}|^{\alpha-1}\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, q = |\mathbf{x}|^{\gamma}, \delta = 0.00001, \Omega = (-1,1)^2$, MINI element

with α such that $\mathbf{F}(\varepsilon(\mathbf{v})) \in W^{1,2}, q \in W^{1,p'}$

Table with EOC of $\|q - q_h\|_{p'}$:

h	p							
	1.25	1.33	1.5	1.67	1.8	2.0	2.5	3.0
1.77e-01	1.02	1.05	1.06	1.01	0.98	1.24	0.90	0.91
8.84e-02	0.58	0.68	0.86	0.96	0.99	1.07	0.99	1.00
4.42e-02	0.49	0.61	0.82	0.95	0.99	1.05	1.01	1.02
2.21e-02	0.45	0.57	0.78	0.94	0.99	1.03	1.01	1.02
1.10e-02	0.42	0.54	0.75	0.93	0.99	1.02	1.01	1.02
5.52e-03	**0.41**	**0.52**	**0.73**	**0.92**	**0.99**	**1.02**	**1.01**	**1.01**
$\min\left\{\frac{2}{p'}, \frac{p'}{2}\right\}$	**0.40**	**0.50**	**0.67**	**0.80**	**0.89**	**1.00**	**0.83**	**0.75**

$$\|q - q_h\|_{p'} \lesssim \inf_{\mu_h \in Q_h} \|q - \mu_h\|_{p'}\, dx + \|\mathbf{S}(\varepsilon(\mathbf{v})) - \mathbf{S}(\varepsilon(\mathbf{v}_h))\|_{p'}$$

Conjecture on the stress

$\mathbf{v} = |\mathbf{x}|^{\alpha-1}\binom{x_2}{-x_1}, q = |\mathbf{x}|^{\gamma}, \delta = 0.00001, \Omega = (-1,1)^2$, MINI element

with α such that $\mathbf{F}(\varepsilon(\mathbf{v})) \in W^{1,2}, q \in W^{1,p'}$

Table with EOC of $\|\mathbf{S}(\varepsilon(\mathbf{v})) - \mathbf{S}(\varepsilon(\mathbf{v}_h))\|_{p'}$:

h	p							
	1.25	1.33	1.5	1.67	1.8	2.0	2.5	3.0
1.77e-01	0.40	0.50	0.63	0.77	0.79	0.85	0.36	0.88
8.84e-02	0.40	0.50	0.65	0.78	0.81	0.88	1.02	0.98
4.42e-02	0.40	0.50	0.66	0.78	0.83	0.90	1.03	1.01
2.21e-02	0.40	0.50	0.66	0.79	0.84	0.91	1.04	1.01
1.10e-02	0.40	0.50	0.67	0.80	0.85	0.92	1.03	1.02
5.52e-03	**0.40**	**0.50**	**0.67**	**0.80**	**0.86**	**0.93**	**1.03**	**1.02**
$\min\{\frac{2}{p'}, 1\}$	**0.40**	**0.50**	**0.67**	**0.80**	**0.89**	**1.00**	**1.00**	**1.00**

Note that $\mathbf{S} \in W^{1,\min\{\frac{2}{p'},1\}}$ in this example.

Variable exponents

Electrorheological fluids require extra stress of form

$$\mathbf{S} = (1 + |\varepsilon(\mathbf{v})|)^{p(x)-2}\varepsilon(\mathbf{v}),$$

where $p(x) = p(|\mathbf{E}(x)|)$ is smooth and $1 < \inf p \leq \sup p < \infty$.

Natural quantity of solution is

$$\int |\varepsilon(\mathbf{v})|^{p(x)}\,dx,$$

which requires Lebesgue spaces with variable exponents.

Variable exponent spaces

Define $\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 \,:\, \int \left|\frac{|f(x)|}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$

Then $L^{p(\cdot)} := \{f \,:\, \|f\|_{p(\cdot)} < \infty\}$ is a Banach space.

Basic properties of $L^{p(\cdot)}$

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 \,:\, \int \left|\frac{|f(x)|}{\lambda}\right|^{p(x)} dx \leq 1\right\} \text{ and } L^{p(\cdot)} := \{f \,:\, \|f\|_{p(\cdot)} < \infty\}.$$

Basic properties

Let $1 < \inf p \leq \sup p < \infty$. Then

1. $L^{p(\cdot)}$ is uniformly convex and reflexive.
2. $C_0^\infty(\mathbb{R}^n)$ is dense.
3. $(L^{p(\cdot)})^* = L^{p'(\cdot)}$ (dual space).
4. Hölder's inequality: $\|f\,g\|_{s(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}$ with $\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$.

Advanced properties of $L^{p(\cdot)}$

Existence theory for electrorheological fluids requires:

1. $\|\nabla \mathbf{v}\|_{p(\cdot)} \leq c\,\|\varepsilon(\mathbf{v})\|_{p(\cdot)}$ (Korn's inequality)
2. sharp embeddings $W^{1,p(\cdot)} \to L^{q(\cdot)}$ with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{1}{n}$.
3. singular integrals on $L^{p(\cdot)}$

This can be reduced to boundedness of maximal operator M on $L^{p(\cdot)}$

$$(Mf)(x) := \sup_{B \ni x} ⨍_B |f|\, dy.$$

We need regularity assumption on p (no jumps!)

log-Hölder continuity

We say that $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$ if and only if

Local: $$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{c_{\log}(p)}{\log\left(e + \frac{1}{|x-y|}\right)}$$ [Zhikov '87, Diening '02]

Global: $$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leq \frac{c_{\log}(p)}{\log(e + |x|)}$$ [Cruz-Uribe+F+N '03],

Less than the "usual" Hölder continuity.

Theorem (Diening '02, Cruz-Uribe+FN '03, DHHMS '09)

$\frac{1}{p} \in C^{\log}$ *and* inf $p > 1$, *then* $\|Mf\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$.

Consequence of log-Hölder

Let $\frac{1}{p} \in C^{\log}$. Then for all balls B with $|B| \leq 1$

$$\|\chi_B\|_{p(\cdot)} \approx |B|^{\frac{1}{p(x)}} \qquad \text{for all } x \in B.$$

Key estimate: [Diening '02, DHHMS '09, DHHR '11, DSch '13]

Let $\frac{1}{p} \in C^{\log}$. Then for all $m > 0$ and all balls B with $|B| \leq 1$,

$$\bigg(\fint_B |f|\, dy \bigg)^{p(x)} \leq c_m \fint_B |f|^{p(y)}\, dy + c\, r_B^m$$

for all f with $\|f\|_{L^{p(\cdot)}+L^\infty} \leq 1$ and all $x \in B$.

This is a substitute for Jensen's inequality!

Projection error

Key estimate: [Diening '02, DHHMS '09, DHHR '11, DSch '13]

Let $\frac{1}{p} \in C^{\log}$. Then for all $m > 0$

$$\bigg(⨍_B |f|\, dy \bigg)^{p(x)} \leq c_m ⨍_B |f|^{p(y)}\, dy + c\, r_B^m.$$

for all B with $|B| \leq 1$, all f with $\|f\|_{L^{p(\cdot)}+L^\infty} \leq 1$ and all $x \in B$.

Projection error – intermediate

Let $\frac{1}{p} \in C^{\log}$ then for $m > 0$

$$\big\|\mathbf{F}(\cdot, \varepsilon(\mathbf{v})) - \mathbf{F}(\cdot, \varepsilon(\Pi_h^{\mathrm{div}} \mathbf{v}))\big\|_2^2 \lesssim \sum_T \int_{S_T} \big|\mathbf{F}(\cdot, \varepsilon(\mathbf{v})) - \mathbf{F}(\cdot, \langle \varepsilon(\mathbf{v}) \rangle_{S_T})\big|^2\, dx + h^m.$$

Projection error – intermediate

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Projection error – final

Let $p \in C^{0,\alpha}$ then

$$\left\|\mathbf{F}(\cdot,\varepsilon(\mathbf{v})) - \mathbf{F}(\cdot,\varepsilon(\Pi_h^{\mathrm{div}}\mathbf{v}))\right\|_2 \lesssim h\,\|\nabla\mathbf{F}(\varepsilon(\mathbf{v}))\|_2 + h^\alpha.$$

- h^α is due to $\mathbf{F}(\cdot,\langle\varepsilon(\mathbf{v})\rangle_{S_T})$ vs. $\langle\mathbf{F}(\cdot,\varepsilon(\mathbf{v}))\rangle_{S_T}$.
- It is possible to freeze $p(\cdot)$ on each triangle.

Conclusion

Combining the above techniques we get:

A priori estimates electrorheological fluids

$$\|\mathbf{F}(\varepsilon(\mathbf{v}_h)) - \mathbf{F}(\varepsilon(\mathbf{v}))\|_2 \lesssim h^{\beta} \qquad \text{with } \beta = \min\left\{1, \frac{\sup p'}{2}, \alpha\right\},$$

$$\|q - q_h\|_{p'(\cdot)} \lesssim h^{\gamma} \qquad \text{with } \gamma = \min\left\{\frac{2}{\sup p'}\frac{\inf p'}{\sup p'}, \frac{\sup p'}{2}, \alpha\right\}.$$

Outlook – Open problems

- Convection + instationary.
- Higher order elements (already problem for p-Laplacian: $C^{1,\alpha}$)
- Optimal estimate for error of extra stress $\mathbf{S}$.

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Combining the above techniques we get:

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$$\|\mathbf{F}(\varepsilon(\mathbf{v}_h)) - \mathbf{F}(\varepsilon(\mathbf{v}))\|_2 \lesssim h^\beta \qquad \text{with } \beta = \min\Big\{1\,,\, \frac{\sup p'}{2}\,,\, \alpha\Big\},$$

$$\|q - q_h\|_{p'(\cdot)} \lesssim h^\gamma \qquad \text{with } \gamma = \min\Big\{\frac{2}{\sup p'}\frac{\inf p'}{\sup p'}\,,\, \frac{\sup p'}{2}\,,\, \alpha\Big\}.$$

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