Multivalued equations motivated by granular flow model

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Program

- The Savage-Hutter model
- Physical motivation for multivalued equations
- Existence of weak solutions to 1-D model
- Idea of the proof
- Problems and extensions

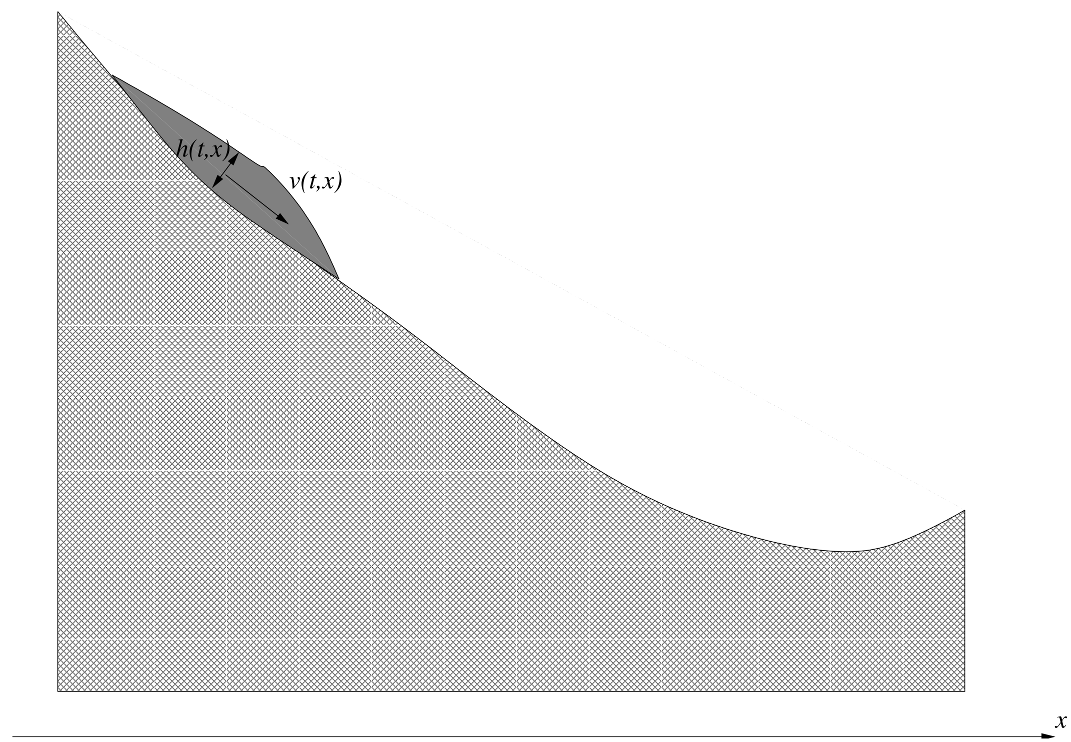
Original 1-D Savage-Hutter Model '89

Find

- the height $h : \mathbb{R} \times \mathbb{R}^{+} \longrightarrow \mathbb{R}$
- the velocity $v : \mathbb{R} \times \mathbb{R}^{+} \longrightarrow \mathbb{R}$

satisfying the system of conservation laws

$$\begin{aligned} \partial_t h + \partial_x (hv) &= 0 \\ \partial_t (hv) + \partial_x \left(hv^2 + \beta h^2 \right) &\ \mathbf{?}\ hg, \end{aligned} \tag{SH}$$


h(t,x)
v(t,x)
x

Here $\beta(x)$, $g(x, v)$ are defined by

$$\begin{aligned} \beta(x) &= k \cos \xi(x), \\ g(x, v) &= \sin\xi(x) - \mathrm{sign}_0(v)\cos \xi(x) \end{aligned}$$

where $\xi(x)-$ inclination angle of bottom topography at point x.

The function g is **not** continuous at $v = 0$.

Therefore extension to a multivalued $\mathrm{sign}(v)$:

$$\mathrm{sign}(v) = \begin{cases} -1 & \text{for} \quad v < 0 \\ [-1, 1] & \text{for} \quad v = 0 \\ 1 & \text{for} \quad v > 0 \end{cases}$$

Physical motivation (Gwiazda '02, Hadeler & Kuttler '03, Bouchut & Westdickenberg '04)

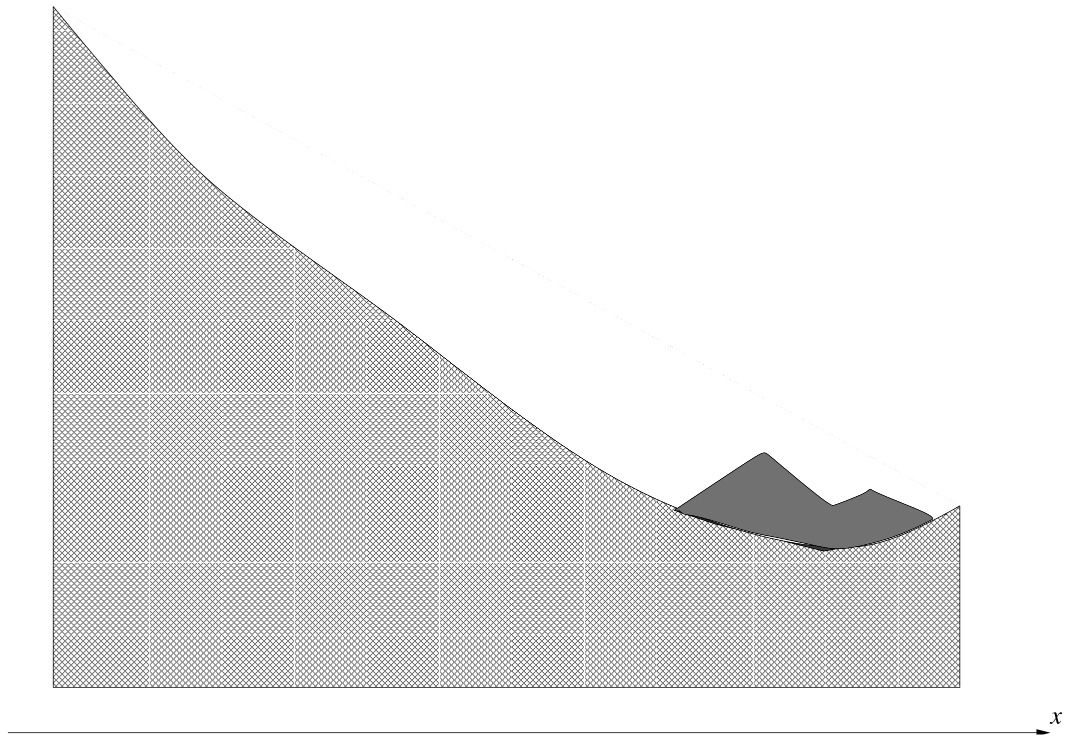
Consider static problem ($v = 0$) with $\xi = 0$

$\Rightarrow$ Governing equation:

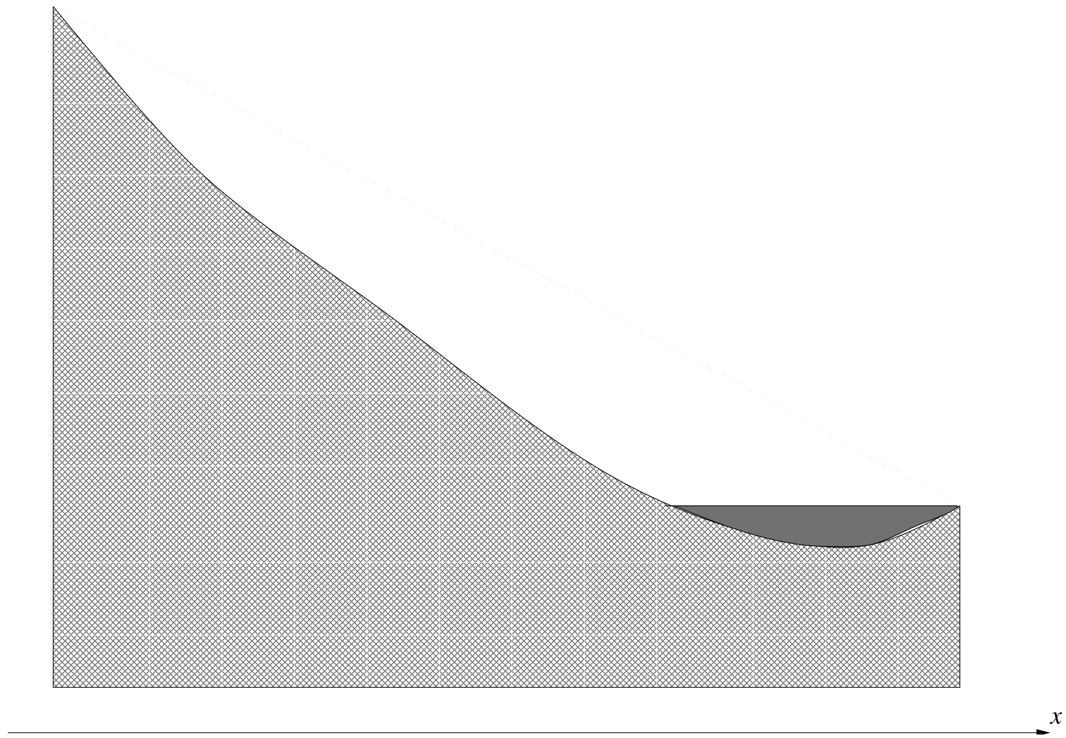
$$kh_x \in [-1, 1]$$

$\Rightarrow$ h is Lipschitz with Lipschitz constant $\leq \frac{1}{k}$

Physically relevant solution!



x


x

Mathematics for 1-D problem (P.G. '02, *Asymptot. Anal.*)

First consider

$$\partial_t u + \partial_x F(u) = G(x,t)$$

Definition 1 (classical)

Suppose that $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ are scalar C^1-functions satisfying

$$\nabla \eta(u_1, u_2) \cdot \nabla F(u_1, u_2) = \nabla q(u_1, u_2).$$

Such functions η, q are called entropy-entropy flux pair . If η is convex, then (η, q) is called a convex entropy-entropy flux pair.

Definition 2 (classical)

We call $u \in L^\infty(\mathbb{R} \times [0,T); \mathbb{R}_+ \times \mathbb{R})$ a weak entropy solution to

$$\partial_t u + \partial_x F(u) = G, \quad u_0 \in L^\infty$$

if u is a weak solution, i.e., $\forall \psi \in C_0^1(\mathbb{R} \times [0,T); \mathbb{R}^2)$ it holds

$$\int_{\mathbb{R}\times[0,T)} (u \cdot \psi_t + F(u) \cdot \psi_x + G \cdot \psi) dx dt = -\int_{\mathbb{R}} u_0 \cdot \psi(0,x) dx,$$ and u

satisfies the entropy inequality, i.e, $\forall \phi \in C_0^1(\mathbb{R} \times [0,T); \mathbb{R}_+)$ and for all convex weak entropy-entropy flux pairs (η, q):

$$\int_{\mathbb{R}\times[0,T)} (\eta\phi_t + q\phi_x + \nabla\eta \cdot G\phi) dx dt \geq -\int_{\mathbb{R}} \eta(0,x)\phi(0,x) dx$$

Definition 2a (nonclassical)

We call $u \in L^\infty(\mathbb{R} \times [0,T); \mathbb{R}_+ \times \mathbb{R})$ a weak entropy solution to the system

$$\partial_t u + \partial_x F(u) \in \tilde{G}(u,x) \qquad (CL)$$

with the initial data $u_0 \in L^\infty$ iff

$$\exists\ G(x,t) \in \tilde{G}(u(x,t),x)$$

such that u is a weak entropy solution to the system

$$\partial_t u + \partial_x F(u) = G.$$

To avoid the problem with $x-$dependent flux

$$u = (u_1, u_2) = (\beta h, \beta h v)$$

and $\tilde{G}$ is a mulitifunction

$$\tilde{G}(u_1, u_2, x) = \begin{pmatrix} \dfrac{\beta'(2\kappa)^{-1}}{\beta^{-1}} u_2 \\ \dfrac{\beta'(2\kappa)^{-1}}{\beta^{-1}} \left(\dfrac{(u_2)^2}{u_1} + \kappa (u_1)^2 \right) + u_1 \tilde{g}\,(u_2, x) \end{pmatrix}$$

where

$$\tilde{g}\,(u_2, x) = \sin \xi(x) - \operatorname{sign}(u_2) \cos \xi(x).$$

Existence of weak solutions (1-D problem)

Theorem 1 (Local in time existence)
Assume $h^0, v^0 \in L^\infty(\mathbb{R})$, $\inf\limits_{x\in\mathbb{R}} h^0(x) \geq 0$ and $\beta \in C^1(\mathbb{R})$ with $\beta(x) \geq \beta_0 > 0$.
Then the equation (SH) possesses a local in time weak entropy solution in the sense of Definition 2a.

Remark
If $\beta(x) = const$, then we have global in time existence.

Proof (idea):

- viscous approximation $(+\varepsilon\partial_x^2 u)$ and approximation of $\tilde{G}$ by single valued Lipschitz function G_ε
- Invariant region estimates (L^∞-estimates)
- limit $\varepsilon \to 0^+$
- Murat Lemma, Div-Curl Lemma, Young measure characterisation of weak limit (fully coressponding to Perthame, Lions & Souganidis '96). Then: $u^\varepsilon \to u$ strongly in L^p.
- weak limits of the terms $G_\varepsilon(x, u_\varepsilon)$ and $G_\varepsilon(x, u_\varepsilon) \cdot \nabla_u \eta(u_\varepsilon)$.

Why do we have a problem?

Answer:

⋆ $\nabla_u \eta$ is not continous at point $(0,0)$ (it possesses directional limits only!)

⋆ Convergence in measure of u_ε doesn't guarantee convergence in measure of $\nabla_u \eta(u_\varepsilon)$!

Problem: both terms $G_\varepsilon(x, u_\varepsilon)$, $\nabla_u \eta(u_\varepsilon)$ converge only weakly!

Lemma Assume

- $u_2^k \to u_2$ strongly in $L^p(\Omega)$ for $1 \leq p < \infty$,
- $\mathrm{sign}_k \to \mathrm{sign}$ uniformly in $\mathbb{R} \setminus [-\frac{1}{n}, \frac{1}{n}]$ for all $n \in \mathbb{N}$
- $|\mathrm{sign}_k| \leq 1.$

Then there exists a function $S \in L^p(\Omega)$ and a subsequence k_j such that

- $\mathrm{sign}_{k_j}(u_2^{k_j}) \rightharpoonup S$ weakly in $L^p(\Omega)$
- $S \in \mathrm{sign}(u_2)$ a.e. in Ω

Problems and extensions

- Well posedness of solutions for related scalar multivalued equation (P.G. & Swierczewska NA TMA 2005)
- Well posedness of solutions for original one ?
- measure-valued solutions to 2-D problem (P.G. MMAS 2004)
- Kinetic scheme for the Savage-Hutter equations. (Kaland &Struckmeier) MMAS 2008 (based on wrong preprint!)
- A. Cattani, R. M. Colombo, and G. Guerra. New model including effect of the exchange of material with a bottom (Cattani, Colombo, Guerra ZAMM, 2012)

Some research in 2-D formulation

⋆ Gray, Wieland, Hutter, Gravity-driven free surface flow of granular avalanches over complex basal topography, *Proc. R. Soc. Lond.* '99

⋆ Wieland, Gray, Hutter, Channelized free-surface flown of cohesionless granular avalanches in a chute with shallow lateral curvature. *J. Fluid Mech.* '99

⋆ Dynamic Response of Granular and Porous Materials under Large and Catastrophic Deformations (Kirchner & Hutter, eds) Lecture Notes in Applied and Computational Mechanics, Springer '03

⋆ Bouchut, Westdickenberg, Gravity driven shallow water models for arbitrary topography '04

Theorem (Fundamental theorem on Young measures)

Let $\Omega \subset \mathbb{R}^d$ and let $z^j : \Omega \to \mathbb{R}^d$ be a sequence of measurable functions. Then there exists (z^{j_k}) and a weakly* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ such that:

(i) $\nu_x \geq 0, \quad \|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} d\nu_x \leq 1$ for a.a. $x \in \Omega$.

(ii) For all $g \in C_0(\mathbb{R}^d)$

$$g(z^{j_k}) \rightharpoonup^* \langle \nu_x, g \rangle \text{ in } L^\infty(\Omega)$$

(iii) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ for a.a. $x \in \Omega \iff$ the 'tightness condition' is satisfied, i.e.

$$\lim_{M\to\infty} \sup_k |\{|z^{j_k}| \geq M\}| = 0.$$

(iv) Tightness condition + $A \subset \Omega$ is measurable, $g \in C(\mathbb{R}^d)$and $g(z^{j_k})$ is relatively weakly compact in $L^1(A)$, then

$$g(z^{j_k}) \rightharpoonup \langle \nu_x, g \rangle \quad \text{in } L^1(A).$$

Remark The map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ is called the Young measure generated by the sequence (z^{j_k}). Every (weakly* measurable map) $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ that satisfies (i) is generated by some sequence (z^k).

Lemma

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be maximal monotone and $g : \mathbb{R}^d \to \mathbb{R}^d$ be any continuous function. Then

$$f \circ (f + Id)^{-1} : \overline{\mathrm{Rang}(f + Id)} \to \mathbb{R}^d$$

and

$$g \circ (f + Id)^{-1} : \mathrm{Rang}(f + Id) \to \mathbb{R}^d$$

are continuous functions.

Proof

Note that

$$(f + Id)^{-1} : \mathrm{Rang}(f + Id) \to \mathbb{R}^d$$

is Lipschitz function with the Lipschitz constant less or equal to one. Then it can be automatically extended to the continuous function

$$(f + Id)^{-1} : \overline{\mathrm{Rang}(f + Id)} \to \mathbb{R}^d.$$

We must prove only that $f \circ (f + Id)^{-1}$ is continuous, but it follows simply from the equality

$$Id = f \circ (f + Id)^{-1} + Id \circ (f + Id)^{-1}.$$