Temporal Discretization of Eulerian Fluid-Structure Interactions

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joint work with Stefan Frei

based on ideas by Thomas Dunne and with support by Rolf Rannacher

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Part I

Eulerian Formulation for Fluid-Structure Interactions

fsi-2

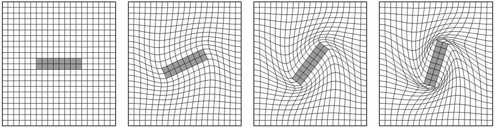
- Elastic ball falling in container with viscous fluid fsi-2

- Elastic vessel walls with active growth
- Pulsating flow

Monolithic Models for FSI

Aims

- We need a fully monolithic model for the coupled problem that allows for large time-step integration with implicit methods, strongly coupled solvers (Newton & multigrid), Galerkin formulation and gradient based methods for error estimation and optimization.
- Model that allows for problems with large deformation, large motion and even contact.



Standard approaches

- Partitioned approaches often fail, as they might require many sub-iterations for stiffly coupled problems. No exact sensitivities.
- ALE (based on transforming the fluid domain to a fixed reference framework) does not do (our) job, as large motion and contact might lead to breakdown (if we do not change the reference frame, which we do not want to do, as it would break the strict monolithic character).

Fully Eulerian Coordinates

- One momentum equation in Eulerian coordinates (complete domain)

$$\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \operatorname{div} \boldsymbol{\sigma} = \rho \mathbf{f} \text{ in } \Omega(t) = \mathcal{F}(t) \cup \Gamma(t) \cup \mathcal{S}(t)$$

- Plus further equations (single domains)

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \mathcal{F}(t), \quad \partial_t \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u} = \mathbf{v} \text{ in } \mathcal{S}(t) \quad (*)$$

- Material law depends on location and time

$$\rho(x,t) = \begin{cases} \rho_f & x \in \mathcal{F}(t) \\ J\rho_s^0 & x \in \mathcal{S}(t) \end{cases}, \ \boldsymbol{\sigma}(x,t) = \begin{cases} \rho_f \nu_f (\nabla \mathbf{v} + \nabla \mathbf{v}^T) - pI & x \in \mathcal{F}(t) \\ J\mathbf{F}^{-1}(2\mu \mathbf{E}_s + \lambda_s \operatorname{tr}(\mathbf{E}_s) I)\mathbf{F}_s^{-T} & x \in \mathcal{S}(t) \end{cases}$$

- Interface-tracking (where is the solid domain at time t, where the fluid part?) with *Initial Point Set* (using Level-Sets is possible, but Initial Point Set gives us $\mathbf{u}$ (equation $(*)$) for free)

$$\partial_t \Phi + \tilde{\mathbf{v}} \cdot \nabla \Phi = 0 \text{ in } \Omega(t), \quad \Phi(x,0) = x \ \Rightarrow \ x \in \begin{cases} \mathcal{F}(t) & \text{if } \Phi(x,t) \notin \mathcal{S}(0) \\ \mathcal{S}(t) & \text{if } \Phi(x,t) \in \mathcal{S}(0) \end{cases}$$

Once again

Summary

- Momentum equation on the whole domain

$$\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \operatorname{div} \boldsymbol{\sigma} = \rho \mathbf{f} \text{ in } \Omega(t) = \mathcal{F}(t) \cup \Gamma(t) \cup \mathcal{S}(t)$$

- Material law depends on the coordinate
- Interface tracking for deciding about $x \in \mathcal{F}(t)$ or $x \in \mathcal{S}(t)$
- Eulerian representation of the deformation to model stress & strain relation

Properties

- Similar to multiphase flows
- But: coupling of two different operators (not just jumping parameters)
- Challenging: coupling of parabolic type equation with hyperbolic
- This is an *interface problem* with a *moving interface*

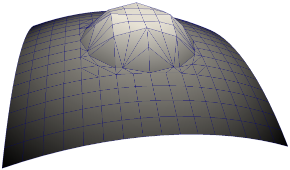
Part II

Discretization of Interface Problems

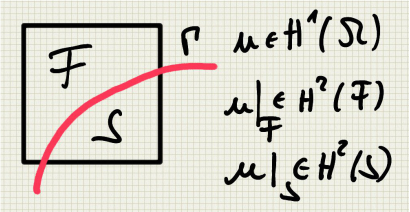
Interface problem

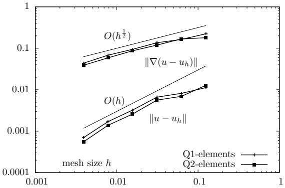
Prototypical interface-problem (fixed interface)

$$-\operatorname{div}(\kappa\nabla u) = f, \quad \kappa(x) = \begin{cases} \kappa_1 & x \in \Omega_1 \\ \kappa_2 & x \in \Omega_s \end{cases}$$



Solution is continuous but not differentiable.





(if the domains have smooth boundaries)

Possible approaches

- Smoothing? Often used for multiphase flows
- Fitted meshes
- Generalized Finite elements (XFEM), enrichment of basis

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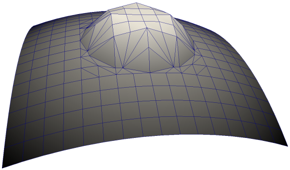

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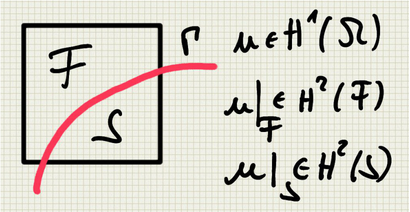

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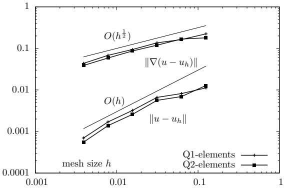

Possible approaches

- Smoothing? Often used for multiphase flows
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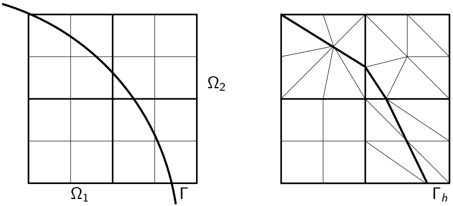
Locally modified parametric finite elements (short summary of another talk)



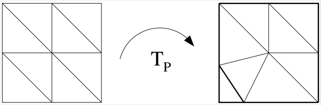
- Organize mesh in patches
- Away from interface:
  Slit patch in 4 quads
- At interface:
  Slit patch in 8 triangles

- Mesh resolves the interface
- Modifications are kept local
- Each patch has the same number of unknowns
- Each patch has the same connectivity in the system matrix

Realization

- Iso-parametric Finite Element approch

$$T_P : \hat{P} \to P$$



- No mesh-nodes are moved
- Discretization can depend on the solution in an implicit way

Locally Modified FEM

Theorem (A priori estimate & condition number)

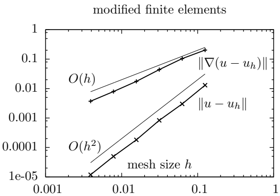
Let $\Gamma \subset \Omega$ *be a smooth interface* $\Gamma \in C^2$ *and let*

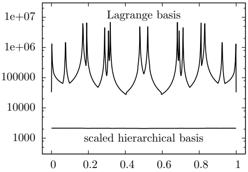
$$u \in H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2), \quad \|u\|_{H^2(\Omega_1 \cup \Omega_2)} \le c_s \|f\|.$$

Then, for the modified finite element solution $u_h \in V_h$ *it holds*

$$\|\nabla(u - u_h)\|_\Omega \le c h_P \|f\|, \quad \|u - u_h\|_\Omega \le c h_P^2 \|f\|$$
$$\mathrm{cond}_2(A) \le c h_P^{-2}$$

with $c > 0$ *not depending on the interface location within the elements.*





[S. Frei, T.R. "A locally modified parametric finite element method for interface problems", SIAM J. Numer. Anal, accepted 2014]

Parabolic interface problem with moving interface

Consider:

$$\partial_t u - \operatorname{div}(\kappa \nabla u) = f \text{ in } Q = Q_1 \cup G \cup Q_2 \subset \mathbb{R}^{d+1}$$

Space-time domain:

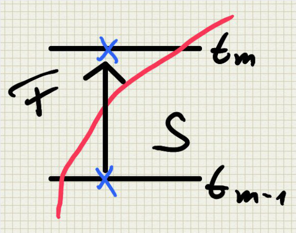
$$Q = \{(x,t),\ t \in [0,T],\ x \in \Omega_1(t) \cup \Gamma(t) \cup \Omega_2(t)\}.$$

Problem of standard discretization

- Limited regularity in time, if $x \in \mathcal{S}(t_{m-1})$ and $x \in \mathcal{F}(t_m)$

$$\frac{u_m(x) - u_{m-1}(x)}{k} \sim ?$$

- What happens at the boundary, if $x \in \Omega(t_{m-1})$ but $x \notin \Omega(t_m)$



Galerkin discretization in time

1. First, derive variational formulation in time

$$u \in \mathcal{X}: \quad \underbrace{(\partial_t u, \phi)_Q + (\kappa \nabla u, \nabla \phi)_Q}_{=:B(u,\phi)} = (f, \phi)_Q, \quad \phi \in \mathcal{Y}$$

where

$$(f, g)_Q := \int_0^T (f, g)_{\Omega(t)}\, \mathrm{d}t.$$

2. Approximate by choosing discrete subspaces $X_k \subset \mathcal{X}$ and $Y_k \subset \mathcal{Y}$. Well known examples:
   - Piece-wise constant (in time) X_k and Y_k leads to variant of backward Euler
   - Piece-wise linear continuous X_k and piece-wise constant (discontinuous) Y_k leads to variant of trapecoidal rule

   What does *variant* mean? Equivalent for linear autonomous problems.

3. Last step, approximate discrete formulation with numerical quadrature rule (of sufficient order, e.g. box-rule for backward Euler, trapecoidal rule for trapecoidal rule):

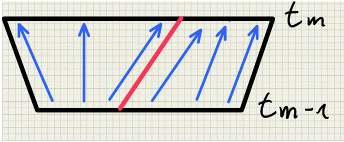
$$B(u_k, \phi_k) \approx B_k(u_k, \phi_k)$$

   Compute (decouples to a time-stepping scheme)

$$B_k(u_k, \phi_k) = (f, \phi_k)_Q$$

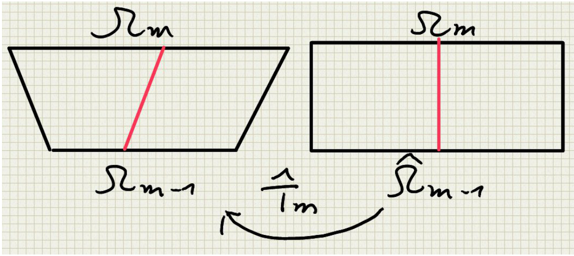
Space-Time Galerkin Approach

- Discretize with continuous piece-wise linear trial- and discontinuous piece-wise constant test-functions
- Use functions, that are linear in alignement with domain



- Temporal basis functions depend on space and time!

Equivalent to a local ALE-approach with standard basis (piece-wise linear plus constant) on reference elements:



Mapping $\hat{T}_m(t_m) = \mathrm{id}$ is identity at time t_m.

(Equivalent) "ALE"-Approach

Introduce mapping:

$$T_m : \hat{Q}_m := [t_{m-1}, t_m] \times \Omega_m \to Q_m := \{(x,t),\ t \in [t_{m-1}, t_m],\ x \in \Omega(t)\}.$$

And gradient

$$F_m := \nabla T_m, \quad J_m := \det(F_m).$$

Transform space-time formulation (now, $\hat{u}$ and $\hat{\phi}$ are standard basis)

$$B_m(u,\phi) = (\partial_t u, \phi)_{Q_m} + (\kappa \nabla u, \nabla \phi)_{Q_m}$$
$$= \boxed{\left(J_m(\partial_t \hat{u} - \partial_t T_m \cdot \hat{\nabla} \hat{u}), \hat{\phi}\right)_{\hat{Q}_m} + \left(J_m \kappa F_m^{-1} \hat{\nabla} \hat{u} F_m^{-T}, \hat{\nabla} \hat{\phi}\right)_{\hat{Q}_m}}$$

Approximate

$$B_{k,m}(u,\phi) = \left(\bar{J}_m(\partial_t \hat{u} - \overline{\partial_t T_m} \cdot \hat{\nabla} \hat{u}), \hat{\phi}\right)_{\hat{Q}_m} + \left(\bar{J}_m \kappa \bar{F}_m^{-1} \hat{\nabla} \hat{u} \bar{F}_m^{-T}, \hat{\nabla} \hat{\phi}\right)_{\hat{Q}_m}$$

With $\bar{J}_m$, $\bar{F}_m$, $\overline{\partial_t T_m}$ piece-wise constant:

$$\bar{F}_m := \frac{1}{2}(F_m(t_{m-1}) + \underbrace{F_m(t_m)}_{=I}), \quad \bar{J}_m := \frac{1}{2}(J_m(t_{m-1}) + \underbrace{J_m(t_m)}_{=1}).$$

Equivalent to time-stepping scheme:

$$\left(\bar{J}_m(u^m - \hat{u}^{m-1}), \phi\right)_{\Omega(t_m)} - \frac{1}{2}\left(\bar{J}_m \overline{\partial_t T_m} \cdot \nabla u^m, \phi\right)_{\Omega(t_m)} - \frac{1}{2}\left(\bar{J}_m \overline{\partial_t T_m} \cdot \nabla \hat{u}^{m-1}, \phi\right)_{\Omega(t_m)} + \dots$$

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Galerkin Discretization in Time

Theorem

[Frei, R., '14] Let the space-time domain $Q \subset \mathbb{R}^{d+1}$ be smooth and splitted $Q = Q_1 \cup \Gamma \cup Q_2$ with smooth interface Γ. Further, let $T_m : \hat{Q}_m \to Q_m$ be such, that

$$\sup_t \left(\|\partial_t^\alpha T_m(t)\|_{W^{2-\alpha,\infty}} + \|\partial_t^\beta T_m^{-1}(t)\|_{W^{2-\beta,\infty}} \right) \le c \quad \alpha, \beta = 0, 1, 2,$$

Next, let the solution to the parabolic interface problem be given with

$$\|\partial_t^2 u\|_{Q_1 \cup Q_2} + \|\partial_t \operatorname{div}(\kappa \nabla u)\|_{Q_1 \cup Q_2} \le c \left(\|f\|_Q + \|\partial_t f\|_{Q_1 \cup Q_2} \right).$$

Then, for the Galerkin approximation of trapecoidal rule type in time it holds

$$\|u - u_k\|_Q \le c k^2 \|\partial_t f\|_{Q_1 \cup Q_2}.$$

- Similar result for $\|u(T) - u_k(T)\|_{\Omega(T)}$ requires

$$\|\partial_t u\|_{H^2(Q_1 \cup Q_2)} + \|\partial_{tt} u\|_{H^1(Q_1 \cup Q_2)} < \infty$$

- Assumptions on mapping limits interface velocity (CFL like condition)

Proof

1. Split error $e_k := u - u_k$ into interpolation error $\eta_k := u - i_k u$ and approximation error $\xi_k := i_k u - u_k$:

$$\|e_k\|_Q^2 = (e_k, \eta_k)_Q + (e_k, \xi_k)_Q$$

2. Interpolation error straightforward

$$(e_k, \eta_k)_Q \leq ch^2 \|f\|_{H^1(Q)} \|e_k\|_Q.$$

3. Introduce dual solutions $z \in \mathcal{Y}$ and $z_k \in Y_k$

$$(e_k, \phi_k)_Q = B_k(\phi_k, z_k), \quad \|\nabla z_k\|_Q \leq \|e_k\|_Q.$$

4. Then,

$$(e_k, \xi_k) = B_k(\xi_k, z_k) = B_k(i_k u, z_k) - \overbrace{B_k(u_k, z_k)}^{=B(u,z_k)\text{ G.O.}} \pm B_k(u, z_k)$$
$$= -\boxed{B_k(u - i_k u, z_k)} + \boxed{B_k(u, z_k) - B(u, z_k)}$$

5. Stability estimate & approximation

$$B_k(\eta_k, z_k) \leq ck^2 \|\partial_t \operatorname{div}(\kappa \nabla u)\|_{L^2(Q_1 \cup Q_2)} \|e_k\|_Q,$$
$$\big[B_k - B\big](u, z_k) \leq ck^2 \|u\|_{H^2(Q_1 \cup Q_2)} \|e_k\|_Q$$

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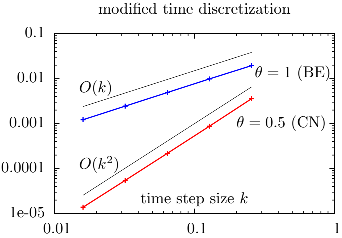
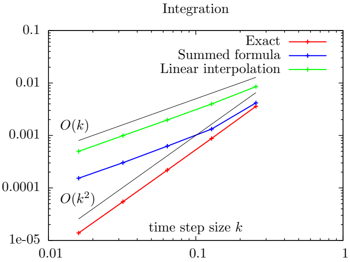
$$\Big[B_k - B\Big](u, z_k) \le ck^2 \|u\|_{H^2(Q_1 \cup Q_2)} \|e_k\|_Q$$

First Results

- Discretization error of space-time Galerkin approach with piece-wise constant (backward Euler like) and piece-wise linears (trapecoidal rule like)





- Correct interpolation is important!

$$(\hat{u}^{m-1}, \phi^m)_{\Omega_m}$$

- Projection only necessary close to interface and for explicit parts!

Conclusion

- The Fully Eulerian model for FSI is monolithic and can handle large deformation, motion and contact
- But, it is of interface-capturing type and an interface problem with the usual difficulties.
- Spatial discretization is handable, but a fitted or generalized finite element technique must be used for optimal order convergence
- Temporal discretization (of high order) is a difficult topic. We require special space-time approaches or implicit transformations that give rise to new nonlinearities and non-standard terms.

T. Richter.
A Fully Eulerian formulation for fluid-structure-interaction problems.
J. Comp. Phys., 233:227–240, 2013.

S. Frei and T. Richter.
A locally modified parametric finite element method for interface problems.
SIAM J. Numer. Anal., accepted, 2014.

S. Frei and T. Richter.
Second order discretization in time for parabolic interface problems with moving interfaces.
To be shortened before submission