On asymptotic behavior of solutions to the compressible Navier-Stokes equation in a periodic layer

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1.Infinite layer with flat boundaries: Stability of parallel flow

- $\rho = \rho(x,t)$, $v = (v^1(x,t), \cdots, v^n(x,t))$, $t \geq 0$, $x \in \mathbf{R}^n$ $(n \geq 2)$.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \operatorname{div} v + \nabla P(\rho) = \rho \boldsymbol{g}. \end{cases}$$

- $P = P(\rho)$: pressure; smooth in ρ,

$$P'(\rho_*) > 0 \ \text{ for } \ \text{ a } \ \text{ constant } \ \rho_* > 0$$

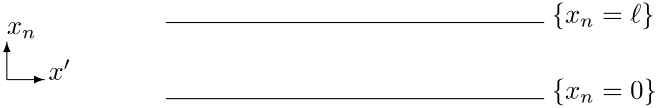
- μ, μ': const's,

$$\mu > 0, \ \frac{2}{n}\mu + \mu' \geq 0$$

- $\rho = \rho(x,t)$, $v = (v^1(x,t), \cdots, v^n(x,t))$, $t \geq 0$, $x \in \mathbf{R}^n$ $(n \geq 2)$.

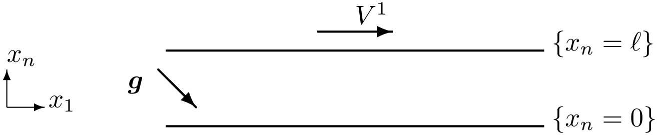
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu') \nabla \operatorname{div} v + \nabla P(\rho) = \rho \boldsymbol{g}. \end{cases}$$

- $\Omega_\ell = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\ 0 < x_n < \ell\}$



Parallel Flow

- $\boldsymbol{g} = (g^1(x_n), 0, \cdots, 0, g^n(x_n))$; bounded smooth
- (B.C) $v|_{x_n=0} = 0$, $v|_{x_n=\ell} = (V^1, 0, \cdots, 0)$ (V^1: const.)



$\Rightarrow \exists$ smooth stationary flow $u_s(x_n) = (\rho_s(x_n), v_s(x_n))$:

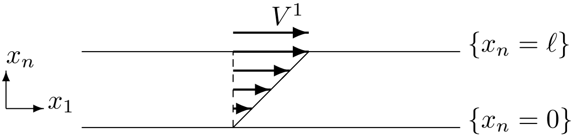
$$\inf_{x_n \in [0,\ell]} \rho_s(x_n) > 0, \quad \frac{1}{\ell} \int_0^\ell \rho_s(x_n)\, dx_n = \rho_*,$$

$$v_s = (v_s^1(x_n), 0, \cdots, 0)$$

E.g.,

plane Couette flow

- $\boldsymbol{g} = \mathbf{0}$,
- $v|_{x_n=\ell} = (V^1, 0, \cdots, 0)$, $v|_{x_n=0} = 0$

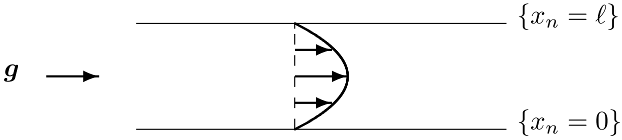


$$\rho_s = \rho_* > 0,$$

$$v_s = \left(\frac{V^1}{\ell} x_n, 0, \cdots, 0 \right).$$

Poiseuille flow

- $\boldsymbol{g} = (g^1, 0, \cdots, 0)$ ($g^1 \equiv \text{const.} \neq 0$)
- $v|_{x_n=0,\ell} = 0$



$$\rho_s = \rho_*,$$

$$v_s = \left(\frac{\rho_* g^1}{2\mu} x_n(\ell - x_n), 0, \cdots, 0 \right).$$

- Stability of parallel flow $u_s = (\rho_s, v_s)$
- Large time behavior of the perturbation :

$$(\phi(t), w(t)) = (\rho(t) - \rho_s, v(t) - v_s).$$

Stability under spatially periodic perturbations

Iooss–Padula (1998):

- Linear stability of parallel flow in a cylindrical domain under spatially periodic perturbations with

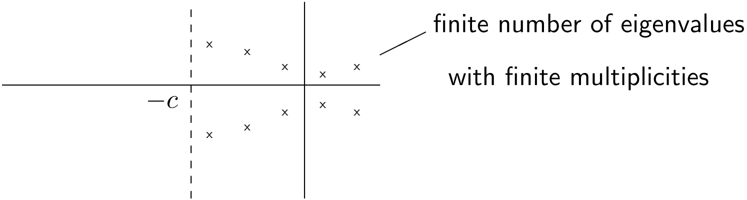
$$\int_{\Omega_{per}} \phi(x,t)\,dx = 0 \quad (\Omega_{per}\text{: the basic periodic domain})$$

- $$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0.$$

  $\mathrm{Re} \leq \exists\, \mathrm{Re}_1 \Rightarrow$ the parallel flow is linearly stable,

$$\|u(t)\|_{L^2(\Omega_{per})} \leq e^{-\delta_1 t}\|u_0\|_{L^2(\Omega_{per})}.$$

- $\sigma(-L) \cap \{\lambda; \mathrm{Re}\,\lambda > -c\} = \{\text{finite number of eigenvalues}\}$.



Stability under local perturbations

Non-dimensionalization

$$x = \ell \tilde{x},\ t = \frac{\ell}{V}\tilde{t},\ v = V\tilde{v},\ \rho = \rho_* \tilde{\rho},\ P = \rho_* V^2 \tilde{P}$$

$$V = \|v_s\|_{C_*^{m+1}[0,\ell]} \equiv \sup_{x_n \in [0,\ell]} \sum_{k=0}^{m+1} \ell^k |\partial_{x_n}^k v_s(x_n)|$$

$$(m \geq [n/2] + 1)$$

$\Longrightarrow$ (omitting tildes)

$\Omega_\ell \to \Omega \equiv \Omega_1$

$\Omega = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\, 0 < x_n < 1\}$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \ \{x_n = 1\}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \ \{x_n = 0\}$$

$u_s = (\rho_s, v_s) \quad \to \quad u_s = (\rho_s, v_s)$:

$$\inf_{x_n \in [0,1]} \rho_s(x_n) > 0, \quad \int_0^1 \rho_s(x_n)\, dx_n = 1,$$
$$v_s = (v_s^1(x_n), 0, \cdots, 0), \quad |v_s|_{C^{m+1}[0,1]} = 1$$

- $u(t) = (\phi(t), w(t)) = (\gamma^2(\rho(t) - \rho_s), v(t) - v_s)$: perturbation

$$\partial_t \phi + v_s \cdot \nabla \phi + \gamma^2 \mathrm{div}\,(\rho_s w) = f^0, \tag{2.1}$$

$$\begin{aligned} \partial_t w - \tfrac{\nu}{\rho_s}\Delta w - \tfrac{\tilde{\nu}}{\rho_s}\nabla \mathrm{div}\, w + \nabla \left(\tfrac{\tilde{P}'(\rho_s)}{\gamma^2 \rho_s}\phi \right) \\ + \tfrac{\nu\phi}{\gamma^2 \rho_s^2}\Delta v_s + v_s \cdot \nabla w + w \cdot \nabla v_s = \boldsymbol{f}, \end{aligned} \tag{2.2}$$

$$w|_{x_n=0,1} = 0, \quad (\phi, w)|_{t=0} = (\phi_0, w_0), \tag{2.3}$$

- f^0, $\boldsymbol{f}$: nonlinearities
- $\gamma = \sqrt{\tilde{P}'(1)} = \frac{\sqrt{P'(\rho_*)}}{V}, \ \ \nu = \frac{\mu}{\rho_* \ell V}, \ \ \tilde{\nu} = \frac{\mu + \mu'}{\rho_* \ell V}$
- $\mathrm{Re} = \dfrac{1}{\nu}$: Reynolds number, $\mathrm{Ma} = \dfrac{1}{\gamma}$: Mach number

Theorem 1 (Y.K., 2012)

Let m *be an integer satisfying* $m \geq [n/2]+1$. *Then* $\exists\ \nu_0 > 0$, $\gamma_0 > 0$, $\omega_0 > 0$ *such that if*

$$\nu \geq \nu_0, \quad \frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_0^2,$$

$$|\rho_s - 1|_{C^{m+1}[0,1]} \leq \omega_0,$$

then the following assertions hold:

$u_0 = (\phi_0, w_0) \in (H^m \cap L^1)(\Omega)$: $|u_0|_{(H^m \cap L^1)(\Omega)} \ll 1$ + *compatibility condition,*

$\Rightarrow \exists 1$ *global solution* $u(t) = (\phi(t), w(t))$ *of (2.1)–(2.3).*

The solution $u(t)$ *has the following properties.*

If $n \geq 3$*, then*

$$\|\partial_{x'}^{\ell} u(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{\ell}{2}}) \quad (t \to \infty)$$

for $\ell = 0, 1$ *and*

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}} \eta_n(t)) \quad (t \to \infty).$$

Here $u^{(0)} = u^{(0)}(x_n)$*;* $\sigma = \sigma(x', t)$*: solution of*

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_1 \partial_{x_1} \sigma = 0,$$

$$\sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n,$$

where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$*;* $\kappa_1 > 0$ $\kappa'' > 0$*,* a_1*: constants;* $\eta_n(t) = 1$ *when* $n \geq 4$ *and* $\eta_n(t) = \log(1+t)$ *when* $n = 3$*.*

If $n=2$*, then*

$$\|\partial_{x_1}^{\ell} u(t)\|_{L^2(\Omega)} = O(t^{-\frac{1}{4}-\frac{\ell}{2}}) \quad (t \to \infty)$$

for $\ell = 0, 1$ *and*

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} = O(t^{-\frac{3}{4}+\varepsilon}), \quad \varepsilon > 0, \quad (t \to \infty).$$

Here $u^{(0)} = u^{(0)}(x_2)$*;* $\sigma = \sigma(x_1, t)$*: solution of*

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + a_1 \partial_{x_1} \sigma + a_2 \partial_{x_1}(\sigma^2) = 0,$$

$$\sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2)\, dx_2,$$

where $\kappa_1 > 0$*,* a_j *(*$j = 1, 2$*): constants.*

Remarks.

(i) Cylindrical Domain of $\mathbf{R}^3$ (R. Aoyama, 2014)

(ii) Time periodic parallel flow (J. Brezina, 2013) :

$\boldsymbol{g} = (g^1(x_n, t), 0, \cdots, 0, g^n(x_n))$, $V^1 = V^1(t)$:

$$g^1(x_n, t+T) = g^1(x_n, t), \quad V^1(t+T) = V^1(t).$$

$\Longrightarrow \exists$ time-periodic parallel flow $(\rho_p, v_p) = (\rho_p(x_n), v_p(x_n, t))$ s.t.

$$v_p(x_n, t) = (v_p^1(x_n, t), 0, \cdots, 0), \;\; v_p^1(x_n, t+T) = v_p^1(x_n, t).$$

Similar results hold

$$u(t) \sim (\sigma u^{(0)})(t) + O(t^{-\frac{n-1}{4}+\varepsilon}) \text{ in } L^2(\Omega) \;(t \to \infty).$$

Here $\sigma = \sigma(x', t)$ as before; $u^{(0)} = u^{(0)}(x_n, t)$: $u^{(0)}(x_n, t+T) = u^{(0)}(x_n, t)$.

Instability of parallel flow

$u_s = {}^\top(\rho_s, v_s)$: Poiseuille flow

Theorem 2 (T. Nishida - Y.K., 2014)

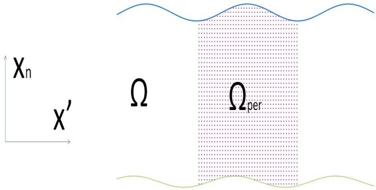
If $\gamma^2 < \frac{1}{280}$ *and* $2\nu^2 + \nu\tilde{\nu} \le 30\gamma^2 \left(\frac{1}{280} - \gamma^2\right)$,
$\Longrightarrow \sigma(-L) \cap \{\mathrm{Re}\,\lambda > 0\} \neq \emptyset$, *i.e., Poiseuille flow* $u_s = {}^\top(\rho_s, v_s)$ *is linearly unstable.*

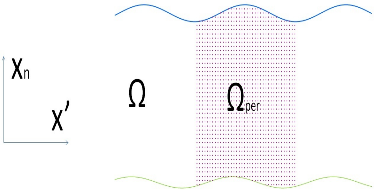
Remarks.

- Incompressible case: Critical Reynolds number $\mathrm{R_c} \sim 5772$.
- Let, for example, $\gamma = 0.05$, $\nu = 1/173$ and $\tilde{\nu} = \nu/3$. Then, by Theorem 2, plane Poiseuille flow is unstable. In this case, the Reynolds number $\mathrm{R} = 1/(16\nu) \sim 10.81$ and the Mach number $\mathrm{M} = 8/\gamma = 160$.

2. Asymptotic behavior on spatially periodic layer

- Ω: Periodic Layer:





$$\Omega = \{x = (x', x_n); x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, \omega_1(x') < x_n < \omega_2(x')\}$$

Here

$$\omega_1(x'),\ \omega_2(x'):\ \text{smooth},\ Q\text{-periodic in } x',\ Q = \Pi_{j=1}^{n-1}\left[-\tfrac{\pi}{\alpha_j}, \tfrac{\pi}{\alpha_j}\right),$$

$$\text{i.e.,}\ \ \omega_1(x' + \frac{2\pi}{\alpha_j}\boldsymbol{e}'_j) = \omega_1(x'),\ \ \omega_2(x' + \frac{2\pi}{\alpha_j}\boldsymbol{e}'_j) = \omega_2(x'),$$

$$\boldsymbol{e}'_j = {}^\top(0, \cdots, \overset{j}{1}, \cdots, 0), \qquad (j = 1, \cdots, n-1),$$

$$\Omega_{per} = \{x = (x', x_n); x' \in Q, \omega_1(x') < x_n < \omega_2(x')\}.$$

- Asymptotic behavior around the motionless state $u_s = {}^\top(\rho_*, 0)$

<u>Linearized Problem</u>

$$\partial_t u + Lu = 0, \quad u|_{t=0} = u_0.$$

Here $u = {}^\top(\phi, w)$, $L : L^2(\Omega) \to L^2(\Omega)$,

$$D(L) = \{u = {}^\top(\phi, w) \in L^2(\Omega); w \in H_0^1(\Omega),\ Lu \in L^2(\Omega)\},$$

$$L = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla & -\nu \triangle - \tilde{\nu} \nabla \mathrm{div} \end{pmatrix}$$

- e^{-tL}: C_0 semigroup on $L^2(\Omega)$

Theorem 4 (N. Makio - Y.K.,2014)

e^{-tL} *is decomposed as*

$$e^{-tL} = e^{-tL}\Pi + e^{-tL}(I - \Pi),$$

where $\Pi^2 = \Pi$, $e^{-tL}\Pi = \Pi e^{-tL}$; *it holds that*

$$\begin{aligned} \|\partial_x^\alpha e^{-tL}\Pi u_0\|_{L^2(\Omega)} &\leq C(1+t)^{-\frac{n-1}{4}-\frac{|\alpha|}{2}}\|u_0\|_{L^1(\Omega)}, \ (|\alpha| = 0, 1), \\ \|e^{-tL}(I - \Pi)u_0\|_{L^2(\Omega)} &\leq Ce^{-\beta t}\|u_0\|_{L^2(\Omega)} \end{aligned}$$

with some $\beta > 0$.

Furthermore,

$$\|e^{-tL}\Pi u_0 - [e^{-tH}\sigma_0]u^{(0)}\|_{L^2(\Omega)} \le Ct^{-\frac{n-1}{4}-\frac{1}{2}}\|u_0\|_{L^1(\Omega)}.$$

Here

$$H\sigma = -\frac{\gamma^2}{\nu}\sum_{j,k=1}^{n-1} a_{jk}\partial_{x_j}\partial_{x_k}\sigma,$$

$$\sum_{j,k=1}^{n-1} a_{jk}\xi_j\xi_k \ge \kappa_0|\xi'|^2 \ (\xi' \in \mathbb{R}^{n-1})$$

with some $\kappa_0 > 0$*,*

$$\sigma_0(x') = \int_{\omega_1(x')}^{\omega_2(x')} \phi_0(x', x_n)dx_n, \quad u^{(0)} = {}^\top(1,0).$$

a_{jk} $(j, k = 1, \cdots, n-1)$ *are given by*

$$a_{jk} = \frac{1}{|\Omega_{per}|} (\nabla w^{(j)}, \nabla w^{(k)})_{L^2(\Omega_{per})},$$

where $w^{(\ell)}$ $(\ell = 1, \cdots, n-1)$ *satisfy:*

$$\begin{cases} \operatorname{div} w^{(\ell)} & = & 0 \ \text{ in } \ \Omega_{per}, \\ -\triangle w^{(\ell)} + \nabla \phi^{(\ell)} & = & \boldsymbol{e}_\ell \ \text{ in } \ \Omega_{per}, \\ w^{(\ell)}, \ \phi^{(\ell)} : & & Q\text{-periodic in } x', \\ w^{(\ell)}|_{x_n = \omega_\ell(x')} & = & 0 \ \ (x' \in Q, \ \ell = 1, 2), \ < \phi^{(\ell)} >= 0 \end{cases}$$

with some $\phi^{(\ell)}$*.*

Nonlinear Problem

$$\partial_t u + Lu = F(u), \quad u|_{t=0} = u_0. \tag{1}$$

Theorem 5

$$m \in \mathbb{Z}, \quad m \geq [n/2] + 1,$$
$$u_0 \in H^m(\Omega) \cap L^1(\Omega) \ + \ \textit{c.c.}, \quad \|u_0\|_{H^m(\Omega)\cap L^1(\Omega)} \ll 1.$$

$\Rightarrow \exists 1 \ u \in C([0,\infty); H^m(\Omega))$ *: solution of* (1);

$$\|\partial_x^\alpha u(t)\|_{L^2(\Omega)} \leq C(1+t)^{-\frac{n-1}{4}-\frac{|\alpha|}{2}} \|u_0\|_{H^m(\Omega)\cap L^1(\Omega)},$$
$$\|u(t) - [e^{-tH}\sigma_0]u^{(0)}\|_{L^2(\Omega)} \leq Ct^{-\frac{n-1}{4}-\frac{1}{2}} \zeta(t) \|u_0\|_{H^m(\Omega)\cap L^1(\Omega)}.$$

Here $|\alpha| = 0, 1$, $\zeta(t) = 1$ $(n \geq 3)$ *and* $\zeta(t) = \log(1+t)$ $(n = 2)$.

Stability of spatially periodic stationary solutions

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div}\, v + \nabla P(\rho) = \rho g, \\ v|_{\partial\Omega} = 0, \\ (\rho, v)|_{t=0} = (\rho_0, v_0). \end{cases}$$

Here $g = g(x', x_n)$: Q-periodic in x'.

If $\|g\|_{H^m(\Omega_{per})} \ll 1$,
$\Longrightarrow \exists$ stationary solution $u_s = {}^\top(\rho_s, v_s)$: Q-periodic, $u_s \sim {}^\top(\rho_*, 0)$.

Theorem 6 (S. Enomoto - Y.K., 2014)

If $\mathrm{Re} \ll 1$ *and* $\mathrm{Ma} \ll 1$, *then similar results hold for the linearized semigroup around* u_s.