Optimal Control of Static Elastoplasticity in Primal and Dual Formulations

Roland Herzog, Gerd Wachsmuth, Christian Meyer, Juan Carlos de los Reyes

Numerical Mathematics

CHEMNITZ UNIVERSITY OF TECHNOLOGY

TU Dortmund

EPN Quito

Modeling, Analysis, and Computing in Nonlinear PDEs

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(Elliptic) VIs of First and Second Kind

Let V be a (real) Hilbert space.

VI of first kind

Find $u \in K$ such that

$$a(u, v - u) \geq \langle \ell, v - u \rangle$$

holds for all $v \in K$.

- $a : V \times V \to \mathbb{R}$ is V-elliptic
- $\ell \in V'$
- $K \subset V$ convex

VI of second kind

Find $u \in V$ such that

$$a(u, v - u) + j(v) - j(u) \geq \langle \ell, v - u \rangle$$

holds for all $v \in V$.

- $a : V \times V \to \mathbb{R}$ is V-elliptic
- $\ell \in V'$
- $j : V \to \mathbb{R} \cup \{\infty\}$ is convex and l.s.c. (often non-differentiable)

Fact #1:

When K is convex and closed, then a VI of 1st kind is also of 2nd kind with $j(v) = I_K(v)$ (indicator function).

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Fact #2:

When $a(\cdot, \cdot)$ is symmetric, both VIs represent an energy minimization principle. (They are necessary and sufficient optimality conditions.)

VIs of First and Second Kind: Examples

VI of first kind

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- obstacle problems, e.g.,

$$a(u, v) = \int_\Omega \varepsilon(u) : \mathbb{C}\, \varepsilon(v)\, \mathrm{d}x$$

$K =$ admissible displacements

- dam problems

VI of second kind

Find $u \in V$ such that

$$a(u, v-u) + j(v) - j(u) \geq \langle \ell, v-u \rangle$$

holds for all $v \in V$.

- frictional contact problems in elasticity, e.g., (Tresca)

$$a(u, v) = \int_\Omega \varepsilon(u) : \mathbb{C}\, \varepsilon(v)\, \mathrm{d}x$$

$$j(u) = \int_{\Gamma_c} g\, |u - (u \cdot n)\, n|\, \mathrm{d}x$$

- Bingham fluids

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- stress-driven (dual) elastoplasticity

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Optimal Control of VIs

Let us use as control, for simplicity, the right hand side data ℓ in a Hilbert space $H \hookrightarrow V'$.

VI of first kind

$$\min_{u \in K,\, \ell \in H} \quad J(u) + \frac{\nu}{2}\|\ell\|_H^2$$

$$\text{s.t.} \begin{cases} a(u, v-u) \ge \langle \ell,\, v-u \rangle \\ \text{for all } v \in K. \end{cases}$$

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Both problems are Mathematical Programs with Equilibrium Constraints (MPECs), so the derivation of optimality conditions (of KKT type) is not straightforward. We care because KKT conditions are the basis for fast optimization algorithms as well as error estimates, ...

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Note that the solution map $\ell \mapsto u$ of the VI is generally non-differentiable.

Optimal Control of VIs of First Kind

Possible approaches:

- exploit conical differentiability of the VI's solution map $\ell \mapsto u$ (implicit programming approach), see for instance [Mignot (1976); Mignot, Puel (1984); Luo, Pang, Ralph (1996)]
  $\rightsquigarrow$ strong stationarity
- exploit (weak) directional differentiability of the VI's solution map, see for instance [Herzog, Meyer, Wachsmuth (2013)]
  $\rightsquigarrow$ B-stationarity (no Lagrange multipliers)
- other approaches leading to B-stationarity conditions
  - exploit the structure of the tangent cone of the feasible set in terms of (u, ℓ), see for instance
  - or apply an exact penalty approach for the VI

  see for instance [Kocvara, Outrata (2004)]

- smooth out the j-term in the VI of 2nd kind

$$a(u, v - u) + j\,(v) - j\,(u) \geq \langle \ell,\, v - u \rangle \quad \text{for all } v \in V$$

Optimal Control of VIs of Second Kind

- smooth out the j-term in the VI of 2nd kind

$$a(u, v - u) + j_\varepsilon(v) - j_\varepsilon(u) \geq \langle \ell, v - u \rangle \quad \text{for all } v \in V$$

Optimal Control of VIs of Second Kind

- smooth out the j-term in the VI of 2nd kind

$$\begin{aligned} & a(u, v-u) + j_\varepsilon(v) - j_\varepsilon(u) \ge \langle \ell, \, v-u \rangle && \text{for all } v \in V \\ \Leftrightarrow \quad & a(u,v) + j_\varepsilon'(u)\, v = \langle \ell, \, v \rangle && \text{for all } v \in V \end{aligned}$$

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- consider the regularized optimal control problem,

$$\begin{aligned} \min_{u \in V,\, \ell \in H} \quad & J(u) + \frac{\nu}{2} \|\ell\|_H^2 \\ \text{s.t.} \quad & a(u, v) + j_\varepsilon'(u)\, v = \langle \ell,\, v \rangle \quad \text{for all } v \in V, \end{aligned}$$

  derive KKT conditions (under suitable constraint qualifications) and pass to the limit

This technique has already been developed by [Barbu (1984)] and used for instance in [Wenbin, Rubio (1991); Bonnans, Tiba (1991); Bonnans, Casas (1995); de los Reyes (2011, 2012)].

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A Second Look at VIs of First Kind

Find $u \in K$ such that $a(u, v-u) \geq \langle \ell, v-u \rangle$ holds for all $v \in K$.

Replacing the VI by a complementarity system

- When K is a convex *cone*, we may re-write the VI equivalently as

$$a(u,v) + \langle \mu, v \rangle = \langle \ell, v \rangle \quad \text{for all } v \in V$$
$$u \in K, \quad \mu \in K^\circ, \quad \langle \mu, u \rangle = 0.$$

- When $K = \{u \in V : g(u) \leq 0\}$, then under some constraint qualification, we get

$$a(u,v) + \langle \mu, g'(u)\, v \rangle = \langle \ell, v \rangle \quad \text{for all } v \in V$$
$$g(u) \leq 0, \quad \mu \geq 0, \quad \langle \mu, g(u) \rangle = 0.$$

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Complementarity conditions arise.

Finite-Dimensional MPCCs

Mathematical Program with a Complementarity Constraint (MPCC):

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \quad \text{with } x \in \mathbb{R}^n \\ \text{subject to} \quad & g(x) \le 0, \quad h(x) = 0 \\ \text{and} \quad & G(x) \ge 0, \quad H(x) \ge 0, \quad G(x)^\top H(x) = 0 \end{aligned}\right\} \qquad \text{(MPCC)}$$

Examples: game theory, robust optimization, bilevel optimization, traffic networks, mechanics, . . .

MPCCs are well known to violate the MFCQ, hence the set of KKT multipliers at a minimizer is either empty or unbounded (redundancy).

MPCC-Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{MPCC}}(x, \lambda, \mu, \nu^G, \nu^H) := f(x) + \mu^\top g(x) + \lambda^\top h(x) \\ - (\nu^G)^\top G(x) - (\nu^H)^\top H(x) \end{aligned}$$

Note: There is no multiplier for $G(x)^\top H(x) = 0$.

Finite-Dimensional MPCCs

There exists a whole zoo of stationarity concepts tailored to MPCCs. [Leyffer, Munson (2007)] coinced the term 'MPCC alphabet soup', which nowadays includes {A, B, C, M, S, T, W}-stationarity

Finite-Dimensional MPCCs

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Strictly (singly) active and biactive sets

The following index sets belong to the complementarity constraint.

$$\begin{aligned} \mathcal{A}_{+0}(x_0) &= \big\{1 \le i \le q : G_i(x_0) > 0,\ H_i(x_0) = 0\big\} \\ \mathcal{A}_{0+}(x_0) &= \big\{1 \le i \le q : G_i(x_0) = 0,\ H_i(x_0) > 0\big\} \\ \mathcal{A}_{00}(x_0) &= \big\{1 \le i \le q : G_i(x_0) = 0,\ H_i(x_0) = 0\big\}. \end{aligned}$$

A feasible point x_0 is called biactive if $\mathcal{A}_{00}(x_0) \neq \emptyset$.

Stationarity Concepts for MPCCs I

A feasible point x_0 of (MPCC) is said to be ...

- **weakly stationary** (**W-stationary**) for (MPCC) if there exist MPCC-multipliers $(\lambda, \mu, \nu^G, \nu^H)$ such that

$$\left.\begin{aligned}
&\nabla_x \mathcal{L}_{\text{MPCC}}(x_0, \lambda, \mu, \nu^G, \nu^H) = \nabla f(x_0) + g'(x_0)^\top \mu \\
&\qquad + h'(x_0)^\top \lambda - G'(x_0)^\top \nu^G - H'(x_0)^\top \nu^H = 0 \\
&h(x_0) = 0 \\
&\mu \geq 0, \quad g(x_0) \leq 0, \quad \mu^\top g(x_0) = 0 \\
&\nu_i^G = 0, \quad \nu_i^H \text{ free} \qquad \text{for } i \in \mathcal{A}_{+0}(x_0) \\
&\nu_i^G \text{ free}, \quad \nu_i^H = 0 \qquad \text{for } i \in \mathcal{A}_{0+}(x_0) \\
&\nu_i^G \text{ free}, \quad \nu_i^H \text{ free} \qquad \text{for } i \in \mathcal{A}_{00}(x_0).
\end{aligned}\right\} \qquad (*)$$

- **alternatively stationary** (**A-stationary**) if, in addition to $(*)$, one has

$$\nu_i^G \geq 0 \quad \text{or} \quad \nu_i^H \geq 0 \quad \text{for } i \in \mathcal{A}_{00}(x_0).$$

Stationarity Concepts for MPCCs II

A feasible point x_0 of (MPCC) is said to be ...

- **Clarke-stationary** (**C-stationary**) if, in addition to $(*)$, one has

$$\nu_i^G \, \nu_i^H \geq 0 \quad \text{for } i \in \mathcal{A}_{00}(x_0).$$

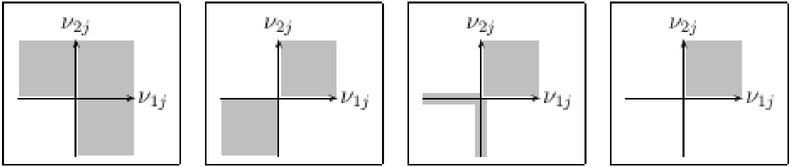
- **Mordukhovich-stationary** (**M-stationary**) if, in addition to $(*)$, one has

$$(\nu_i^G \geq 0 \quad \text{and} \quad \nu_i^H \geq 0) \quad \text{or} \quad \nu_i^G \, \nu_i^H = 0 \quad \text{for } i \in \mathcal{A}_{00}(x_0).$$

- **strongly stationary** (**S-stationary**) if, in addition to $(*)$, one has

$$\nu_i^G \geq 0 \quad \text{and} \quad \nu_i^H \geq 0 \quad \text{for } i \in \mathcal{A}_{00}(x_0).$$

Stationarity Concepts for MPCCs III



(a) A-stationär (b) C-stationär (c) M-stationär (d) S-stationär

$$\text{S-stationarity} \quad \Rightarrow \quad \text{M-stat.} \quad \Rightarrow \quad \left\{ \begin{array}{c} \text{A-stat.} \\ \text{C-stat.} \end{array} \right\} \quad \Rightarrow \quad \text{W-stat.}$$

$$\text{M-stat.} \quad \Leftrightarrow \quad \text{A-stat. and C-stat.}$$

- The above mentioned stationary concepts differ only in terms of the conditions for ν_i^G and ν_i^H on the biactive set $\mathcal{A}_{00}(x_0)$.
- Consequently, they all agree when $\mathcal{A}_{00}(x_0) = \emptyset$.
- S-, A- and W-stationarity are related to auxiliary NLPs.

Intermediate Conclusions

- Optimal control of VIs (of first or second kind) leads to Mathematical Programs with Equilibrium Constraints (MPECs) in function space.
- Deriving optimality conditions is not straightforward; techniques may differ from problem to problem.
- VIs of first kind can be reformulated as complementarity conditions, hence optimal control problems become Mathematical Programs with Complementarity Constraints (MPCCs).
- A hierarchy of stationarity concepts (and appropriate constraint qualifications) tailored for MPCCs exists in the literature. Hence optimality conditions for MPECs involving VIs of first kind can be classified according to strength.
- Such a classification seems to be lacking for MPECs with VIs of second kind.

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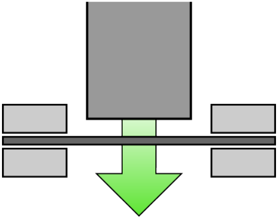
Application Background



Deep drawing

- car body parts
- plane body parts
- packings (thermoforming)

Springback

- release of stored elastic energy once the loads are withdrawn
- partial restoration away from the desired shape

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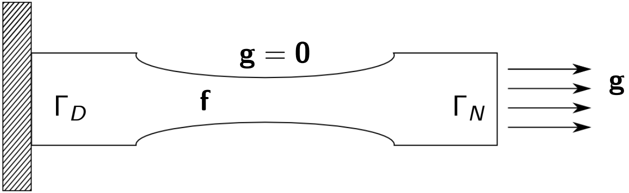
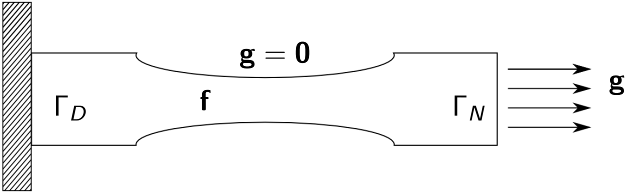


$\mathbf{u} = \mathbf{0}$ on Γ_D

$\boldsymbol{\sigma}\,\mathbf{n} = \mathbf{g}$ on Γ_N

displacement function space: $V = \{\mathbf{u} \in H^1(\Omega)^d : \mathbf{u} = \mathbf{0} \text{ on } \Gamma_D\}$
stress function space: $S = L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$
strain function space: $Q = S \cap L^2(\Omega; \mathbb{R}^{d\times d}_{\text{trace}=0})$



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We will consider a model with linear kinematic hardening.

- displacement $\mathbf{u} \in V$

dual (stress-based) formulation:

- stress $\boldsymbol{\sigma} \in S$
- back-stress $\boldsymbol{\chi} \in S$

primal (strain-based) formulation:

- plastic strain $\mathbf{p} \in Q$

The Dual Formulation of Elastoplasticity

Formulation: VI of first kind

$$a(\mathbf{\Sigma}, \mathbf{T} - \mathbf{\Sigma}) + b(\mathbf{T} - \mathbf{\Sigma}, \mathbf{u}) \geq 0 \quad \text{for all } \mathbf{T} \in K,$$
$$b(\mathbf{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V$$

Variables: $(\mathbf{u}, \mathbf{\Sigma}) = (\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\chi}) \in V \times S \times S$ (stress-based)

Forms:

$$a((\boldsymbol{\sigma}, \boldsymbol{\chi}), (\boldsymbol{\tau}, \boldsymbol{\mu})) = \int_\Omega \boldsymbol{\tau} : \mathbb{C}^{-1} \boldsymbol{\sigma} \, \mathrm{d}x + \int_\Omega \boldsymbol{\mu} : \mathbb{H}^{-1} \boldsymbol{\chi} \, \mathrm{d}x$$
$$b((\boldsymbol{\sigma}, \boldsymbol{\chi}), \mathbf{v}) = -\int_\Omega \boldsymbol{\sigma} : \varepsilon(\mathbf{v}) \, \mathrm{d}x$$
$$\langle \ell, \mathbf{v} \rangle = -\int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \, \mathrm{d}s$$

$K = \{\mathbf{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi}) \in S \times S : \phi(\mathbf{\Sigma}) \leq 0\}$ admissible generalized stresses

$\phi(\mathbf{\Sigma}) = |\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D|_F^2 - \widetilde{\sigma}_0^2$ yield function

The Primal Formulation of Elastoplasticity

Formulation: VI of second kind

$$a\big((\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q})-(\mathbf{u},\mathbf{p})\big)+j(\mathbf{q})-j(\mathbf{p})\geq\langle\ell,\,\mathbf{v}-\mathbf{u}\rangle\quad\text{for all }(\mathbf{v},\mathbf{q})\in V\times Q$$

Variables: $(\mathbf{u},\mathbf{p})\in V\times Q$ (strain-based)

Forms:

$$a\big((\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q})\big)=\int_\Omega\big[\underbrace{(\varepsilon(\mathbf{u})-\mathbf{p})}_{\text{elastic strain}}:\mathbb{C}\,(\varepsilon(\mathbf{v})-\mathbf{q})\big]\,\mathrm{d}x+\int_\Omega\mathbf{p}:\mathbb{H}\,\mathbf{q}\,\mathrm{d}x$$

$$\langle\ell,\,\mathbf{v}\rangle=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x+\int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}s$$

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Both static (incremental) problems represent one time step of their instationary (quasi-static) counterparts.

Relation Between Primal and Dual Models

The unique solutions $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\chi})$ of the dual and $(\mathbf{u}, \mathbf{p})$ of the primal problem are related by

$$\boldsymbol{\sigma} = \mathbb{C}(\underbrace{\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{p}}_{\text{elastic strain}}) \quad \text{and} \quad \boldsymbol{\chi} = -\mathbb{H}\,\mathbf{p}.$$

This can be shown using, e.g., Fenchel or Lagrange duality, see [Temam (1983); Han, Reddy (1999); Herzog, Meyer (2011)] .

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equivalence

$$\min_{(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\chi},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2}$$

s.t. dual VI (first kind)

$$\min_{(\mathbf{u},\mathbf{p},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2}$$

s.t. primal VI (second kind)

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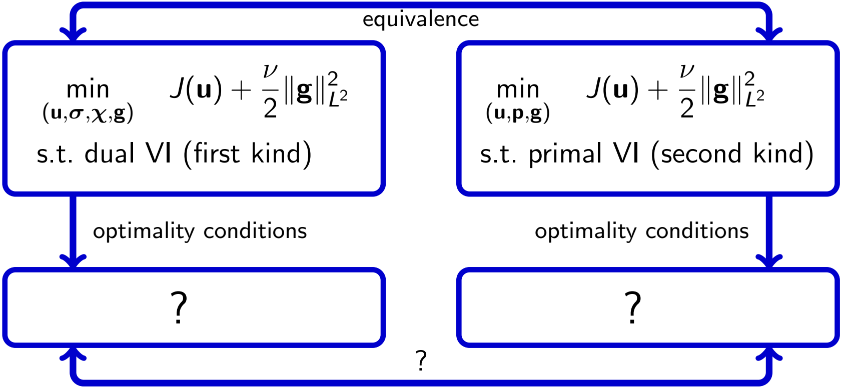
$$\boldsymbol{\sigma} = \mathbb{C}(\underbrace{\varepsilon(\mathbf{u}) - \mathbf{p}}_{\text{elastic strain}}) \quad \text{and} \quad \chi = -\mathbb{H}\,\mathbf{p}.$$

This can be shown using, e.g., Fenchel or Lagrange duality, see [Temam (1983); Han, Reddy (1999); Herzog, Meyer (2011)] .



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Optimality System for the Dual Model

- introduce a plastic multiplier $\lambda \in L^2(\Omega)$ and convert the VI of first kind into a complementarity system:

$$
\begin{aligned}
a(\boldsymbol{\Sigma}, \mathbf{T} - \boldsymbol{\Sigma}) + b(\mathbf{T} - \boldsymbol{\Sigma}, \mathbf{u}) &\geq 0 \quad \text{for all } \mathbf{T} \in K \\
b(\boldsymbol{\Sigma}, \mathbf{v}) &= \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V
\end{aligned}
$$

[Herzog, Meyer, Wachsmuth (ZAMM, 2011; SICON, 2012)]

Optimality System for the Dual Model

- introduce a plastic multiplier $\lambda \in L^2(\Omega)$ and convert the VI of first kind into a complementarity system:

$$\begin{aligned} a(\mathbf{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + \lambda\, \phi'(\mathbf{\Sigma})\, \mathbf{T} &= 0 \quad \text{for all } \mathbf{T} \in S^2 \\ b(\mathbf{\Sigma}, \mathbf{v}) &= \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V \\ \lambda \geq 0, \quad \phi(\mathbf{\Sigma}) \leq 0, \quad \lambda\, \phi(\mathbf{\Sigma}) &= 0 \quad \text{a.e. in } \Omega \end{aligned}$$

(proof by auxiliary problem which satisfies a constraint qualification, or by abstract convex analysis arguments)

[Herzog, Meyer, Wachsmuth (ZAMM, 2011; SICON, 2012)]

Optimality System for the Dual Model

- introduce a plastic multiplier $\lambda \in L^2(\Omega)$ and convert the VI of first kind into a complementarity system:

$$a(\mathbf{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + \lambda\,\phi'(\mathbf{\Sigma})\,\mathbf{T} = 0 \quad \text{for all } \mathbf{T} \in S^2$$

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$$\lambda \geq 0, \quad \phi(\mathbf{\Sigma}) \leq 0, \quad \lambda\,\phi(\mathbf{\Sigma}) = 0 \quad \text{a.e. in } \Omega$$

  (proof by auxiliary problem which satisfies a constraint qualification, or by abstract convex analysis arguments)

- treat $\phi(\mathbf{\Sigma}) \leq 0$ by a quadratic penalty method (exploiting that the forward model represents a principle of minimal energy),

$$\text{Minimize} \quad \frac{1}{2} a(\mathbf{\Sigma}, \mathbf{\Sigma}) + \frac{\gamma}{2} \|\mathbf{\Sigma} - P_K(\mathbf{\Sigma})\|_S^2$$

$$\text{s.t.} \quad b(\mathbf{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V.$$

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$$\text{Minimize} \quad \frac{1}{2} a(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}) + \frac{\gamma}{2} \int_\Omega \max\big\{0, |\underbrace{\mathcal{D}\boldsymbol{\Sigma}}_{\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D}| - \widetilde{\sigma}_0\big\}^2 \, dx$$
$$\text{s.t.} \quad b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V.$$

[Herzog, Meyer, Wachsmuth (ZAMM, 2011; SICON, 2012)]

Optimality System for the Dual Model Cont'd

- the optimality system for the penalized forward problem needs to be smoothed for further differentiation:

$$a(\mathbf{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + \gamma \int_\Omega \max \left\{0, 1 - \frac{\widetilde{\sigma}_0}{|\mathcal{D}\mathbf{\Sigma}|}\right\} \mathcal{D}\mathbf{\Sigma} : \mathcal{D}\mathbf{T} \, dx = 0,$$

$$b(\mathbf{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle$$

[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

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- the optimality system for the penalized forward problem needs to be smoothed for further differentiation:

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- the derivation of the KKT system for the penalized and smoothed problem requires a $W^{1,p}$ result (due to differentiation of Nemytzki operators) for nonlinear elasticity problems

[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

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- the derivation of the KKT system for the penalized and smoothed problem requires a $W^{1,p}$ result (due to differentiation of Nemytzki operators) for nonlinear elasticity problems
- finally, passage to the limit requires a rather close look at the particular problem ...

[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

Optimality System for the Dual Model Cont'd

- the optimality system for the penalized forward problem needs to be smoothed for further differentiation:

$$a(\boldsymbol{\Sigma}, \mathbf{T}) + b(\mathbf{T}, \mathbf{u}) + \gamma \int_\Omega \max{}_\varepsilon \Big\{ 0, 1 - \frac{\widetilde{\sigma}_0}{|\mathcal{D}\boldsymbol{\Sigma}|} \Big\} \mathcal{D}\boldsymbol{\Sigma} : \mathcal{D}\mathbf{T} \, \mathrm{d}x = 0,$$

$$b(\boldsymbol{\Sigma}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle$$

- the derivation of the KKT system for the penalized and smoothed problem requires a $W^{1,p}$ result (due to differentiation of Nemytzki operators) for nonlinear elasticity problems
- finally, passage to the limit requires a rather close look at the particular problem ...
- **Note:** in the quasi-static situation, penalization and smoothing leads to an elastoviscoplastic model.

[Herzog, Meyer, Wachsmuth (JMAA, 2011)]

Optimality System for the Dual Model

$$
\begin{aligned}
A\boldsymbol{\Sigma} + \lambda\, \mathcal{D}^\star \mathcal{D}\boldsymbol{\Sigma} + B^\star \mathbf{u} &= 0 \\
B\boldsymbol{\Sigma} &= R\, \mathbf{g} \\
0 \le \lambda \quad \perp \quad \phi(\boldsymbol{\Sigma}) &\le 0
\end{aligned}
$$

$$\mathcal{D}\boldsymbol{\Sigma} := \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D, \quad \langle R\, \mathbf{g}, \mathbf{v}\rangle := -\int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}\, \mathrm{d}s$$

[Herzog, Meyer, Wachsmuth (SICON, 2012)]

Optimality System for the Dual Model

$$
\begin{aligned}
A\boldsymbol{\Upsilon} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Upsilon} + \theta\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{w} &= 0 \qquad \mathcal{L}_{\boldsymbol{\Sigma}} = 0\\
B\boldsymbol{\Upsilon} &= \mathbf{u} - \mathbf{u}_d \qquad \mathcal{L}_{\mathbf{u}} = 0\\
\mathcal{D}\boldsymbol{\Upsilon} : \mathcal{D}\boldsymbol{\Sigma} - \mu &= 0 \qquad \mathcal{L}_{\lambda} = 0
\end{aligned}
$$

$$
\begin{aligned}
A\boldsymbol{\Sigma} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{u} &= 0\\
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[Herzog, Meyer, Wachsmuth (SICON, 2012)]

Optimality System for the Dual Model

$$A\boldsymbol{\Upsilon} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Upsilon} + \theta\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{w} = 0 \qquad \mathcal{L}_{\boldsymbol{\Sigma}} = 0$$

$$B\boldsymbol{\Upsilon} = \mathbf{u} - \mathbf{u}_d \qquad \mathcal{L}_{\mathbf{u}} = 0$$

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$$A\boldsymbol{\Sigma} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{u} = 0$$

$$B\boldsymbol{\Sigma} = R\,\mathbf{g}$$

$$0 \le \lambda \quad \perp \quad \phi(\boldsymbol{\Sigma}) \le 0$$

$$\perp \qquad\qquad \perp$$

$$\mu \qquad\qquad \theta$$

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[Herzog, Meyer, Wachsmuth (SICON, 2012)]

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$$A\boldsymbol{\Upsilon} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Upsilon} + \theta\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{w} = 0 \qquad \mathcal{L}_{\boldsymbol{\Sigma}} = 0$$

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$$B\boldsymbol{\Sigma} = R\,\mathbf{g}$$

$$0 \le \lambda \quad \perp \quad \phi(\boldsymbol{\Sigma}) \le 0$$

$$\perp \qquad \perp$$

$$\mu \qquad \theta$$

$$R^\star\mathbf{w} + \nu\,\mathbf{g} = 0 \qquad \mathcal{L}_{\mathbf{g}} = 0$$

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[Herzog, Meyer, Wachsmuth (SICON, 2012)]

Optimality System for the Dual Model

$$A\Upsilon + \lambda\,\mathcal{D}^\star\mathcal{D}\Upsilon + \theta\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{w} = 0 \qquad \mathcal{L}_{\boldsymbol{\Sigma}} = 0$$

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$$A\boldsymbol{\Sigma} + \lambda\,\mathcal{D}^\star\mathcal{D}\boldsymbol{\Sigma} + B^\star\mathbf{u} = 0$$

$$B\boldsymbol{\Sigma} = R\,\mathbf{g}$$

$$0 \le \lambda \quad \perp \quad \phi(\boldsymbol{\Sigma}) \le 0$$

$$\perp \qquad \perp$$

$$\mu \quad \cdot \quad \theta \quad \ge 0 \qquad \text{C-stationarity}$$

$$R^\star\mathbf{w} + \nu\,\mathbf{g} = 0 \qquad \mathcal{L}_{\mathbf{g}} = 0$$

$$\mathcal{D}\boldsymbol{\Sigma} := \boldsymbol{\sigma}^D + \chi^D, \qquad \langle R\,\mathbf{g}, \mathbf{v}\rangle := -\int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}s$$

[Herzog, Meyer, Wachsmuth (SICON, 2012)]

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Optimality System for the Primal Model

$$\text{Minimize} \quad \frac{1}{2}\|\mathbf{u} - \mathbf{u}_d\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2(\Gamma_N)}$$

$$\text{s.t.} \quad \begin{cases} a\big((\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q}) - (\mathbf{u},\mathbf{p})\big) + j(\mathbf{q}) - j(\mathbf{p}) \ge \langle \ell, \mathbf{v} - \mathbf{u}\rangle \\ \hfill \text{for all } (\mathbf{v},\mathbf{q}) \in V \times Q \end{cases}$$

Optimality System for the Primal Model

- Smooth out the j-term

$$\text{Minimize} \quad \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2(\Gamma_N)}$$

$$\text{s.t.} \quad \begin{cases} a\big((\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q})-(\mathbf{u},\mathbf{p})\big) + j_\varepsilon(\mathbf{q}) - j_\varepsilon(\mathbf{p}) \ge \langle \ell, \mathbf{v}-\mathbf{u}\rangle \\ \hfill \text{for all } (\mathbf{v},\mathbf{q}) \in V \times Q \end{cases}$$

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$$\text{s.t.} \quad \begin{cases} a\big((\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q})\big) + j_\varepsilon'(\mathbf{p})\,\mathbf{q} = \langle \ell,\, \mathbf{v}\rangle \\ \qquad \text{for all } (\mathbf{v},\mathbf{q}) \in V \times Q \end{cases}$$

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Recall $j(\mathbf{p}) = \widetilde{\sigma}_0 \int_\Omega |\mathbf{p}|\,\mathrm{d}x \quad \Rightarrow \quad j'(\mathbf{p})\,\mathbf{q} = \widetilde{\sigma}_0 \int_\Omega \frac{\mathbf{p}:\mathbf{q}}{|\mathbf{p}|}\,\mathrm{d}x.$

We used a Huber-type smoothing, where

$$j_\varepsilon'(\mathbf{p})\,\mathbf{q} = \frac{\widetilde{\sigma}_0}{\varepsilon} \int_\Omega \frac{\mathbf{p}:\mathbf{q}}{m_\varepsilon(|\mathbf{p}|)}\,\mathrm{d}x.$$

Optimality System for the Primal Model

- Smooth out the j-term

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$$j_\varepsilon'(\mathbf{p})\,\mathbf{q} = \frac{\widetilde{\sigma}_0}{\varepsilon} \int_\Omega \frac{\mathbf{p}:\mathbf{q}}{m_\varepsilon(|\mathbf{p}|)}\,\mathrm{d}x.$$

- Again, the $W^{1,p}$ result (necessary for the differentiation of the Nemytzki operator above) for nonlinear elasticity problems plays a role in deriving optimality conditions for the regularized problems.

Optimality System for the Primal Model

$$a\big((\mathbf{v},\mathbf{q}),(\mathbf{w},\mathbf{r})\big) + \int_\Omega \boldsymbol{\pi} : \mathbf{q} \, \mathrm{d}x = -\int_\Omega (\mathbf{u}-\mathbf{u}_d)\cdot \mathbf{v} \, \mathrm{d}x \quad \text{for all } (\mathbf{v},\mathbf{q}) \in V \times Q$$

$$a\big((\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q})\big) + \int_\Omega \boldsymbol{\varrho} : \mathbf{q} \, \mathrm{d}x = \langle \ell, \mathbf{v}\rangle \quad \text{for all } (\mathbf{v},\mathbf{q}) \in V \times Q$$

$$\underbrace{\boldsymbol{\varrho} : \mathbf{p} = \widetilde{\sigma}_0 \, |\mathbf{p}|, \quad |\boldsymbol{\varrho}| \le \widetilde{\sigma}_0}_{\boldsymbol{\varrho} \in \widetilde{\sigma}_0 \partial |\mathbf{p}|} \quad \text{a.e. in } \Omega$$

$$\boldsymbol{\pi} : \mathbf{p} = 0 \quad \text{a.e. in } \Omega$$

$$\widetilde{\sigma}_0 \, \mathbf{r} = |\mathbf{p}| \, \boldsymbol{\pi} + \vartheta \boldsymbol{\varrho} \quad \text{a.e. in } \Omega$$

$$\vartheta \boldsymbol{\varrho} : \boldsymbol{\pi} \ge 0 \quad \text{a.e. in } \Omega$$

$$\vartheta = 0 \text{ where } |\boldsymbol{\varrho}| < \widetilde{\sigma}_0$$

$$\nu \, \mathbf{g} - \mathbf{w} = \mathbf{0} \quad \text{a.e. on } \Gamma_N$$

[de los Reyes, Herzog, Meyer (submitted)]

Optimality System for the Primal Model

$$a\big((\mathbf{v},\mathbf{q}),(\mathbf{w},\mathbf{r})\big) + \int_\Omega \boldsymbol{\pi} : \mathbf{q}\,\mathrm{d}x = -\int_\Omega (\mathbf{u}-\mathbf{u}_d)\cdot\mathbf{v}\,\mathrm{d}x \quad \text{for all } (\mathbf{v},\mathbf{q}) \in V \times Q$$

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[de los Reyes, Herzog, Meyer (submitted)]

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Completing the Picture

equivalence

$$\min_{(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\chi},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|_{L^2}^2$$

s.t. dual VI (first kind)

$$\min_{(\mathbf{u},\mathbf{p},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|_{L^2}^2$$

s.t. primal VI (second kind)

optimality conditions

optimality conditions

?

?

?

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s.t. primal VI (second kind)

optimality conditions

optimality conditions

C-stationarity

$$(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\chi}, \lambda, \mathbf{g}, \ \mathbf{w}, \zeta, \psi, \theta)$$

?

?

Completing the Picture

equivalence

$$\min_{(\mathbf{u},\boldsymbol{\sigma},\chi,\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|_{L^2}^2$$

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s.t. primal VI (second kind)

optimality conditions

optimality conditions

C-stationarity

$$(\mathbf{u}, \boldsymbol{\sigma}, \chi, \lambda, \mathbf{g},\ \mathbf{w}, \zeta, \psi, \theta)$$

system just seen

$$(\mathbf{u}, \mathbf{p}, \varrho, \mathbf{g},\ \mathbf{w}, \mathbf{r}, \boldsymbol{\pi}, \vartheta)$$

?

Completing the Picture

equivalence

$$\min_{(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\chi},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|_{L^2}^2$$

s.t. dual VI (first kind)

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$$(\mathbf{u}, \mathbf{p}, \varrho, \mathbf{g}, \ \mathbf{w}, \mathbf{r}, \boldsymbol{\pi}, \vartheta)$$

?

$$\mathbf{p} := \varepsilon(\mathbf{u}) - \mathbb{C}^{-1}\boldsymbol{\sigma}, \quad \varrho := \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D$$

$$\mathbf{r} := \varepsilon(\mathbf{w}) - \mathbb{C}^{-1}\zeta, \quad \boldsymbol{\pi} := \zeta^D + \psi^D, \quad \vartheta := \widetilde{\sigma}_0\,\theta$$

Completing the Picture

equivalence

$$\min_{(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\chi},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|_{L^2}^2$$

s.t. dual VI (first kind)

$$\min_{(\mathbf{u},\mathbf{p},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|_{L^2}^2$$

s.t. primal VI (second kind)

optimality conditions

optimality conditions

C-stationarity

$$(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\chi}, \lambda, \mathbf{g}, \ \mathbf{w}, \boldsymbol{\zeta}, \boldsymbol{\psi}, \theta)$$

system just seen

$$(\mathbf{u}, \mathbf{p}, \boldsymbol{\varrho}, \mathbf{g}, \ \mathbf{w}, \mathbf{r}, \boldsymbol{\pi}, \vartheta)$$

equivalence!

$$\boldsymbol{\sigma} := \mathbb{C}(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{p}), \quad \boldsymbol{\chi} := -\mathbb{H}\,\mathbf{p}, \quad \lambda := \frac{|\mathbf{p}|}{\widetilde{\sigma}_0}$$

$$\boldsymbol{\zeta} := \mathbb{C}\,(\boldsymbol{\varepsilon}(\mathbf{w}) - \mathbf{r}), \quad \boldsymbol{\psi} := -\mathbb{H}\,\mathbf{r}, \quad \theta := \frac{\vartheta}{\widetilde{\sigma}_0}.$$

Completing the Picture

equivalence

$$\min_{(\mathbf{u},\boldsymbol{\sigma},\chi,\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2}$$

s.t. dual VI (first kind)

$$\min_{(\mathbf{u},\mathbf{p},\mathbf{g})} \quad J(\mathbf{u}) + \frac{\nu}{2}\|\mathbf{g}\|^2_{L^2}$$

s.t. primal VI (second kind)

optimality conditions

optimality conditions

C-stationarity

$$(\mathbf{u},\boldsymbol{\sigma},\chi,\lambda,\mathbf{g},\ \mathbf{w},\zeta,\psi,\theta)$$

system just seen

$$(\mathbf{u},\mathbf{p},\varrho,\mathbf{g},\ \mathbf{w},\mathbf{r},\boldsymbol{\pi},\vartheta)$$

equivalence!

Recall: A classification scheme seems to be lacking for MPECs with VIs of second kind. Through the equivalence with C-stationarity, we can compare now.

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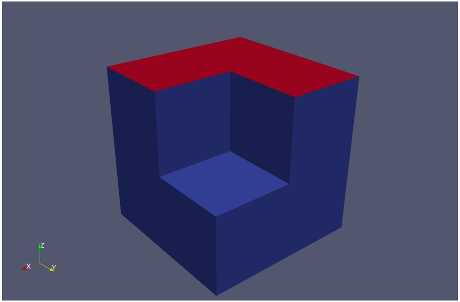
Example I: Static Primal Model

- control and observation at upper boundary
- discretization with P_1 (displacements) and P_0 (stresses, strains) elements
- total number of unknowns in (primal, Huber-regularized) optimality system $\approx 415\,000$



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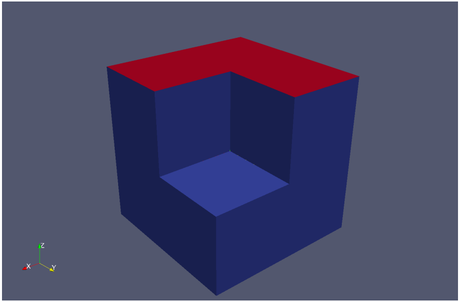
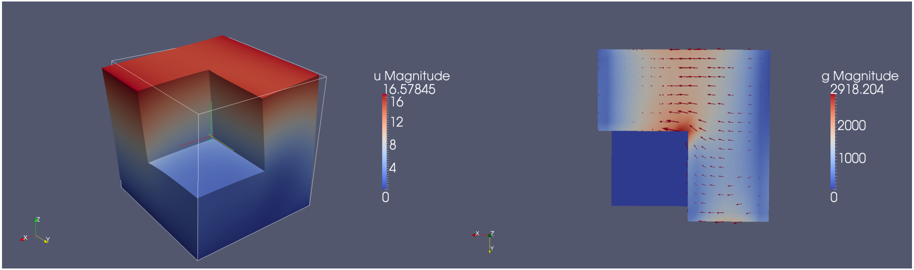
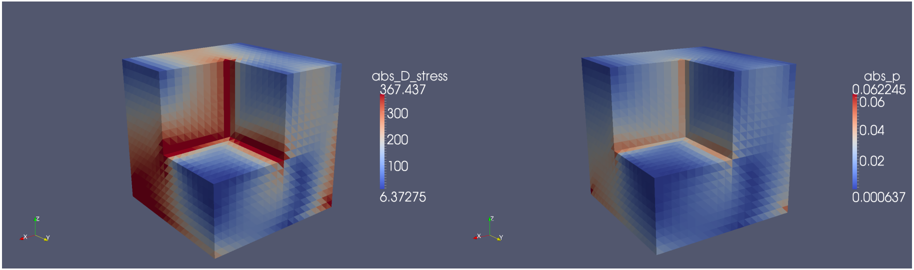


γ	Newton	$\lvert \boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \rvert$	infeasibility
10^0	2	$6.3374 \cdot 10^{-2}$	$3.6736 \cdot 10^2$
10^1	2	$6.3371 \cdot 10^{-1}$	$3.6679 \cdot 10^2$
10^2	2	$6.3350 \cdot 10^0$	$3.6109 \cdot 10^2$
10^3	2	$6.3132 \cdot 10^1$	$3.0429 \cdot 10^2$
10^4	3	$3.6742 \cdot 10^2$	$4.7411 \cdot 10^{-6}$
10^5	did not converge		

Example I: Static Primal Model

displacement $|\mathbf{u}|$ boundary traction $\mathbf{g}$



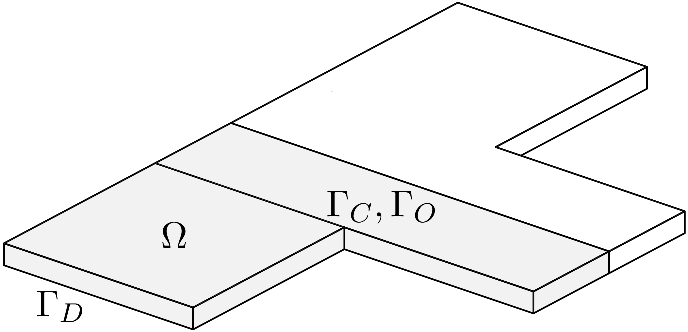


von Mises stress $|\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D|$ plastic strain $|\mathbf{p}|$

Example II: Quasi-Static Dual Model

Control and observation at upper boundary, $\mathbf{u}_{T,d} = (0, 0, 0.05)\,\text{mm}$,
Left: Dirichlet boundary conditions.

Setup:

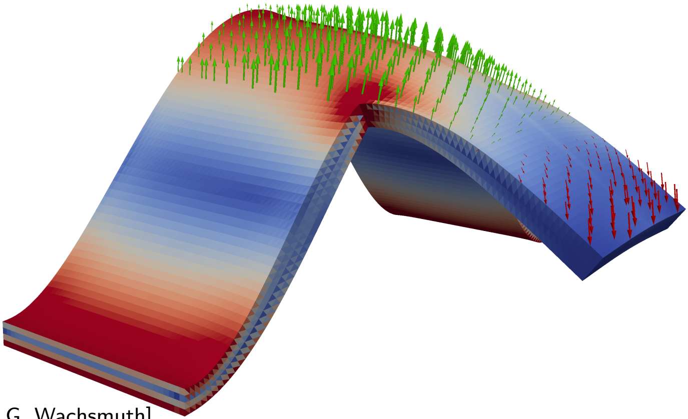


[Dissertation G. Wachsmuth]

Example II: Quasi-Static Dual Model

Control and observation at upper boundary, $\mathbf{u}_{T,d} = (0, 0, 0.05)\,\text{mm}$,
Left: Dirichlet boundary conditions.

von Mises stress in the mid of the time intervall (displacement $\times 50$):

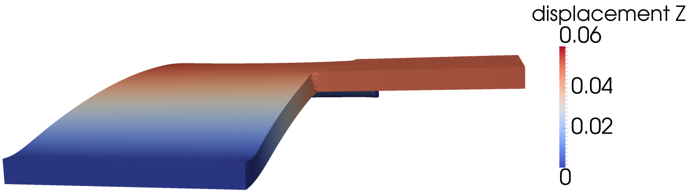


[Dissertation G. Wachsmuth]

Example II: Quasi-Static Dual Model

Control and observation at upper boundary, $\mathbf{u}_{T,d} = (0, 0, 0.05)\,\mathrm{mm}$,
Left: Dirichlet boundary conditions.

Displacement u_z at the final time T (displacement $\times 500$):



[Dissertation G. Wachsmuth]

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Summary and Conclusions

- pointed out challenges in deriving optimality conditions for problems which involve VIs of first and second kind
- relaxation/penalty is often a good way to go
- optimization problems involving VIs of first kind can often be formulated as MPCCs
- for MPCCs, a classification of optimality conditions exists (MPCC alphabet soup)
- for static elastoplastic control problems, penalty/smoothing (dual formulation) and Huber-type regularization (primal formulation) lead to equivalent optimality systems

DFG SPP 1253 funding is gratefully acknowledged.

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Thank You

DFG SPP 1253 funding is gratefully acknowledged.

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